Short extenders forcings together with Supercompact Prikry and with Levy Collapses.

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Abstract

We study the PCF structure of models obtained as a combination of the gap 2 short extenders forcing with the supercompact Prikry forcing. Effects of combinations of the gap 2 short extenders forcing with Levy collapses are studied as well.

1 Basic setting for the gap 2 case.

Assume GCH. Let κ be a limit of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega \rangle$ so that for every $n < \omega$, κ_n carries an extender E_n such that

1. if $j_n : V \to M_n \simeq Ult(V, E_n)$, then $M_n \supseteq H(\kappa_n^{+n+2})$,
2. the normal measure $E_n(\kappa_n)$ of E_n concentrates on the set $\{\nu < \kappa_n \mid \nu$ is a ν^{+n+1}–supercompact cardinal$\}$.

Note that the above conditions imply that κ_n is a κ_n^{+n+1}–supercompact cardinal. Fix a normal ultrafilter U_n over $\mathcal{P}_{\kappa_n}(\kappa_n^{+n+1})$ and for every ν^{+n+1}–supercompact cardinal $\nu < \kappa_n$ fix a normal ultrafilter $U_{n\nu}$ over $\mathcal{P}_\nu(\nu^{+n+1})$.

We would like simultaneously blow up the power of κ to κ^{++} and to use the supercompact Prikry forcing to change cofinality of every ρ_n^{+k} to ω, for every $n < \omega$ and $k \leq n+1$, where $\rho_n (n < \omega)$ is the one element Prikry sequence for $E_n(\kappa_n)$. Our special attention will be to the resulting PCF–configuration.

2 Definition of the forcing at level n.

Fix $n < \omega$. As usual we first define Q_{n0} and Q_{n1}. The final Q_n will be their union.

Definition 2.1 Q_{n0} consists of all p of the form $\langle a, A, f, F\rangle$, where

1. the triple $\langle a, A, f\rangle$ is as in the gap 2 short extenders forcing,
2. $F : A \to V_{\kappa_n}$ is so that $F(\nu) \in U_{n\nu^0}$, for every $\nu \in A$.
   Note that then $[F]_{E_n} \in U_n$.

Define the order on Q_{n0}.

Definition 2.2 Let $\langle a, A, f, F\rangle, \langle a', A', f', F'\rangle \in Q_{n0}$. Set $\langle a, A, f, F\rangle \geq_{Q_{n0}} \langle a', A', f', F'\rangle$ iff

1. $a \supseteq a'$
2. $f \supseteq f'$,
3. $\pi_{\max(\mathrm{rng}(a)),\max(\mathrm{rng}(a'))}{}''A \subseteq A'$,
4. $F'(\pi_{\max(\mathrm{rng}(a)),\max(\mathrm{rng}(a'))}(\nu) \supseteq F(\nu)$, for every $\nu \in A$.

Definition 2.3 Q_{n1} consists of all triples $\langle f, \rho, t\rangle$ such that

1. $f : \kappa^{++} \to \kappa_n$ is a partial function of cardinality $\leq \kappa$,
2. $\rho < \kappa_n$ is a ρ^{+n+1}–supercompact cardinal,
3. t is a condition in the supercompact Prikry forcing with $U_{n\rho}$.

The order over Q_{n1} is defined in obvious fashions (following the orders of gap 2 short extenders and the supercompact Prikry).

Definition 2.4 $Q_n = Q_{n0} \cup Q_{n1}$.

Define now two orders $\leq, \leq^*$ over Q_n.

Definition 2.5 Set $\leq^* = \leq_{Q_{n0}} \cup \leq_{Q_{n1}}$.

Definition 2.6 Set $p \leq q$ iff

1. $p, q \in Q_{n0}$ and $p \leq_{Q_{n0}} q$,
   or
2. $p, q \in Q_{n1}$ and $p \leq_{Q_{n1}} q$,
   or

3. $p = \langle a, A, f, F\rangle \in Q_{n0}$, $q = \langle g, \rho, t\rangle \in Q_{n1}$ and the following holds for some $\nu \in A$

   (a) $\nu^0 = \rho$,

   (b) t is stronger than $F(\nu)$ in the supercompact Prikry forcing,

   (c) g is stronger than $\langle a, A, f\rangle^\frown \nu$ in the gap 2 short extender forcing.

3 Main forcing.

Combine now all Q_n's together.

Definition 3.1 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega\rangle$ such that there is $\ell(p) < \omega$

1. for every $n < \ell(p)$, $p_n \in Q_{n1}$,

2. for every $n \geq \ell(p)$, $p_n \in Q_{n0}$.

   We modify the forcing $\mathcal{P}$ from above a bit as it is done usually. In addition to previous properties demanded of $a_n(p)$ we require the following:

3. for all $\alpha \in dom(a_n(p)), a_n(p)(\alpha)$ is at least 2-good

4. if for some $\alpha < \kappa^{++}$ there is $i \geq \ell(p)$ such that $\alpha \in \mathrm{dom}(a_i(p))$, then there is a nondecreasing sequence converging to infinity sequence $\langle k_m \mid i \leq m < \omega\rangle$ such that for every $m \geq i, a_m(p)(\alpha)$ is k_m-good.

Define the orders $\leq, \leq^*$ on $\mathcal{P}$.

Definition 3.2 Let $p, q \in \mathcal{P}$. Set $p \geq q$ iff for every $n < \omega$ we have $p_n \geq_{Q_n} q_n$, and $p \geq^* q$ iff in addition $\ell(p) = \ell(q)$.

Lemma 3.3 *The forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. The argument is like those for gap 2 short extenders with a small addition which uses the Prikry condition of the supercompact Prikry.
□

By standard means now the following holds:

Lemma 3.4 *The cardinal structure after the forcing with $\langle \mathcal{P}, \leq \rangle$ is as follows:*

1. *all the cardinals above κ^{++} are preserved,*

2. *κ, κ^+ are preserved,*

3. *if $\langle \rho_n \mid n < \omega \rangle$ is the generic sequence corresponding to the normal measures of extenders E_n's, then, for every $n < \omega$, each of the cardinals in the interval $[\rho_n, \rho_n^{+n+1}]$ changes its cofinality to ω and all the cardinals of the interval $(\rho_n, \rho_n^{+n+1}]$ fall down to ρ_n.*

4. *No other cardinal below κ changes its cofinality.*

Define the equivalence relation $\longleftrightarrow$ on $\mathcal{P}$ following [3].

Definition 3.5 Let $p, p' \in \mathcal{P}$. Set $p \longleftrightarrow p'$ iff

1. $\ell(p) = \ell(p')$,

2. for every $n < \ell(p)$, $p_n = p'_n$,

3. there is a non-decreasing converging to infinity sequence $\langle k_n \mid n \geq \ell(p) \rangle$ such that for every $n \geq \ell(p)$ the following holds:

   (a) $\langle A_n(p), f_n(p), F_n(p) \rangle = \langle A_n(p'), f_n(p'), F_n(p') \rangle$,

   (b) $\mathrm{dom}(a_n(p)) = \mathrm{dom}(a_n(p'))$,

   (c) $\mathrm{rng}(a_n(p)), \mathrm{rng}(a_n(p'))$ realize the same k_n-type.

Remark 3.6 1. A crucial component of the construction is that F_n's are conditions in the supercompact Prikry forcing (actually sets of measures one) with the choice of ultrafilter decided by measures over maximal coordinates. So once we have two conditions with same measures over maximal coordinates, then the corresponding F_n's are compatible.

2. It is nothing special here in the particular choice of the supercompact Prikry forcing. Any Prikry type forcing notion $\langle Q, \leq, \leq^* \rangle$ can be used instead as well provided that it satisfies the following compatibility condition:

(*) for every $s \in Q$ and for every $s_1, s_2 \geq^* s$ there is $t \geq^* s_1, s_2$.

For example, Magidor, Radin forcings and their supercompact versions fall under this category.

3. It is important that the supercompact measures act over indiscernibles and not over κ_n's them self, since otherwise the number of types will be too big, and which in turn will prevent to define an equivalence relation $\longleftrightarrow$ effectively.

Define next the main forcing relation $\longrightarrow$ as in [3].

Definition 3.7 Let $p, q \in \mathcal{P}$. Define $p \longrightarrow q$ if and only if there is an $m < \omega$ and a sequence of elements $\langle r_i \mid i < m\rangle$ such that

1. $r_0 = p$,

2. $r_{m-1} = q$,

3. for each $i < m-1$, either $r_i \leq r_{i+1}$ or $r_i \longleftrightarrow r_{i+1}$.

By [3], $\langle \mathcal{P}, \longrightarrow \rangle$ is a projection of $\rangle \mathcal{P}, \leq \rangle$.
Now we have the following:

Lemma 3.8 *$\langle \mathcal{P}, \longrightarrow \rangle$ satisfies κ^{++}–c.c.*

Proof. Repeats the standard argument see [3] or [5]. The part of the supercompact Prikry forcing is defined on sets of measures one of extenders and stay the same in equivalent conditions.
□

4 The PCF structure.

Let us study now the resulting PCF structure in a generic extension $V[G]$ of V by $\langle \mathcal{P}, \longrightarrow \rangle$. Denote by $\langle \rho_n \mid n < \omega\rangle$ the generic sequence corresponding to the normal measures of extenders E_n's.

We obtain the following:

Theorem 4.1 *In $V[G]$ there is an increasing sequence $\langle \eta_n \mid n < \omega\rangle$ such that*

1. *η_n is the immediate successor of a cardinal of cofinality ω, for every $n < \omega$,*

2. *$\bigcup_{n<\omega} \eta_n = \kappa$,*

3. *$tcf(\prod_{n<\omega} \eta_n / finite) = \kappa^{++}$.*

Proof. As usual $tcf(\prod_{n<\omega}(\rho_n^{+n+2})^V/finite) = \kappa^{++}$, but now $(\rho_n^{+n+2})^V = (\rho_n^+)^{V^{\langle \mathcal{P}, \longrightarrow \rangle}}$ and $\mathrm{cof}(\rho_n) = \omega$, by 3.4. Set $\eta_n = \rho_n^+$.

$\square$

Let us refer to $(\rho_n^+)^{V^{\langle \mathcal{P}, \longrightarrow \rangle}}$ as ρ_n^+. Hence a final segment of cardinals below each ρ_n will be in the generator $\mathfrak{b}_{\rho_n^+}$, since GCH holds below κ. By closure (transitivity) of the PCF generators (see [6] or [1]) then this final segments will be in $\mathfrak{b}_{\kappa^{++}}$.

Let us analyze the situation more carefully.

Denote by $\langle P_{nm} \mid m < \omega \rangle$ the generic supercompact Prikry sequence of members of $\mathcal{P}_{\rho_n}((\rho_n^{+n+1})^V)$, for every $n < \omega$. Assume for simplicity that $P_{nm} \cap \rho_n$ is an inaccessible,$|P_{nm}| < P_{nm+1} \cap \rho_n$, $\mathrm{otp}(P_{nm}) = (P_{nm} \cap \rho_n)^{+n+1}$ and $P_{nm} \subset P_{nm+1}$, for every $n, m < \omega$.

Lemma 4.2 $\mathrm{cof}(\prod_{n<\omega} \eta_n/finite) = \kappa^+$*, for any sequence of regular cardinals* $\langle \eta_n \mid n < \omega \rangle \in \prod_{0<n<\omega}(\rho_{n-1}, \mathrm{otp}(P_{n0})]$.

Proof. Thus, let, for example, $\eta_n = (P_{n0} \cap \rho_n)^{+n+1}$. Then it corresponds to ρ_n^{+n+1} (represented by $P \mapsto \mathrm{otp}(P)$), which in turn corresponds to κ_n^{+n+1}. Also $\mathrm{cof}(\prod_{n<\omega} \kappa_n^{+n+1}/finite) = \kappa^+$.

$\square$

Lemma 4.3 $\mathrm{cof}(\prod_{n<\omega} \eta_n/finite) = \kappa^{++}$*, for any sequence of regular cardinals* $\langle \eta_n \mid n < \omega \rangle \in \prod_{n<\omega}[\mathrm{otp}(P_{n0})^+, \rho_n)$.

Proof. The point is that every such η_n is a value of a function which represents a regular cardinal η'_n between ρ_n^{+n+2} and $j'_n(\rho_n)$ in a finite iterated ultrapower with $U_{n\rho_n}$ or its finite power used as the largest ultrafilter, where j'_n denotes the corresponding elementary embedding. The real cofinality (i.e. the one computed in V) η'_n must be ρ_n^{+n+2}, since $U_{n\rho_n}$ is a normal ultrafilter on $\mathcal{P}_{\rho_n}(\rho_n^{+n+1})$.

Recall that $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+2}/finite) = \kappa^{++}$.

Combine corresponding scales.

$\square$

It was remarked in 3.6(2) that the supercompact Prikry forcing can be replaced by the supercompact Magidor. So the following analog of Theorem 4.4 holds:

Theorem 4.4 *Let* $\langle \delta_n \mid n < \omega \rangle$ *be a sequence of regular cardinals in* $\prod_{n<\omega} \kappa_n$*. Then there is a forcing extension with an increasing sequence* $\langle \eta_n \mid n < \omega \rangle$ *such that*

1. η_n *is the immediate successor of a cardinal of cofinality* δ_n*, for every* $n < \omega$*,*

2. $\bigcup_{n<\omega} \eta_n = \kappa$,

3. $tcf(\prod_{n<\omega} \eta_n / finite) = \kappa^{++}$.

Proof. Just use at a level $n < \omega$ the supercompact Magidor forcing which changes cofinality to δ_n instead of the supercompact Prikry forcing.
$\square$

5 A bit different setting.

Let us now collapse not only ρ_n^{+n+1}, but also ρ_n^{+n+2} (which corresponds to κ^{++}) by applying the supercompact Prikry forcing.

Use now a normal ultrafilter U_n over $\mathcal{P}_{\kappa_n}(\kappa_n^{+n+2})$ and for every ν^{+n+2}–supercompact cardinal $\nu < \kappa_n$ a normal ultrafilter $U_{n\nu}$ over $\mathcal{P}_\nu(\nu^{+n+2})$. Define Q_n's, $\mathcal{P}, \leq, \leq^*, \longleftrightarrow$ etc. as before, but with new $U_{n\nu}$'s. Lemmas of Section 3 are valid in the present context only with an obvious change in the last item of Lemma 3.4.

Let us analyze the resulting PCF structure. Note that now each ρ_n^{+n+2} has cofinality ω, so the question is which cardinals correspond to κ^{++}.

Lemma 5.1 $\mathrm{cof}(\prod_{n<\omega} otp(P_{n0})/finite) = \kappa^{++}$.

Proof. Let $\langle g_\alpha \mid \alpha < \kappa^{++}\rangle$ be the generic increasing mod finite sequence of functions (ω–sequences) added by G in $\prod_{n<\omega}(\rho_n^{+n+2})^V$. Define $\langle h_\alpha \mid \alpha < \kappa^{++}\rangle$ in $\prod_{n<\omega} otp(P_{n0})$.

Set $h_\alpha(n) = \begin{cases} otp(P_{n0} \cap g_\alpha(n)), & \text{if } g_\alpha(n) \in P_{n0}; \\ 0, & \text{otherwise.} \end{cases}$

Note that for all but finitely many n's the former possibility holds. Just we can always shrink the supercompact sets of measure one of P's such that $a_n(\alpha) \in P$.

Let us argue that such defined sequence $\langle h_\alpha \mid \alpha < \kappa^{++}\rangle$ is a scale.
Suppose $f \in \prod_{n<\omega} otp(P_{n0})$.
Using the standard arguments on Prikry forcings find a function $\langle f_n^* \mid n < \omega\rangle \in V$ such that for all but finitely many $n < \omega$ the following holds:

1. $\mathrm{dom}(f_n^*) \in U_n$,

2. for every $P \in \mathrm{dom}(f_n^*)$, $f_n^*(P) \in otp(P)$,

3. $f(n) \leq f^*_n(P_{n0})$.

Denote by $\bar{P}$ the transitive collapse of a set P of ordinals and by $\pi_P : P \to \bar{P}$ the collapsing function.
Set $\hat{f}^*_n(P) = \pi_P^{-1}(f^*_n(P))$, for all $P \in \mathrm{dom}(f^*_n)$. Let $\gamma_n = [\hat{f}^*_n]_{U_n}$. Then $\langle \gamma_n \mid n < \omega \rangle$ will be a sequence in $\prod_{n<\omega} \kappa_n^{+n+2}$ and in V. So there will be $\alpha < \kappa^{++}$ and $p = \langle p_k \mid k < \omega \rangle \in G$ with $\alpha \in \mathrm{dom}(a_n(p))$ and $a_n(p)(\alpha) > \gamma_n$, for every $n \geq \ell(p)$.
This already insures $h_\alpha(n) > f(n)$, for all but finitely many $n < \omega$.
$\square$

The next lemma is similar to 4.2

Lemma 5.2 $\mathrm{cof}(\prod_{n<\omega} \eta_n / finite) = \kappa^+$, *for any sequence of regular cardinals* $\langle \eta_n \mid n < \omega \rangle \in \prod_{0<n<\omega} [\mathrm{otp}(P_{n-1,0})^+, \mathrm{otp}(P_{n0}))$.

6 Levy collapses instead of Supercompact Prikry.

Let us briefly discuss effects of Levy collapses on PCF structure.

Conditions in Q_{n0} are of the form $\langle a, A, f, F \rangle$ as in Definition 2.1 with the second item changed as follows:

(2)' $F : A \to V_\kappa$ is so that $F(\nu) \in Col(\xi_n(\nu), < \zeta_n(\nu))$, for every $\nu \in A$, where $\xi_n(\nu), \zeta_n(\nu)$ are some cardinals below κ.

We will see below how a choice of $\xi_n(\nu), \zeta_n(\nu)$ effects the resulting PCF structure.
Let us assume that always $\zeta_n(\nu) \leq \rho^0_{n+1}$, i.e. $F(\nu)$ may be in $Col(\xi_n(\nu), < \kappa_{n+1})$, but once ρ_n is picked then all the members of the measure one set A_{n+1} of the next level should be above $F(\rho_n)$.
This restriction insures the Prikry condition.

Lemma 6.1 *The forcing* $\langle \mathcal{P}, \leq, \leq^* \rangle$ *satisfies the Prikry condition.*

6.1 Dependence on normal measures only.

We start with the case when $\xi_n(\nu), \zeta_n(\nu)$ depend only on ν^0, i.e. the projection of ν to the normal measure $E(\kappa_n)$ of the extender E_n.
The main case here involves $Col(\rho_n^{+n+3}, < \rho_{n+1})$, for all $n < \omega$. In particular, $\kappa_n, \kappa_n^+, ..., \kappa_n^{+n+2}$ are collapsed to ρ_n^{+n+3}.

This corresponds to $Col((\kappa_n^{+n+3})^{N_n}, < \kappa_{n+1})$ in N_n, where N_n is the ultrapower by the normal measure of the extender E_n. Note that $\longleftrightarrow$ is not effected since the collapses depend only on the normal measures and they are fixed.
By standard arguments only the desired cardinals are collapsed here and
$\mathfrak{b}_{\kappa^{++}} = \{\rho_n^{+n+2} \mid n < \omega\}$.

It is problematic to allow $\xi_n(\nu) < \rho_n^{+n+3}$. Thus the Prikry condition will brake down if one requires that all the elements of Col depend on the normal measure. In general, once every (or even unboundedly many of them) ρ_n^{+n+2} is collapsed to ρ_n^{+n+1} or even if they have the same cardinality in the extension, then κ^{++} will collapse as well. Just similarity of types (used in $\longleftrightarrow$) would be lost due to conditions in the collapses. Thus for example suppose that we would like to combine two conditions over a level n such that the assignment functions are have the same range, domain of the first consists of γ, α of the second of γ, β with the image of γ moved to the image of α (which is the same as those of β) by the collapse part of both conditions. It is impossible to put together such conditions.

6.2 Dependence on generators $\leq \kappa_n^{+n+1}$–the principal case.

The typical case here is $Col(\rho_n^{+n+3}, \rho_{n+1}^{+n+2})$. This corresponds to $Col(\rho_n^{+n+3}, \kappa_{n+1}^{+n+2})$ in the ultrapower M_{n+1} of V by the extender E_{n+1}. The strength of E_{n+1} implies that
$(Col(\rho_n^{+n+3}, \kappa_{n+1}^{+n+2}))^{M_{n+1}} = (Col(\rho_n^{+n+3}, \kappa_{n+1}^{+n+2})^V$.
It is tempting to argue that the forcing $\langle \mathcal{P}, \longrightarrow \rangle$ still satisfies κ^{++}–c.c. The point is that the part of collapses inside conditions does not reach κ_{n+1}^{+n+1+2}. So conditions with same collapsing parts can still be combined together as in the usual proof of κ^{++}–c.c. of the forcing $\langle \mathcal{P}, \longrightarrow \rangle$. But actually there is a problem with $\longleftrightarrow$ at an early stage. Let us illustrate it in a simpler but a principal case.

Deal with a simpler setting. Thus let us collapse ρ_n^{+n+1} to ρ_n^+, for every n. Again, this corresponds to $Col(\rho_n^+, \kappa_n^{+n+1})$ in the ultrapower M_n of V by the extender E_n. The strength of E_n implies that $(Col(\kappa_n^+, \kappa_n^{+n+1}))^{M_n} = (Col(\kappa_n^+, \kappa_n^{+n+1})^V$. Let us review possible ways to define $\longleftrightarrow$.
First try likely to be to require two equivalent conditions have the same collapsing parts. But then it would be impossible to combine together an equivalent conditions unless they are identical or collapsing parts are trivial, since the collapses start with κ_n^+'s and there are no ways to identify ordinals below κ_n^+.
Another, seemingly more promising approach, would be to require that equivalent conditions are completely identical below κ_n^{+n+1}'s. The problem then will be that the forcing $\langle \mathcal{P}, \longrightarrow \rangle$

would not then a (complete) subforcing of $\langle \mathcal{P}, \leq \rangle$. Thus let $p = \langle p_n \mid n < \omega \rangle$ be a condition with $\ell(p) = 0$ such that for every $n < \omega$, we have $\mathrm{dom}(a_n(p)) = \{\gamma', \gamma, \alpha\}$, for some $\gamma' < \kappa^+, \gamma, \alpha \in (\kappa^+, \kappa^{++}), \gamma < \alpha$. Denote $\gamma'_n := a_n(p)(\gamma')$,$\gamma_n := a_n(p)(\gamma)$ and $\alpha_n := a_n(p)(\alpha)$. Suppose that γ_n is the γ'_n-th member of the least (in some fixed well ordering) cofinal in α_n sequence.

Now suppose that we like to replace p by an equivalent condition by replacing only γ to some δ. So we should replace each of γ_n's to some δ_n's. But remember that γ'_n cannot be changed since it is below κ_n^{+n+1} and no changes are allowed there. This makes the tusk into an impossible one.

6.3 Dependence on generators $\leq \kappa_n^{+n+2}$.

This is a problematic situation. Consider a likely simplest case – $Col(\rho_n^{+n+2}, \rho_n^{+n+3})$. It corresponds to $(Col(\kappa_n^{+n+2}, \kappa_n^{+n+3}))^{M_n}$. But note that $(\kappa_n^{+n+3}))^{M_n}$ is an ordinal of cofinality κ_n^+. So, forcing with $(Col(\kappa_n^{+n+2}, \kappa_n^{+n+3}))^{M_n}$ over V will collapse everything to κ_n^+.

In our setting, a generic for $(Col(\kappa_n^{+n+2}, \kappa_n^{+n+3}))^{M_n}$ over V will not be created since only $Col(\rho_n^{+n+2}, \rho_n^{+n+3})$ is used, once ρ_n was decided. However, κ^{++} will be collapsed since $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+3}/finite) = \kappa^+$.

7 Cohen forcings and $\longleftrightarrow$.

Let us analyze the effects of adding Cohen subsets to $\rho_n^{+k_n}, n < \omega, k_n \leq n+2$.

First note that once $k_n \leq n+1$, then $\longleftrightarrow$ is not effected. It is similar to the situation in Levy Collapses case.

Suppose now that we add a Cohen subset to ρ_n^{+n+2}. Then the forcing $\langle \mathcal{P}, \longrightarrow \rangle$ will not satisfy κ^{++}–c.c. anymore by the same reason as in Subsection 6.1. Moreover it is not hard to find a function from κ^+ onto $(\kappa^{++})^V$ in $V^{\langle \mathcal{P}, \longrightarrow \rangle}$. Proceed as follows:

let $\langle t_\alpha \mid \alpha < (\kappa^{++})^V \rangle$ be the generic scale in $\prod_{n<\omega} \rho_n^{+n+2}/finite$. Define $h : \kappa^+ \to (\kappa^{++})^V$. Let $h(\gamma) = \min\{\alpha < (\kappa^{++})^V \mid \forall n < \omega \quad s_n(t_\gamma(n)) < t_\alpha(n)\}$, where $s_n : \rho_n^{+n+2} \to \rho_n^{+n+2}$ is a Cohen generic function. Then, using density arguments, it follows that $\mathrm{rng}(h)$ is unbounded in $(\kappa^{++})^V$.

We need to produce M_n-generic subset of the corresponding forcing over κ_n in order to gain κ^{++}–c.c. of $\langle \mathcal{P}, \longrightarrow \rangle$. Actually forcing over V is needed for this, but may be with a different forcing notion. Thus, for example, once dealing in M_n with adding κ_n^{+n+3}–Cohen subsets to κ_n^{+n+2}, the corresponding forcing over V will have a different support, since

$\mathrm{cof}((\kappa_n^{+n+3})^M) = \kappa_n^+$.
Define $\longleftrightarrow$ only after having such generic object by including it into the structure over which types are defined.

8 Collapses as subforcings of the supercompact Prikry.

We would like to argue here that it is still possible to implement Levy collapses (like those of 6.2) into the frame of short extenders forcings without unwanted effects. The limit taken here will be different. We will use the assumptions of Section 1 on existence of supercompacts.
The point is that once ν is a supercompact cardinal then any ν-distributive forcing Q is a subforcing of the supercompact Prikry forcing with a normal ultrafilter over $\mathcal{P}_\nu(2^{|Q|})$, see for example [2].
So suppose that for every $n < \omega$ we a have a κ_n–distributive forcing Q_n of relatively small cardinality. Define the set of conditions as in Section 1, only once a non-direct extension is made- say ρ_n is decided, then instead of applying the supercompact forcing over ρ_n– we use the its projection which corresponds to Q_n.
Still, for a final segment of $k < \omega$, supercompact parts over κ_k are pure, i.e. of the form the empty sequence with a measure one set. This allows to define $\longleftrightarrow$ and argue that $\langle \mathcal{P}, \longrightarrow \rangle$ is a subforcing of $\langle \mathcal{P}, \leq \rangle$ exactly as in Section 1.

In particular, we can use $Q_n = Col(\kappa_n^+, \kappa_n^{+n+1})$ and even $Q_n = Col(\kappa_n^+, \kappa_n^{+n+2})$.
This last forcing will produce a bit mysterious Pcf-configuration. Thus $\prod_{n<\omega} \rho^{+n+2}/finite$ would not give anymore κ^{++}. Rather, as in Section 5, indiscernibles for supercompacts below ρ_n's will be generated and they will correspond to κ^{++}.

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