A model with a precipitous ideal, but no normal precipitous ideal.

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Abstract

Starting with a measurable cardinal κ of the Mitchell order κ^{++} we construct a model with a precipitous ideal on $\aleph_1$ but without normal precipitous ideals. This answers a question by T. Jech and K. Prikry. In the constructed model there are no Q–point precipitous filters on $\aleph_1$, i.e. those isomorphic to extensions of $Cub_{\aleph_1}$.

1 Introduction and Basic ideas

Precipitous ideals were introduced by T. Jech and K. Prikry [10]. A κ–complete ideal I on κ is *precipitous* if the generic ultrapower $V \cap {}^{\kappa}V/G$ is well-founded for every generic ultrafilter $G \subseteq I^+$. Precipitousness can be viewed as a weakening of measurability which is compatible with small cardinals.

Given a κ–complete ultrafilter U over a measurable κ there always exists a normal ultrafilter U^* over κ as well. Just take a function $f : \kappa \to \kappa$ which represents κ in the ultrapower by U, i.e. $[f]_U = \kappa$, and project U using f, which yields the normal ultrafilter $U^* := \{X \mid f^{-1\prime\prime}X \in U\}$ over κ. There are two obstacles that prevent implementation of the same approach to a precipitous filter F. The first is that there does not necessary exist a *single* function that represents κ in a generic ultrapower (the choice of such function may depend on particular condition, i.e. a set in F^+. In [5] an example of a precipitous filter without a normal filter below it in the Rudin-Keisler order was given. It is easy to fix this by simply restricting F to its positive set that decides a function f which represents κ in the ultrapower. The second much more serious obstacle is that the projection of F (or a

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restriction that decides f) by f need not in general be precipitous. The first example of this type was given by R. Laver [11] using a supercompact cardinal. Later in [5] we gave an example using only a measurable cardinal.

Let us briefly explain the idea used in this construction since it will be relevant for the present one. We started with a GCH model with a measurable κ and a normal ultrafilter U over it. Using the Backward Easton iteration (in order to preserve the measurability of κ) κ^+–many Cohen functions $\langle f_\beta \mid \beta < \kappa^+\rangle$ from κ to κ were added. A precipitous filter F was defined over κ^2 and its generic embedding extended i_{02} the second iterated ultrapower of U, i.e. $i_{01} : V \to M_1 \simeq {}^\kappa V/U, \kappa_1 = i_{01}(\kappa), i_{12} : M_1 \to M_2 \simeq {}^{\kappa_1} M_1/i_{01}(U)$, and $i_{02} = i_{12} \circ i_{01} : V \to M_2$. The projection F^* of F to a normal filter was not precipitous because for no one of the Cohen functions f_β could it be forced that $[f_\beta]$ is minimal in the generic ultrapower among the set $\{[f_\beta] \mid \beta < \kappa^+\}$. In the proof of these, it was critical that all of the functions were candidates to represent κ_1. It is not by chance that this f_β's were candidates for the function that represents κ_1. Further results starting with Section 4 of [5], then 2.4 of [6] and [7] suggest that the only ordinals which have a chance to produce an ill-foundness must be of the form κ_α (i.e. critical points of iterated ultrapowers).

On the other hand, if the number of critical points is too small (i.e. the length of the iteration is too short), say at most κ^+, then results of [6], [7] imply (at least under GCH-type assumptions and in absence of too large cardinals) that there will be always normal precipitous filters.

So it is natural to try the following:

Start with a normal ultrafilter U over κ and iterate it κ^{++}–many times. This will create critical points $\langle \kappa_\alpha \mid \alpha < \kappa^{++}\rangle$. Next add κ^{++} many blocks, each consisting of κ^+–many Cohen function from κ to κ. Arrange this (say by adding clubs) so that the functions of α-th block are the candidates to represent κ_α. Note that by J.-P. Levinski [12] no assumptions beyond measurability are needed in order to blow up the power and to preserve precipitousness. A problematic point is that his arguments and their extensions in [4] produce large (size κ^{++}) antichains which allow using the method of [6] to construct normal precipitous ideals.

A way around this obstacle will be to collapse in advance a measurable κ to $\aleph_1$ and to relay on $Col(\omega, <\kappa^{++})$ (which satisfies κ^{++}–c.c.) in order to generically extend the relevant embeddings (namely $i_{0\kappa^{++}}$).

An additional problem with this approach is that the models of the iteration (iterated ultrapowers) $M_\alpha, \alpha \leq \kappa^{++}$ are very unclosed. Thus already starting with M_ω we lose closure even under ω–sequences. For example $\langle \kappa_n \mid n < \omega\rangle \notin M_\alpha$ for every $\alpha, \omega \leq \alpha \leq \kappa^{++}$. This

turns out to be bad once we try to change values of Cohen functions in order to insure the right representations. Remember that α–th block should provide potential candidates for functions representing κ_α. Which makes values changing a crucial issue.
The way to gain the missing closure will be to switch from dealing with a single normal ultrafilter and its iterated ultrapowers to κ^{++}–many ultrafilters. We will use as an initial model the model from [3] which has a Rudin-Keisler increasing sequence of ultrafilters of length κ^{++}.

The actual construction will be as follows. We start with a model of [3] (assuming that there is a measurable κ of Mitchell order κ^{++}). Collapse κ to $\aleph_1$ and add κ^{++} many blocks of Cohen functions. Organize suitable generics using $Col(\omega, < \kappa^{++})$ and use them to define an ∞–semi precipitous filter over $\aleph_1$. Add clubs in order to turn it into $Cub_{\aleph_1}$ restricted to a certain set. Next argue that there are no normal precipitous filters on $\aleph_1$ (and, hence, if the construction was started with the core model for $o(\kappa) = \kappa^{++}$, no normal precipitous filters at all). Finally a precipitous filter on $\aleph_1$ will be constructed using methods of [6].

2 Construction of the model

Start with GCH model W and assume that for some κ there exists a coherent sequence of ultrafilters $\vec{U}$ with $o^{\vec{U}}(\kappa) = \kappa^{++}$ and $o^{\vec{U}}(\alpha) < \alpha^{++}$, for every $\alpha < \kappa$. We assume further for the purpose of the main result that W is the minimal model $L[\vec{U}]$ having a cardinal κ such that $o(\kappa) = \kappa^{++}$; however this assumption will not be used in most of the arguments below. The conclusion of such general setting will be only that there are no precipitous filters which extend $Cub_{\aleph_1} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$. [1]
By a coherent sequence $\vec{U}$ of ultrafilters in W we mean a function with domain of the form

$$\{(\alpha, \beta) \mid \alpha < \ell^{\vec{U}} \text{ and } \beta < o^{\vec{U}}(\alpha)\}.$$

For each pair $(\alpha, \beta) \in \mathrm{dom}(\vec{U})$,

1. $U(\alpha, \beta)$ is a normal ultrafilter on α, and
2. if $j^\alpha_\beta : W \to N^\alpha_\beta \simeq W^\alpha / U(\alpha, \beta)$ is the canonical embedding, then

$$j^\alpha_\beta(\vec{U}) \restriction \alpha + 1 = \vec{U} \restriction (\alpha, \beta).$$

[1] If F is a filter on a set X and A is F–positive set then $F + A$ denotes the extension of F generated by A, i.e. $F + A = \{B \subseteq X \mid \exists S \in F \quad B \supseteq A \cap S\}$.

We assume that $\ell^{\vec{U}} = \kappa + 1, o^{\vec{U}}(\kappa) = \kappa^{++}$ and for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \alpha^{++}$.

Force with the forcing of [3] and turn the sequence $\langle U(\kappa, \beta) \mid \beta < \kappa^{++} \rangle$ into a Rudin - Keisler increasing commutative sequence of Q–point ultrafilters $\langle U_\beta \mid \beta < \kappa^{++} \rangle$ over κ in a GCH cardinal preserving generic extension V of W.

Which means the following:

1. U_β is a κ–complete ultrafilter over κ in V,
2. U_0 is a normal ultrafilter over κ in V,
3. $U_\beta \supseteq U(\kappa, \beta)$,
4. $U_\beta \supseteq Cub_\kappa$ (this means that U_β is a Q–point),
5. if $\beta < \alpha < \kappa^{++}$, then there is *a projection function* $\pi_{\alpha\beta} : \kappa \to \kappa$, $U_\beta = \{\pi_{\alpha\beta}{}''X \mid X \in U_\alpha\}$ (this means that U_β is below U_α in the Rudin - Keisler order).

Denote by $M_{\kappa^{++}}$ the direct limit of the ultrapowers of $U_\beta, \beta < \kappa^{++}$. Let $i_{0\kappa^{++}} : V \to M_{\kappa^{++}}$ be the corresponding elementary embedding.

We have by [3], ${}^\kappa M_{\kappa^{++}} \subseteq M_{\kappa^{++}}$.

By elementarity, $M_{\kappa^{++}}$ is a generic extension of a model $\tilde{M}_{\kappa^{++}}$ such that $i_{0\kappa^{++}} \restriction W : W \to \tilde{M}_{\kappa^{++}}$. The model $\tilde{M}_{\kappa^{++}}$ is the complete iterated ultrapower of W by measures from $\vec{U}$.[2]

Denote by $\langle \kappa_\alpha \mid \alpha < \kappa^{++} \rangle$ the sequence of all critical points of such iteration. It is a closed unbounded subset of κ^{++}. For every $\alpha < \kappa^{++}$ define an ultrafilter $U'_\alpha = \{X \subseteq \kappa \mid \kappa_\alpha \in i_{0\kappa^{++}}(X)\}$.

For every $\alpha < \kappa^{++}$ let $M_{\alpha+1}$ be the transitive collapse of V^κ/U'_α and $i_{0\alpha+1}$ the corresponding elementary embedding. Set $M_0 = V$, $i_{00} = id$.

For a limit $\alpha \leq \kappa^{++}$ let M_α be the direct limit of $\langle M_\gamma \mid \gamma < \alpha \rangle$ and $\langle i_{\gamma\alpha} \mid \gamma < \alpha \rangle$ the corresponding elementary embeddings, i.e. $i_{\gamma\alpha} : M_\gamma \to M_\alpha$.

We have by [3], ${}^\kappa M_{\kappa^{++}} \subseteq M_{\kappa^{++}}$.

For every $\alpha \leq \kappa^{++}$, by elementarity, M_α is a generic extension of a model $\tilde{M}_\alpha$ such that $i_{0\alpha} \restriction W : W \to \tilde{M}_\alpha$. Models $\tilde{M}_\alpha$ are iterated ultrapowers of W by measures from $\vec{U}$. If $W = \mathcal{K}$, then $\tilde{M}_\alpha$ is the core model of M_α. Let us denote further $(o^{i_{0\alpha}(\vec{U})}(\delta))^{\tilde{M}_\alpha}$ simply by $(o(\delta))^{\tilde{M}_\alpha}$, for any ordinal δ and $\alpha \leq \kappa^{++}$.

[2]This means that we start with $U(\kappa, 0)$ then apply its image and so one ω–many times. At the next stage (i.e. at the stage ω the image of $U(\kappa, 1)$ is applied. Then again the image of $U(\kappa, 0)$ for ω–many steps and at the stage $\omega + \omega$ the image of $U(\kappa, 1)$ is used again and so on. The image of $U(\kappa, 2)$ is applied only after $U(\kappa, 1)$ was used ω–many times etc.

Collapse κ to $\aleph_1$ by $Col(\omega, <\kappa)$. Then let us add κ^{++} blocks of functions from κ to κ as follows: in $V^{Col(\omega,<\kappa)}$ set

$$Cohen(\kappa, \kappa^{++} \times \kappa^+) = \{f \mid |f| < \kappa, f \text{ is a partial function from } \kappa \times (\kappa^{++} \times \kappa^+) \text{ to } \kappa\}.$$

Let $G_1 \subseteq Col(\omega, <\kappa)$ be generic over V and $G_2 \subseteq Cohen(\kappa, \kappa^{++} \times \kappa^+)$ be a generic over $V[G_1]$. Set $\bar{G}_2 = \bigcup G_2$ and for every $\alpha < \kappa^{++}, \beta < \kappa^+, \nu < \kappa$ let $f_{\alpha\beta}(\nu) = \bar{G}_2(\nu, \alpha, \beta)$. Denote by

$$F_\alpha = \{f_{\alpha\beta} \mid \beta < \kappa^+\}.$$

This will be our α-th block of functions.

2.1 Constructing generics

Let show that the elementary embedding $i_{0\kappa^{++}} : V \to M_{\kappa^{++}}$ extends (generically). It is possible to use [4], but the construction there collapses κ^{++} which is bad for our purposes here. We will need to extend the embedding in a different fashion. One of the issues will be to generate an $M_{\kappa^{++}}^{Col(\omega,<\kappa^{++})}$-generic subset of $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$ using $Col(\omega, <\kappa^{++})$.

For each $\alpha < \kappa^{++}$ let us add only $(o(\kappa_\alpha))^{\tilde{M}_{\kappa^{++}}} < \kappa^{++}$ blocks of Cohen functions over $M_{\alpha+1}^{Col(\omega,<i_{0\alpha+1}(\kappa))}$. More generally $M_{\alpha+1}^{Col(\omega,<i_{0\alpha+1}(\kappa))}$–generic subsets of iterations of length $(o(\kappa_\alpha))^{\tilde{M}_{\kappa^{++}}}$ need to be constructed, since we will add also certain clubs further. Dealing with them is very similar, so let us concentrate on blocks of Cohen functions.

Let $P = i'_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$ or the image of a κ–support iteration of forcings of cardinality κ with closure properties, where $i'_{0\kappa^{++}} : V^{Col(\omega,<\kappa)} \to M_{\kappa^{++}}^{Col(\omega,<(\kappa^{++})^V)}$ is the obvious extension of $i_{0\kappa^{++}}$. Note that in $V^{Col(\omega,<\kappa)}$ we have $\kappa = \aleph_1$ and $\kappa^{++} = \aleph_3$. In order to simplify the notation, let us use $i_{0\kappa^{++}}$ to denote also $i'_{0\kappa^{++}}$ and by κ^{++} we will mean $(\kappa^{++})^V$.

We would like to construct an $M_{\kappa^{++}}^{Col(\omega,<\kappa^{++})}$-generic subset of $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$ in $V^{Col(\omega,<\kappa)*Cohen(\kappa,\kappa^{++}\times\kappa^+)*Col(\omega,[\kappa,\kappa^{++}))}$.

Let us first do some warm ups.

2.1.1 A single Cohen function.

Let us deal with a single Cohen function. Namely we would like to construct $f : \kappa^{++} \to \kappa^{++}$ which is a Cohen generic over $M_{\kappa^{++}}^{Col(\omega,<\kappa^{++})}$.

The construction will proceed by recursion, building $M_\alpha^{Col(\omega,<\kappa_\alpha)}$–generic Cohen function $f^\alpha : \kappa_\alpha \to \kappa_\alpha$, for every $\alpha \le \kappa^{++}$.

Set $f^0 = f_{00}$. Let $\alpha \le \kappa^{++}$. Assume that $\langle f^\gamma \mid \gamma < \alpha\rangle$ is defined and for every $\gamma' < \gamma < \alpha$ we have $f^\gamma \restriction \kappa_{\gamma'} = f^\gamma$. Define f^α.

Case 1. α is a limit ordinal.
Set then $f^\alpha = \bigcup_{\gamma<\alpha} f^\gamma$. Let us argue that such defined f^α is $M_\alpha^{Col(\omega,<\kappa_\alpha)}$–generic Cohen function. Let D be a dense set. Recall that M_α is a direct limit of $\langle M_\gamma \mid \gamma < \alpha\rangle$, since α is a limit ordinal. Then, for some $\gamma < \alpha$ and a dense subset D_γ of the Cohen forcing for κ_γ in $M_\gamma^{Col(\omega,<\kappa_\gamma)}$, $i_{\gamma\alpha}(D_\gamma) = D$. But $D_\gamma \subseteq (H(\kappa_\gamma))^{M_\gamma^{Col(\omega,<\kappa_\gamma)}}$ and κ_γ is the critical point of $i_{\gamma\alpha}$, hence $D_\gamma = D \cap (H(\kappa_\gamma))^{M_\gamma^{Col(\omega,<\kappa_\gamma)}}$. The function f^γ is Cohen generic, so it extends an element of D_γ. Then also f^α extends it and we are done.
We can assume using induction that f^α is definable from the sequence $\langle \kappa_\gamma \mid \gamma < \alpha\rangle$. This sequence belongs to $M_{\alpha+1}$.[3] Hence $f^\alpha \in M_{\alpha+1}^{Col(\omega,<\kappa_\alpha)}$.

Case 2. α is a successor ordinal.
Use $Col(\omega, ((\kappa_\alpha)^+)^{M_\alpha} + \kappa^+))$ to find $M_\alpha^{Col(\omega,<\kappa_\alpha)}$–generic Cohen function $f'^\alpha : \kappa_\alpha \to \kappa_\alpha$ in some canonical way. Then replace in it $f'^\alpha \restriction \kappa_{\alpha-1}$ by $f^{\alpha-1}$. Set f^α to be the result. Clearly f^α will be $M_\alpha^{Col(\omega,<\kappa_\alpha)}$–generic Cohen function, since $f^{\alpha-1} \in M_{\alpha+1}^{Col(\omega,<\kappa_\alpha)}$, and so it is a condition in the Cohen forcing.

2.1.2 The first block of Cohen functions.

Let us deal with the κ^+–Cohen function of the first block $F_0 = \{f_{0\beta} \mid \beta < \kappa^+\}$. Namely we would like to construct $f_\beta : \kappa^{++} \to \kappa^{++}, \beta < i_{0\kappa^{++}}(\kappa^+)$ which is a Cohen generic for $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^+)$ over $M_{\kappa^{++}}^{Col(\omega,<\kappa^{++})}$. We would like also to have $f_{i_{0\kappa^{++}}}(\beta) \restriction \kappa = f_{0\beta}$, for every $\beta < \kappa^+$, in order to be able to lift the embedding. Also we would like to spread generating parts of collapses a bit.

The construction will proceed by recursion, building $M_\alpha^{Col(\omega,<\kappa_\alpha)}$–generic Cohen functions $f_\beta^\alpha : \kappa_\alpha \to \kappa_\alpha$ for the forcing $i_{0\alpha}(Cohen(\kappa, \kappa^+)$, for every $\alpha \le \kappa^{++}, \alpha \ne 1$ and $\beta < \kappa^+$.

Case 1. $\alpha = 0$.
Set $f_\beta^0 = f_{0\beta}$, for every $\beta < \kappa^+$.

Case 2. $\alpha = 2$.
Define $f_\beta^2 : \kappa_2 \to \kappa_2$, for every $\beta < i_{02}(\kappa^+)$.
Clearly, $i_{02}(Cohen(\kappa, \kappa^+)) = (Cohen(\kappa_2, \kappa_2^+))^{M_2^{Col(\omega,<\kappa_2)}}$. It is a κ_2^+–c.c. forcing of size κ_2^+ in $M_2^{Col(\omega,<\kappa_2)}$. Use $Col(\omega, (\kappa_2^+)^{M_2})$ to build an $M_2^{Col(\omega,<\kappa_2)}$–generic subset G_2'. Denote the Cohen functions produced by G_2' by $\langle f'^2_\beta \mid \beta < (\kappa_2^+)^{M_2}\rangle$.
Now we define f_β^2 to be f'^2_β unless $\beta = i_{02}(\gamma)$, for some $\gamma < \kappa^+$. If $\beta = i_{02}(\gamma)$, for some

[3] It is (up to an initial segment) the Magidor-Radin generic sequence for κ_α in $M_{\alpha+1}$.

$\gamma < \kappa^+$, then let us proceed as follows.

First use $Col(\omega, \{(\kappa_1^+)^{M_1} + \kappa^+ + \gamma\})$ to pick genericly an ordinal $\gamma^* \in [\kappa_1, \kappa_2)$. Then set $f_\beta^2 = f_{0\gamma} \cup \{(\kappa, \gamma^*)\} \cup f_\beta'^2 \restriction [\kappa + 1, \kappa_2)$. I.e. the value at κ is changed to some rather random value $\geq \kappa_1$.

The intuition behind is that we would like that the values $\langle i_{0\kappa^{++}}(f_{0\gamma})(\kappa) \mid \gamma < \kappa^+\rangle$ will be kind of independent. Also note that for every $f : \kappa \to \kappa$ in V, $i_{0\kappa^{++}}(f)(\kappa) < \kappa_1$, so each function from the first block will dominate every old function.

Let G_2 be the resulting transformation of G_2'.

Note that for every $X \in M_2$ of size at most κ_2 there, we have $|i_{02}{}''\kappa^+ \cap X| \leq \kappa$. So G_2 is still $(Cohen(\kappa_2, \kappa_2^+))^{M_2^{Col(\omega,<\kappa_2)}}$–generic.

Case 3. α is a limit ordinal.

Then for every $\beta \in i_{0\alpha}(\kappa^+)$ there is $\gamma < \alpha$ such that $\beta \in i_{\gamma\alpha}{}''\kappa^+$. Denote the least such γ by γ_β and let β^* denotes the pre-image of β under $i_{\gamma_\beta\alpha}$.

Now set $f_\beta^\alpha = \bigcup_{\gamma_\beta \leq \gamma < \alpha} f^\gamma_{i_{\gamma_\beta\gamma}(\beta^*)}$.

It is not hard to check (similar to 2.1.1, Case 1) that $\langle f_\beta^\alpha \mid \beta < i_{0\alpha}(\kappa^+)\rangle$ is as desired.

Let us emphasize the following which is crucial for further successor stages. Suppose that $X \subseteq i_{0\alpha}(\kappa^+)$ of cardinality at most κ_α in M_α. Then there is $\gamma < \alpha$ such that $X \in \mathrm{rng}(i_{\gamma\alpha})$. Denote the least such γ by γ_X and let $X^* \subseteq i_{0\gamma_X}(\kappa^+)$ denotes the pre-image of X under $i_{\gamma_X\alpha}$. Clearly, γ_X is a successor ordinal. Also $|X^*|^{M_{\gamma_X}}$ is at most κ_{γ_X}. Consider a function $h_{X^*} : \kappa_{\gamma_X} \to M_{\gamma_X}$ such that $i_{\gamma_X\gamma_X+1}(h_{X^*})(\kappa_{\gamma_X}) = X^*$.

Then $h_X := i_{\gamma_X\alpha}(h_{X^*}) \in M_\alpha$ and $h_X(\kappa_\gamma) = i_{\gamma_X\gamma}(X^*)$, for every $\gamma, \gamma_X < \gamma \leq \alpha$.

Case 4. α is a successor ordinal with $\alpha - 1 > 1$.

Use $Col(\omega, ((\kappa_\alpha)^+)^{M_\alpha} + \kappa^+))$ to find $M_\alpha^{Col(\omega,<\kappa_\alpha)}$–generic set G'^α for $i_{0\alpha}(Cohen(\kappa, \kappa^+)) = (Cohen(\kappa_\alpha, \kappa_\alpha^+))^{M_\alpha^{Col(\omega,<\kappa_\alpha)}}$ in some canonical way. Let $f_\beta'^\alpha : \kappa_\alpha \to \kappa_\alpha, \beta < i_{0\alpha}(\kappa^+)$ be the Cohen functions defined by G'^α.

Now we define f_β^α to be $f_\beta'^\alpha$ unless $\beta = i_{0\alpha}(\gamma)$, for some $\gamma < \kappa^+$. If $\beta = i_{0\alpha}(\gamma)$, for some $\gamma < \kappa^+$, then let f_β^α be the function obtained from $f_\beta'^\alpha$ by replacing in it $f_\beta'^\alpha \restriction \kappa_{\alpha-1}$ by $f_\gamma^{\alpha-1}$, i.e. $f_\beta^\alpha = f_\gamma^{\alpha-1} \cup f_\beta'^\alpha \restriction [\kappa_{\alpha-1}, \kappa_\alpha)$.

We need to check that the changed sequence $\langle f_\beta^\alpha \mid \beta < i_{0\alpha}(\kappa^+)\rangle$ is still $M_\alpha^{Col(\omega,<\kappa_\alpha)}$–generic for $(Cohen(\kappa_\alpha, \kappa_\alpha^+))^{M_\alpha^{Col(\omega,<\kappa_\alpha)}}$.

It is enough to show that for every $\xi < (\kappa_\alpha^+)^{M_\alpha}$, $\langle f_\beta^\alpha \restriction \kappa_{\alpha-1} \mid \beta < \xi\rangle$ is in $M_\alpha^{Col(\omega,<\kappa_\alpha)}$.

Let $\xi < (\kappa_\alpha^+)^{M_\alpha}$. Pick some $\rho < (\kappa_{\alpha-1}^+)^{M_{\alpha-1}}$ such that $i_{\alpha-1\alpha}(\rho) \geq \xi$. Note that $i_{\alpha-1\alpha}{}''(\kappa_{\alpha-1}^+)^{M_{\alpha-1}}$ is unbounded in $(\kappa_\alpha^+)^{M_\alpha}$ so it is possible. Set $X = \rho$. Then $X \subseteq (\kappa_{\alpha-1}^+)^{M_{\alpha-1}}$ of cardinality at most $\kappa_{\alpha-1}$. Let h_X be as in the previous case. The sequence $\langle \kappa_\delta \mid \delta < \alpha\rangle$ is in M_α as

well as h_X. Then the sequence $\langle f^{\alpha-1}_\mu \mid \mu \in X\rangle$ will be in $M_\alpha^{Col(\omega,<\kappa_\alpha)}$. But also the function $\{(\nu, i_{\alpha-1\alpha}(\nu)) \mid \nu < \rho\}$ is in M_α. Hence $\langle f^\alpha_\beta \restriction \kappa_{\alpha-1} \mid \beta \in i_{\alpha-1\alpha}{''}\rho\rangle$ is in $M_\alpha^{Col(\omega,<\kappa_\alpha)}$. So, $\langle f^\alpha_\beta \restriction \kappa_{\alpha-1} \mid \beta < \xi\rangle$ is in $M_\alpha^{Col(\omega,<\kappa_\alpha)}$.
Note that $\langle f^{\alpha-1}_\mu \mid \mu < (\kappa^+_{\alpha-1})^{M_{\alpha-1}}\rangle$ is not in $M_\alpha^{Col(\omega,<\kappa_\alpha)}$.

2.1.3 An arbitrary block of Cohen functions.

Let $\eta < \kappa^{++}$. We deal now with η's block $F_\eta = \{f_{\eta\beta} \mid \beta < \kappa^+\}$ of Cohen functions. Repeat the construction of 2.1.2, but only start from $\eta + 1$ instead of 2.

2.1.4 Dealing with all blocks of Cohen functions simultaneously.

Now we will deal simultaneously with all κ^{++} blocks.

Namely we would like to construct functions $f^{\kappa^{++}}_{\alpha\beta} : \kappa^{++} \to \kappa^{++}, \alpha < i_{0\kappa^{++}}(\kappa^{++}), \beta < i_{0\kappa^{++}}(\kappa^+)$ which are a Cohen generic for $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$ over $M_{\kappa^{++}}^{Col(\omega,<\kappa^{++})}$. We would like also to have $f^{\kappa^{++}}_{i_{0\kappa^{++}}(\alpha) i_{0\kappa^{++}}(\beta)} \restriction \kappa = f_{\alpha\beta}$, for every $\alpha < \kappa^{++}, \beta < \kappa^+$, in order to be able to lift the embedding.

Note that $i_{0\kappa^{++}}(\kappa^{++}) = i_{0\kappa^{++}}(o(\kappa)) = \bigcup i_{0\kappa^{++}}{''}\kappa^{++}$.

The construction will proceed by recursion, building $M_\eta^{Col(\omega,<\kappa_\eta)}$–generic Cohen functions $f^\eta_{\alpha\beta} : \kappa_\eta \to \kappa_\eta$ for the forcing $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times i_{0\eta}(\kappa^+))$, for every successor $\eta \le \kappa^{++}$ and $\beta < \kappa^+$. We define some of $f^\eta_{\alpha\beta}$ for limit $\eta, 0 < \eta < \kappa^{++}$ as well, but in this case they will not always be $M_\eta^{Col(\omega,<\kappa_\eta)}$–Cohen generic for the forcing $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times i_{0\eta}(\kappa^+))$.

Let $\eta, 0 < \eta < \kappa^{++}$. We deal at this stage with the forcing $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times (\kappa^+_\eta)^{M_\eta})$.

Note that $o^{\tilde{M}_{\kappa^{++}}}(\kappa_n) = 0$, for every $n < \omega$ and $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\omega) = 1$. So the first non-trivial case will be $\eta = \omega + 1$.

Case 1. $\eta = \omega + 1$.
So we have $Cohen(\kappa, 1 \times \kappa^+)$. It is just a single Cohen function. Proceed as in 2.1.2.

Case 2. η is a limit ordinal.
Set

$$Z_\eta = \{\alpha < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \mid \exists \xi < \eta \quad \exists \alpha_\xi < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi) \quad i_{\xi\eta}(\alpha_\xi) = \alpha\}.$$

Note that Z_η may be a proper subset of $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta)$, if $\eta < \kappa^{++}$, but for $\eta = \kappa^{++}$ we have the equality.

Claim 1 $Z_{\kappa^{++}} = o^{\tilde{M}_{\kappa^{++}}}(\kappa^{++})$.

Proof. Let $\alpha < o^{\tilde{M}_{\kappa^{++}}}(\kappa^{++})$. Pick some $\xi, \rho' < \xi < \kappa^{++}$ and α_ξ such that $i_{\xi\kappa^{++}}(\alpha_\xi) = \alpha$. By elementarity, $M_\xi \models \alpha_\xi < o(\kappa_\xi)$. Then at some stage δ of the iteration from M_ξ to $M_{\kappa^{++}}$ a measure $i_{\xi\delta}(U(\kappa_\xi, \alpha_\xi + 1))$ should be used, and then $i_{\xi\delta+1}(\alpha_\xi) < o^{\tilde{M}_{\delta+1}}(\kappa_\delta) = o^{\tilde{M}_{\kappa^{++}}}(\kappa_\delta)$, or already $\alpha_\xi < o^{\tilde{M}_{\xi+1}}(\kappa_\xi) = o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi)$.
$\square$ of the claim.

Let $\alpha \in Z_\eta$ and $\beta < i_{0\eta}(\kappa^+)$. Find the least $\xi < \eta$ such that for some $\alpha_\xi < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi)$ and β_ξ we have $i_{\xi\eta}(\alpha_\xi) = \alpha$ and $i_{\xi\eta}(\beta_\xi) = \beta$. Denote the least such ξ by $\xi_{\alpha\beta}$. Set

$$f^\eta_{\alpha\beta} = \bigcup\{f^\xi_{\alpha_\xi\beta_\xi} \mid \xi_{\alpha\beta} \leq \xi < \eta \text{ and } \alpha_\xi < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi)\}.$$

Case 3. η is a successor ordinal > 1.
Note that $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) = o^{\tilde{M}_{\eta+1}}(\kappa_\eta) < (\kappa_\eta^{++})^{M_{\eta+1}} = (\kappa_\eta^{++})^{M_{\kappa^{++}}}$.
So, $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times (\kappa_\eta^+)^{M_\eta})$ is a κ_η^+-c.c. forcing of cardinality κ_η^+ in $M_{\eta+1}$. Use $Col(\omega, (\kappa_\eta^+)^{M_{\kappa_\eta}} + \kappa^+)$ to find M_η–generic subset G'_η of it in some canonical way. Denote by $\langle f'^\eta_{\alpha\beta} \mid \alpha < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta), \beta < (\kappa_\eta^+)^{M_\eta}\rangle$ the Cohen functions generated by G'_η.
Next let us change some of this functions restricted to $\kappa_{\eta-1}$.

If there is no $\xi \leq \eta - 1$ such that for some $\alpha_\xi < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi)$ and β_ξ we have $i_{\xi\eta}(\alpha_\xi) = \alpha$ and $i_{\xi\eta}(\beta_\xi) = \beta$, then set $f^\eta_{\alpha\beta} = f'^\eta_{\alpha\beta}$.
Otherwise let $\check{\eta}$ be the maximal $\xi \leq \eta - 1$ such that for some $\alpha_\xi < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi)$ and β_ξ we have $i_{\xi\eta}(\alpha_\xi) = \alpha$ and $i_{\xi\eta}(\beta_\xi) = \beta$.
Set $f^\eta_{\alpha\beta} = f'^\eta_{\alpha\beta} \restriction [\kappa_{\check{\eta}}, \kappa_\eta) \cup f^{\check{\eta}}_{\alpha_{\check{\eta}}\beta_{\check{\eta}}}$.
Let G_η be the corresponding changed G'_η. Let us argue that such changes do not effect genericity, i.e. G_η remains generic.

Suppose that $Y \subseteq o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times (\kappa_\eta^+)^{M_\eta}$ of cardinality at most κ_η in M_η. Consider

$$X = \{(\alpha, \beta) \in (\kappa_{\eta-1}^{++})^{M_{\eta-1}} \times (\kappa_{\eta-1}^+)^{M_{\eta-1}} \mid i_{\eta-1\eta}((\alpha, \beta)) \in Y\}.$$

Then $|X|^{M_{\eta-1}} \leq \kappa_{\eta-1}$.
If $\eta - 1$ is a successor ordinal then we can use the induction and argue that $\langle f^\eta_{i_{\eta-1\eta}(\alpha)i_{\eta-1\eta}(\beta)} \restriction \kappa_{\eta-1} \mid (\alpha, \beta) \in X\rangle$ is in $M_{\eta-1}$. Now this set will be also in M_η, due to the size of X. So $G_\eta \restriction Y$ will be generic since in is obtained from G'_η by basically changing a single condition.
Suppose now that $\eta - 1$ is a limit ordinal. Then there is $\gamma < \eta - 1$ such that $X \in \text{rng}(i_{\gamma\eta-1})$. Denote the least such γ by γ_X and let X^* be the pre-image of X under $i_{\gamma_X\eta-1}$. Clearly, γ_X is a successor ordinal. Also $|X^*|^{M_{\gamma_X}}$ is at most κ_{γ_X}. Consider a function $h_{X^*} : \kappa_{\gamma_X} \to M_{\gamma_X}$ such that $i_{\gamma_X\gamma_X+1}(h_{X^*})(\kappa_{\gamma_X}) = X^*$.

Then $h_X := i_{\gamma_X \eta-1}(h_{X^*}) \in M_{\eta-1}$ and $h_X(\kappa_\gamma) = i_{\gamma_X\gamma}(X^*)$, for every $\gamma, \gamma_X < \gamma \leq \eta - 1$. Now using h_X and $\langle \kappa_\gamma \mid \gamma_X < \gamma \leq \eta - 1\rangle$ which are both in M_η it is possible to define there $\langle f^\eta_{i_{\eta-1\eta}(\alpha) i_{\eta-1\eta}(\beta)} \restriction \kappa_{\eta-1} \mid (\alpha,\beta) \in X\rangle$. So again $G_\eta \restriction Y$ will be generic since in is obtained from G'_η by basically changing a single condition.
This completes the construction.

Let us argue that the final $G_{\kappa^{++}}$ is generic over $M^{Col(\omega,<\kappa^{++})}_{\kappa^{++}}$.

Claim 2 $G_{\kappa^{++}}$ is a generic subset of $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$ over $M^{Col(\omega,<\kappa^{++})}_{\kappa^{++}}$.

Proof. It is enough to show that for every $X \subseteq \kappa^{++} \times i_{0\kappa^{++}}(\kappa^+), X \in M_{\kappa^{++}}$ of cardinality at most κ^{++} in $M_{\kappa^{++}}$ the restriction $G_{\kappa^{++}} \restriction X$ is $Cohen(\kappa^{++}, X)$–generic over $M^{Col(\omega,<\kappa^{++})}_{\kappa^{++}}$.
Fix such X. Then there is $\gamma < \kappa^{++}$ such that $X \in \mathrm{rng}(i_{\gamma\kappa^{++}})$. Denote the least such γ by γ_X and let X^* be the pre-image of X under $i_{\gamma_X\kappa^{++}}$. Clearly, γ_X is a successor ordinal. Also $|X^*|^{M_{\gamma_X}}$ is at most κ_{γ_X}. Then there are arbitrary large successor ordinals $\delta, \gamma_X \leq \delta < \kappa^{++}$ such that every coordinate of $i_{\xi_X\delta}(X^*)$ appears in G_δ, i.e. for every $(\alpha, \beta) \in i_{\xi_X\delta}(X^*)$ we have $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\delta) > \alpha$.
Supose now that in $M^{Col(\omega,<\kappa^{++})}_{\kappa^{++}}$ we have a dense open subset D of $Cohen(\kappa^{++}, X)$. Define γ_D and D^* as before. Pick δ as above with $\gamma_D < \delta$. Then $i_{\gamma_D\delta}(D^*)$ will be a dense open subset of $Cohen(\kappa_\delta, i_{\gamma_X\delta}(X^*))$ in M_δ. So $(G_\delta \restriction i_{\gamma_X\delta}(X^*)) \cap i_{\gamma_D\delta}(D^*) \neq \emptyset$. Then, by the construction, also $G_{\kappa^{++}} \cap D \neq \emptyset$.
□ of the claim.

2.1.5 Dealing with all blocks of Cohen functions simultaneously revised.

In previous settings only values of Cohen functions on κ were addressed with a special care (see 2.1.2). Here we would like revise a previous construction (2.1.4) and to deal with all κ_α's.

We construct functions $f^{\kappa^{++}}_{\alpha\beta} : \kappa^{++} \to \kappa^{++}, \alpha < i_{0\kappa^{++}}(\kappa^{++}), \beta < i_{0\kappa^{++}}(\kappa^+)$ which are a Cohen generic for $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$ over $M^{Col(\omega,<\kappa^{++})}_{\kappa^{++}}$. Still we would like to have $f^{\kappa^{++}}_{i_{0\kappa^{++}}(\alpha) i_{0\kappa^{++}}(\beta)} \restriction \kappa = f_{\alpha\beta}$, for every $\alpha < \kappa^{++}, \beta < \kappa^+$, in order to be able to lift the embedding.

The construction will proceed by recursion, building $M^{Col(\omega,<\kappa_\eta)}_\eta$–generic Cohen functions $f^\eta_{\alpha\beta} : \kappa_\eta \to \kappa_\eta$ for the forcing $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times i_{0\eta}(\kappa^+))$, for every successor $\eta < \kappa^{++}$ and $\beta < \kappa^+$. We define some of $f^\eta_{\alpha\beta}$ for limit $\eta, 0 < \eta < \kappa^{++}$ as well, but in this case they will not form always $M^{Col(\omega,<\kappa_\eta)}_\eta$–Cohen generic for the forcing $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times i_{0\eta}(\kappa^+))$.

Let $\eta, 0 < \eta < \kappa^{++}$. We deal at this stage with the forcing $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times (\kappa_\eta^+)^{M_\eta})$.
The first non-trivial case is $\eta = \omega + 1$.

Case 1. $\eta = \omega + 1$.
So we have $Cohen(\kappa, 1 \times \kappa^+)$.
Define $f^{\omega+1}_{0\beta} : \kappa_{\omega+1} \to \kappa_{\omega+1}$, for every $\beta < i_{0\omega+1}(\kappa^+)$.
Clearly, $i_{0\omega+1}(Cohen(\kappa, \kappa^+)) = (Cohen(\kappa_{\omega+1}, \kappa^+_{\omega+1}))^{M^{Col(\omega,<\kappa_{\omega+1})}_{\omega+1}}$. It is a $\kappa^+_{\omega+1}$–c.c. forcing of size $\kappa^+_{\omega+1}$ in $M^{Col(\omega,<\kappa_{\omega+1})}_{\omega+1}$. Use $Col(\omega, (\kappa^+_{\omega+1})^{M_{\omega+1}})$ to build an $M^{Col(\omega,<\kappa_{\omega+1})}_{\omega+1}$–generic subset $G'_{\omega+1}$. Denote the Cohen functions produced by $G'_{\omega+1}$ by $\langle f'^{\omega+1}_\beta \mid \beta < (\kappa^+_{\omega+1})^{M_{\omega+1}}\rangle$.
Now we define $f^{\omega+1}_{0\beta}$ to be $f'^{\omega+1}_\beta$ unless $\beta = i_{0\omega+1}(\gamma)$, for some $\gamma < \kappa^+$. If $\beta = i_{0\omega+1}(\gamma)$, for some $\gamma < \kappa^+$, then let us proceed as follows.
First use $Col(\omega, \{(\kappa^+_\omega)^{M_\omega} + \kappa^+ + \gamma\})$ to pick generically an ordinal $\gamma^* \in [\kappa_\omega, \kappa_{\omega+1})$. Then set $f^2_{0\beta} = f_{0\gamma} \cup \{(\kappa, \gamma^*)\} \cup f'^2_\beta \restriction [\kappa + 1, \kappa_{\omega+1})$. I.e. the value at κ is changed to some rather random value $\geq \kappa_\omega$. It is possible to change the values at each of κ_n's but let us make changes in values only at places where the relevant forcing appears.
The next stage for the forcing $Cohen(\kappa, 1 \times \kappa^+)$ will be $\eta = \omega + \omega + 1$. At this stage the value given to κ will be preserved and the value at κ_ω will be changed to some ordinal in $[\kappa_{\omega+\omega}, \kappa_{\omega+\omega+1})$.

The first place when the second block of Cohen functions will come into the play will be at stage $\omega \cdot \omega + 1$, since $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\alpha) < 2$, for every $\alpha < \omega \cdot \omega + 1$, and $o^{\tilde{M}_{\kappa^{++}}}(\kappa_{\omega\cdot\omega}) = 2$.
At the stage $\omega \cdot \omega$ we will have $\langle f^{\omega\cdot\omega}_{0\beta} \mid \beta < i_{0\omega\cdot\omega}(\kappa^+)\rangle$. Let us describe the construction at the next stage.

Case 2. $\eta = \omega \cdot \omega + 1$.
Define $f^{\omega\cdot\omega+1}_{\alpha\beta} : \kappa_{\omega+1} \to \kappa_{\omega+1}$, for every $\alpha < 2, \beta < i_{0\omega+1}(\kappa^+)$.
Clearly, $i_{0\omega\cdot\omega+1}(Cohen(\kappa, 2\times\kappa^+)) = (Cohen(\kappa_{\omega\cdot\omega+1}, 2\times\kappa^+_{\omega\cdot\omega+1}))^{M^{Col(\omega,<\kappa_{\omega\cdot\omega+1}))}_{\omega\cdot\omega+1}}$. It is a $\kappa^+_{\omega\cdot\omega+1}$–c.c. forcing of size $\kappa^+_{\omega\cdot\omega+1}$ in $M^{Col(\omega,<\kappa_{\omega\cdot\omega+1})}_{\omega\cdot\omega+1}$. Use $Col(\omega, (\kappa^+_{\omega\cdot\omega+1})^{M_{\omega\cdot\omega+1}})$ to build an $M^{Col(\omega,<\kappa_{\omega\cdot\omega+1})}_{\omega\cdot\omega+1}$–generic subset $G'_{\omega\cdot\omega+1}$. Denote the Cohen functions produced by $G'_{\omega\cdot\omega+1}$ by $\langle f'^{\omega\cdot\omega+1}_{\alpha\beta} \mid \alpha < 2, \beta < (\kappa^+_{\omega\cdot\omega+1})^{M_{\omega\cdot\omega+1}}\rangle$.
Define $f^{\omega\cdot\omega+1}_{\alpha\beta}$ to be $f'^{\omega\cdot\omega+1}_{\alpha\beta}$, $\alpha < 2$, unless $\beta = i_{\omega\cdot\omega\omega\cdot\omega+1}(\gamma)$, for some $\gamma < i_{0\omega\cdot\omega}(\kappa^+)$. If $\beta = i_{\omega\cdot\omega\omega\cdot\omega+1}(\gamma)$, for some $\gamma < i_{0\omega\cdot\omega+1}(\kappa^+)$, then set $f^{\omega\cdot\omega+1}_{0\beta} = f^{\omega\cdot\omega}_{0\gamma} \cup f'^{\omega\cdot\omega+1}_\beta \restriction [\kappa_{\omega\cdot\omega}, \kappa_{\omega\cdot\omega+1})$.

Set $f^{\omega\cdot\omega+1}_{1\beta}$ to be $f''^{\omega\cdot\omega+1}_{1\beta}$, unless $\beta = i_{0\omega\cdot\omega+1}(\delta)$, for some $\delta < \kappa^+$. If $\beta = i_{0\omega\cdot\omega+1}(\delta)$, for some $\delta < \kappa^+$, then let us proceed as follows.
First use $Col(\omega, \{(\kappa^+_{\omega\cdot\omega})^{M_{\omega\cdot\omega}} + \kappa^+ \cdot 2 + \delta\})$ to pick generically an ordinal $\delta^*_1 \in [\kappa_{\omega\cdot\omega}, \kappa_{\omega\cdot\omega+1})$.
Then set $f^{\omega\cdot\omega+1}_{1\beta} = (f''^{\omega\cdot\omega+1}_{1\beta} \setminus \{(\kappa, f''^{\omega\cdot\omega+1}_{1\beta}(\kappa))\}) \cup \{(\kappa, \delta^*_1)\}$.

I.e. the value at κ is changed to some rather random value $\geq \kappa_{\omega\cdot\omega}$.
Note that the value $f^{\omega\cdot\omega+1}_{0\beta}(\kappa_{\omega\cdot\omega})$ stays unchanged here. It will be changed further at the first relevant stage, i.e. at $\omega\cdot\omega+\omega+1$.

Let us deal with a general situation now.

Case 3. $\eta > 0$ is a limit ordinal.
We proceed exactly as in the corresponding case of 2.1.4.

Case 4. η is a successor ordinal.
Assume that $\eta > \omega+1$.

Note that $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) = o^{\tilde{M}_{\eta+1}}(\kappa_\eta) < (\kappa^{++}_\eta)^{M_{\eta+1}} = (\kappa^{++}_\eta)^{M_{\kappa^{++}}}$.
So, $Cohen(\kappa_\eta, o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \times (\kappa^+_\eta)^{M_\eta})$ is a κ^+_η-c.c. forcing of cardinality κ^+_η in $M_{\eta+1}$. Use $Col(\omega, (\kappa^+_\eta)^{M_{\kappa_\eta}} + \kappa^+)$ to find M_η–generic subset G'_η of it in some canonical way. Denote by $\langle f'^{\eta}_{\alpha\beta} \mid \alpha < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta), \beta < (\kappa^+_\eta)^{M_\eta}\rangle$ the Cohen functions generated by G'_η.
Next let us change some of this functions restricted to $\kappa_{\eta-1}$.

Set $A = \{\alpha < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \mid \exists \alpha' < \kappa^{++} \quad i_{0\eta}(\alpha') = \alpha\}$.
If $\alpha \in o^{\tilde{M}_{\kappa^{++}}}(\kappa_\eta) \setminus A$, then no change is made and we set $f^{\eta}_{\alpha\beta} = f'^{\eta}_{\alpha\beta}$, for every $\beta < (\kappa^+_\eta)^{M_\eta}$.
Suppose now that $\alpha \in A$. Let $\beta < (\kappa^+_\eta)^{M_\eta}$. If there is no β' such that $i_{\eta-1\eta}(\beta') = \beta$, then again set $f^{\eta}_{\alpha\beta} = f'^{\eta}_{\alpha\beta}$.
Suppose that $i_{\eta-1\eta}(\delta) = \beta$, for some δ.
If there is no $\xi \leq \eta-1$ such that for some $\alpha_\xi < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi)$ we have $i_{\xi\eta}(\alpha_\xi) = \alpha$. Then use $Col(\omega, \{(\kappa^+_{\eta-1})^{M_{\eta-1}} + \kappa^+ \cdot \alpha + \delta\})$ to pick genericly an ordinal $\delta^*_\alpha \in [\kappa_{\eta-1}, \kappa_\eta)$.
Set $f^{\eta}_{\alpha\beta} = (f''^{\eta}_{\alpha\beta} \setminus \{(\kappa, f''^{\eta}_{\alpha\beta}(\kappa))\}) \cup \{(\kappa, \delta^*_\alpha)\}$.
Otherwise let $\check{\eta}$ be the maximal $\xi \leq \eta-1$ such that for some $\alpha_\xi < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\xi)$ we have $i_{\xi\eta}(\alpha_\xi) = \alpha$.
Set $f''^{\eta}_{\alpha\beta} = f'^{\eta}_{\alpha\beta} \restriction [\kappa_{\check{\eta}}, \kappa_\eta) \cup f^{\check{\eta}}_{\alpha_{\check{\eta}}\beta_{\check{\eta}}}$.
If $\check{\eta} = \eta-1$, then set $f^{\eta}_{\alpha\beta} = f''^{\eta}_{\alpha\beta}$.
Suppose that $\check{\eta} < \eta-1$. Then use $Col(\omega, \{(\kappa^+_{\eta-1})^{M_{\eta-1}} + \kappa^+ \cdot \alpha + \delta\})$ to pick genericly an ordinal $\delta^*_\alpha \in [\kappa_{\eta-1}, \kappa_\eta)$.
Set $f^{\eta}_{\alpha\beta} = (f''^{\eta}_{\alpha\beta} \setminus \{(\kappa_{\check{\eta}}, f''^{\eta}_{\alpha\beta}(\kappa_{\check{\eta}}))\}) \cup \{(\kappa_{\check{\eta}}, \delta^*_\alpha)\}$.

Let G_η be the corresponding changed G'_η. The argument that G_η remains generic is similar to those of 2.1.4.

This completes the construction.

Finally, the following holds exactly as in 2.1.4.

Claim 3 $G_{\kappa^{++}}$ is a generic subset of $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$ over $M^{Col(\omega,<\kappa^{++})}_{\kappa^{++}}$.

2.2 ∞–semi precipitous filter

Recall that $G_1 \subseteq Col(\omega, <\kappa)$ is generic over V and $G_2 \subseteq Cohen(\kappa, \kappa^{++} \times \kappa^+)$ is generic over $V[G_1]$. For every $\alpha < \kappa^{++}$ we denote the α-th block of Cohen functions by $F_\alpha = \{f_{\alpha\beta} \mid \beta < \kappa^+\}$.

Next we would like to arrange that the functions in F_α are those that have a chance to represent κ_α in a generic ultrapower. For this purpose let us add clubs by forcing over $V[G_1, G_2]$.

Force with $< \kappa$–support iteration a club into

$$\{\nu < \kappa \mid f_{\alpha\beta}(\nu) < f_{\alpha'\beta'}(\nu)\},$$

for every $\alpha < \alpha' < \kappa^{++}$ and $\beta, \beta' < \kappa^+$.

In addition for each $f \in {}^\kappa\kappa \cap V$ and $\beta < \kappa^+$ force a club into

$$\{\nu < \kappa \mid f(\nu) < f_{0\beta}(\nu)\}.$$

Also, for each $n < \omega, f \in {}^{[\kappa]^n}\kappa \cap V, \alpha_1 < ... < \alpha_n < \alpha < \kappa^{++}, \beta_1, ..., \beta_n, \beta < \kappa^+$ we force a club into

$$\{\nu < \kappa \mid f(f_{\alpha_1\beta_1}(\nu), ..., f_{\alpha_n\beta_n}(\nu)) < f_{\alpha\beta}(\nu)\}.$$

This insures that in any normal filter the block F_α of functions will be strictly above each of the blocks $F_{\alpha'}$ with $\alpha' < \alpha$.

Note that each ordinal in the interval $[\kappa_\alpha, \kappa_{\alpha+1})$ is of a form $i_{0,\alpha+1}(f)(\kappa_{\alpha_1}, ..., \kappa_{\alpha_n})$ for some $f \in {}^{[\kappa]^n}\kappa \cap V$ and $\alpha_1 < ... < \alpha_n \leq \alpha$.

Let G_3 be a corresponding generic object. Note that it is easy to reorganize the forcing to add both of the blocks of Cohen functions and the clubs in a single iteration of length κ^{++}.

Let us define a filter F over κ in $V[G_1, G_2, G_3]$ as follows:

$$F = \{X \subseteq \kappa \mid 0_{Col(\omega, < i_{0\kappa^{++}}(\kappa))/G_1 * G_2 * G_3} \Vdash \kappa \in i_{0\kappa^{++}}(\underset{\sim}{X})\}.$$

Then

$$F^+ = \{X \subseteq \kappa \mid \exists p \in Col(\omega, < i_{0\kappa^{++}}(\kappa))/G_1 * G_2 * G_3 \quad p \Vdash \kappa \in i_{0\kappa^{++}}(\underset{\sim}{X})\}.$$

The next lemma is immediate.

Lemma 2.1 *F is ∞–semi precipitous*[4] *filter with a witnessing forcing $Col(\omega, < \kappa^{++})/G_1 * G_2 * G_3$ and with a generic embedding which extends $i_{0\kappa^{++}}$.*

[4]We refer to [1] and [2] for this notion. The meaning is that after forcing with $Col(\omega, < \kappa^{++})/G_1 * G_2 * G_3$ it is possible to extend $i_{0\kappa^{++}}$ to an elementary embedding of $V[G, G_2, G_3]$ into a transitive model.

2.3 Adding clubs

Next we would like to add clubs to sets in F, and then to extensions of F as it was done in Jech-Magidor-Mitchell-Prikry [9], but picking generics over $M_{\kappa^{++}}$ using the procedure above. It should be done a bit more carefully in order to keep a resulting generic embedding to extend $i_{0\kappa^{++}}$. Just note that the set $\{\kappa_\alpha \mid \alpha < \kappa^{++}\}$ is a club, so we cannot force a club for example into the set $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\} \in F$ and still to extend $i_{0\kappa^{++}}$, since for every $\alpha, 0 < \alpha < \kappa^{++}$, $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\alpha) = \alpha > 0$.

So let us add clubs only to subsets X of κ in F such that for a final segment of α's below κ^{++},

$$0_{Col(\omega,<i_{0\kappa^{++}}(\kappa))/G_1*G_2*G_3} \Vdash \kappa_\alpha \in i_{0\kappa^{++}}(\underset{\sim}{X}).$$

Then, in particular, the set $\{\nu < \kappa \mid \nu$ is an accessible ordinal in $V\}$ will be nonstationary.

We would like to arrange a situation where each filter which extends $Cub_{\aleph_1}$ concentrates on the set $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$. The simplest way to guarantee this is to shoot a club into $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$. But doing it will destroy $i_{0\kappa^{++}}$ completely, since $\{\kappa_\alpha \mid 0 < \alpha < \kappa^{++}\}$ is a club in κ^{++} and it is disjoint to the image of a club in $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$. So adding such a club will change the cofinality of κ^{++} to ω and eventually will produce a normal precipitous filter. An other way is to shoot clubs disjoint from $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 1\}$, $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 2\}$ etc., and this way prevent the ground model ultrafilters U_2, U_3, etc. to have a normal extensions. This works nicely, but unfortunately not for all ground model ultrafilters. Remember that we have a sequence of κ^{++}–many of them. So up–repeat points must be on the sequence, i.e. for some $\alpha < \kappa^{++}$, for every $X \in U_\alpha$ there will be $\beta > \alpha$ (even κ^{++} many of them) with $X \in U_\beta$. Shooting clubs for them will not work.
The actual approach will be as follows. We add together with blocks F_α's of Cohen functions an additional sequence $\langle g_\alpha \mid \alpha < \kappa^{++}\rangle$ of Cohen functions from κ to κ (it is possible just to to use the first function of each block instead). Require that for each $\nu < \kappa$ with $o(\nu) > 0$, $g_\alpha(\nu) < \nu^{++}$.
Now, as in 2.1, by changing values of generics, we insure that

1. $\langle i_{0\kappa^{++}}(g_\alpha)(\kappa) \mid \alpha < \kappa^{++}\rangle$ is an increasing sequence,

2. for every $\alpha < \kappa^{++}$, $\langle i_{0\kappa^{++}}(g_\gamma)(\kappa_\alpha) \mid \gamma < o^{\tilde{M}_{\kappa^{++}}}(\kappa_\alpha)\rangle$ is an increasing sequence of ordinals below $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\alpha)$.

Now, for every $\gamma < \alpha < \kappa^{++}$, we force clubs into the set

$$\{\nu < \kappa \mid g_\gamma(\nu) < g_\alpha(\nu)\}$$

and into the complement of the set

$$\{\nu < \kappa \mid o^{\vec{U}}(\nu) > 0 \wedge o^{\vec{U}}(\nu) \leq g_\alpha(\nu)\}.$$

Note κ is in the image of each of these sets under $i_{0\kappa^{++}}$, as is κ_ξ for sufficiently large $\xi < \kappa^{++}$. We will show further in Lemma 3.1 that this does the job.

Clubs will be added to any set of the form $A \cup \{\nu < \kappa \mid o^{\vec{U}}(\nu) > 0\}$ with $A \in U_0$.

Denote by $\tilde{F}$ the extension of F obtained by adding all the clubs.

Let $\tilde{V}$ be a generic extension obtained by this forcing.

Note that a bit nicer (but less intuitive) way to organize the iteration used here will be an inductive definition of an iteration of the length κ^{++}. Thus suppose that at a stage $\alpha < \kappa^{++}$ we have a forcing $P_{<\alpha}$ defined with a generic subset $G_{<\alpha}$. Force the α–th block of Cohen functions F_α. Now over $V[G_1, G_{<\alpha}, F_\alpha]$ we add clubs relevant for the blocks of Cohen functions $\langle F_\gamma \mid \gamma \leq \alpha\rangle$. This will be Q_α. Its length is below κ^{++}. Set $P_{<\alpha+1} = P_{<\alpha} * Q_\alpha$.

Let us point out the following basic property:

Lemma 2.2 *Let $\langle \alpha_n \mid n < \omega\rangle, \langle \beta_n \mid n < \omega\rangle$ be ω-sequences which consist of different elements of κ^{++} and of κ^+ respectively. Then the following set contains a club*

$$\bigcup_{n<\omega} \{\nu < \kappa \mid f_{\alpha_n\beta_n}(\nu) > f_{\alpha_n\beta_n+1}(\nu)\}.$$

Proof. By the construction of $f_{i_{0\kappa^{++}}(\alpha_n) i_{0\kappa^{++}}(\beta_n)}(\kappa)$'s in 2.1.5 and genericity of the collapse there always will be p in a generic object such that for some $n < \omega$

$$p \Vdash f_{i_{0\kappa^{++}}(\alpha_n) i_{0\kappa^{++}}(\beta_n)}(\kappa) > f_{i_{0\kappa^{++}}(\alpha_n) i_{0\kappa^{++}}(\beta_n+1)}(\kappa).$$

□

In general, suppose that we have a sequence $\langle A_\eta \mid \eta < \kappa^{++}\rangle$ of $\tilde{F}$-positive sets. Let $\langle p_\eta \mid \eta < \kappa^{++}\rangle$ be a sequence of conditions in $Col(\omega, <\kappa)$ such that for every $\eta < \kappa^{++}$, $p_\eta \Vdash \kappa \in \underset{\sim}{i}(A_\eta)$. Shrink if necessary the sequence $\langle p_\eta \mid \eta < \kappa^{++}\rangle$ in order to form a Δ-system. If the kernel of it is empty, then for any sequence $\langle \eta_n \mid n < \omega\rangle$ of different ordinals below κ^{++} the set $\bigcup_{n<\omega} A_{\eta_n}$ contains a club.

Similarly, if p is a kernel and for some $A \subseteq \kappa$ we have $p = \|\kappa \in \underset{\sim}{i}(\underset{\sim}{A})\|^{Col(\omega,<\kappa^{++})}$, then the set $(\kappa \setminus A) \cup \bigcup_{n<\omega} A_{\eta_n}$ contains a club.

3 No normal precipitous ideals

We will prove a slightly more general statement– in $\tilde{V}$ there is no precipitous filter on $\aleph_1$ which contains $Cub_{\aleph_1}$, i.e. which is a Q–point filter. If the initial ground model had no large cardinals above κ (say $o(\kappa) = \kappa^{++}$ but nothing more), then there will be no normal precipitous filters at all.
Suppose otherwise. Let H be a precipitous filter over $\kappa = \aleph_1^{\tilde{V}}$ which includes $Cub_{\aleph_1}$.

Lemma 3.1 *Assume that there is no inner model with a strong cardinal. Then $H \supseteq U_0$.*

Remark. Here is actually the only place where the core model is used in an essential way. If we restrict ourself initially to filters which extend $Cub_{\aleph_1} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$, then no $\mathcal{K}$ is needed.
Proof. Let $G \subseteq H^+$ be a generic ultrafilter and $j : \tilde{V} \to N$ be the corresponding generic elementary embedding. Now $j \restriction \mathcal{K}$ is an iterated ultrapower of $\mathcal{K}$. Let E_α be the extender (actually a measure) used to move κ in this iteration. If $\alpha = 0$, then we are done. Suppose otherwise. Consider $\delta = (o(\kappa))^{\mathcal{K}^N}$. Then $\delta = (o(\kappa))^{\mathcal{K}_\alpha} = \alpha$, where $\mathcal{K}_\alpha = Ult(\mathcal{K}, E_\alpha)$. Now $\alpha < (\kappa^{++})^{\mathcal{K}_\alpha} < (\kappa^{++})^{\mathcal{K}} = (\kappa^{++})^{\tilde{V}}$, since a club was forced into $\{\nu < \kappa \mid o(\nu) < (\nu^{++})^{\mathcal{K}}\}$. Consider now the sequence $\langle j(g_\xi)(\kappa) \mid \xi < (\kappa^{++})^{\tilde{V}} \rangle$. It is an increasing sequence of ordinals of order type $(\kappa^{++})^{\tilde{V}}$. But $\delta < (\kappa^{++})^{\tilde{V}}$, hence there is $\eta < (\kappa^{++})^{\tilde{V}}$ with $\delta \leq j(g_\eta)(\kappa)$. By elementarity, then $\{\nu < \kappa \mid o(\nu) > 0 \wedge o(\nu) \leq g_\eta(\nu)\} \in H^+$. This is impossible since we added a club into its compliment and $H \supseteq Cub_{\aleph_1}$.
□
Note that if δ (the Mitchell order of κ as computed in the ground model of N) is less than $(\kappa^{++})^{\tilde{V}}$, then the argument above still provides the desired conclusion.

By the lemma we have in particular that $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$ is in H. Hence $H \supseteq Cub_{\aleph_1} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$, since $\tilde{F} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$ is $Cub_{\aleph_1} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$. Let us further assume that every $A \in H^+$ under consideration is automatically a subset of $\{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$.

For each $\alpha < \kappa^{++}$, pick a set $A_\alpha \in H^+$ and an ordinal $\beta_\alpha < \kappa^+$ such that A_α forces that $f_{\alpha\beta_\alpha}$ represents in the generic ultrapower, the smallest ordinal among the functions in F_α, i.e.

$$A_\alpha \Vdash_{H^+} \forall \beta < (\kappa^+)^V \quad [f_{\alpha\beta_\alpha}]_{\mathcal{G}(H^+)} \leq [f_{\alpha\beta}]_{\mathcal{G}(H^+)}.$$

It is tempting to assume that $f_{\alpha\beta_\alpha}$ represents κ_α, but this need not be true, since functions from lower blocks may represent κ_α. Thus $f_{\alpha\beta_\alpha}$ will represent κ_γ for some $\gamma \geq \alpha$.

Note that at most finitely many of the sets A_α's are in H, since otherwise, by the countable completeness of H, we will have in $\tilde{V}$ a countable sequence $\langle \alpha_n \mid n < \omega \rangle$ with $f_{\alpha_n \beta_{\alpha_n}}$ being the least function of the block F_{α_n}. By countable completeness of H then the set

$$\{\nu < \kappa \mid \forall n < \omega \quad f_{\alpha_n \beta_{\alpha_n}}(\nu) \leq f_{\alpha_n \beta_{\alpha_n+1}}(\nu)\}$$

is in H. But its complement contains a club, by Lemma 2.2. Contradiction.
We can assume that for every $\alpha < \kappa^{++}$ the set A_α is not in H. Actually the argument below will not be effected even if some of A_α's are in H.

We will use now the fact that the iteration $P_{<\kappa^{++}}$ over $V[G_1]$ (adding blocks of Cohen functions and clubs) satisfies κ^+–c.c. So each of A_α's depends only on at most κ–many Cohen functions and clubs.

Let $\alpha < \kappa^{++}$. Consider the characteristic function $\chi_\alpha : \kappa \to 2$ of A_α.
There are Cohen functions $\{f_{\eta,\xi} \mid \langle \eta, \xi \rangle \in a_\alpha\}$, clubs $\{c_\eta \mid \eta \in \mathrm{dom}(a_\alpha)\}$ [5] and a continuous function $t_\alpha \in V[G_1]$, such that $|a_\alpha| \leq \kappa$ and $\chi_\alpha = t_\alpha(\langle f_{\eta,\xi} \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \{c_\eta \mid \eta \in \mathrm{dom}(a_\alpha)\} \rangle)$.
We can assume, by shrinking if necessary, that for some t each $t_\alpha = t$, and that $\langle \mathrm{dom}(a_\alpha) \mid \alpha < \kappa^{++} \rangle$ forms a Δ-system.
Now, for every $\alpha < \kappa^{++}$, pick some $\beta^*_\alpha \in (\kappa^+)^V \setminus \mathrm{rng}(a_\alpha) \cup \{\beta_\alpha\}$. Consider the set

$$B_\alpha = \{\nu < \kappa \mid o^{\vec{U}}(\nu) > 0 \text{ or } (o^{\vec{U}}(\nu) = 0 \text{ and } f_{\alpha \beta^*_\alpha}(\nu) < f_{\alpha \beta_\alpha}(\nu))\}.$$

Then B_α and even $A_\alpha \cap B_\alpha$ are $\tilde{F}$–positive, by the choice of β^*_α. Recall that we have

$$H \supseteq \tilde{F} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\} = Cub_{\aleph_1} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}.$$

So each of A_α's is $\tilde{F} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\}$–positive.
On the other hand the set $A_\alpha \cap B_\alpha$ is in the ideal dual to H, since A_α forces in the forcing with H^+) that $f_{\alpha\beta}$ is the least function of the block F_α.

Case 1. *The kernel of the Δ–system is empty.*
For each $\alpha < \kappa^{++}$ pick a condition p_α in the collapse $Col(\omega, < \kappa^{++})$ of the smallest size which forces "$\kappa \in i_{0\kappa^{++}}(\underset{\sim}{A}_\alpha \cap \underset{\sim}{B}_\alpha)$" which means more explicitly:

$$i_{0\kappa^{++}}(t)(\langle \underset{\sim}{i}(f_{\eta,\xi}) \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \underset{\sim}{i}(c_\eta) \mid \eta \in \mathrm{dom}(a_\alpha) \rangle)(\kappa) = 1 \text{ and } \underset{\sim}{i}(f_{\alpha \beta^*_\alpha})(\kappa) < \underset{\sim}{i}(f_{\alpha \beta_\alpha})(\kappa)).$$

The value of $i_{0\kappa^{++}}(t)(\langle \underset{\sim}{i}(f_{\eta,\xi}) \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \underset{\sim}{i}(c_\eta) \mid \eta \in \mathrm{dom}(a_\alpha) \rangle)$ on κ depends only on an initial segment of $\langle \underset{\sim}{i}(f_{\eta,\xi}) \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \underset{\sim}{i}(c_\eta) \mid \eta \in \mathrm{dom}(a_\alpha) \rangle$. Assume that p_α

[5] a_α is a binary relation, $\mathrm{dom}(a_\alpha)$ refers to the set of its first coordinates and $\mathrm{rng}(a_\alpha)$ refers to the set of its second coordinates.

already decides it, i.e. there is $\xi_\alpha < \kappa^{++}$ such that p_α forces "$i_{0\kappa^{++}}(t)(\langle \underset{\sim}{i}(f_{\eta,\xi}) \restriction \xi_\alpha \mid \langle \eta, \xi\rangle \in a_\alpha\rangle, \langle \underset{\sim}{i}(c_\eta) \restriction \xi_\alpha \mid \eta \in \mathrm{dom}(a_\alpha)\rangle)(\kappa) = 1$". Let $S \subseteq \kappa^{++}$ such that whenever $\alpha < \alpha'$ are in S we have $\xi_\alpha < \alpha'$.
Now, if $\alpha < \alpha'$ are in S, then $\min(\mathrm{dom}(p_{\alpha'}))$ is above $\max(\mathrm{dom}(p_\alpha))$. This follows from the way of constructing generics from the collapse in 2.1 and Δ–system.

Let $\langle \alpha_n | n < \omega\rangle$ be an increasing sequence of elements of S (in $\tilde{V}$). Then one of $\langle p_{\alpha_n} \mid n < \omega\rangle$ always will in any generic subset of the collapse. But this means that $\bigcup_{n<\omega} A_{\alpha_n} \cap B_{\alpha_n} \in \tilde{F} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\} = Cub_{\aleph_1} + \{\nu < \kappa \mid o^{\vec{U}}(\nu) = 0\} = H$. But remember that $A_\alpha \cap B_\alpha$ is in the dual to H ideal, for every $\alpha < \kappa^{++}$. This contradicts the σ–completeness of H.

Case 2. *The kernel of the Δ–system is not empty.*
We may assume that all $\mathrm{rng}(a_\alpha)$, $\alpha < \kappa^{++}$ are the same, since there are only κ^+ many possibilities for them. Pick $\eta < \kappa^+$ to be an ordinal which includes all the ranges. Assume for simplicity that they are η. Also assume that all β^*_α are the same and are equal to η.
Let a be the kernel of $\langle \mathrm{dom}(a_\alpha) \mid \alpha < \kappa^{++}\rangle$. Then $|a| \leq \kappa$. Suppose that $|a| = \kappa$. The case $|a| < \kappa$ is similar. Let $a = \{\rho_\tau \mid \tau < \kappa\}$ be its enumeration in $\tilde{V}$. Denote $\{\rho_\tau \mid \tau < \nu\}$ by $a \restriction \nu$, for every $\nu < \kappa$.
Now we have

$$A_\alpha = \{\nu < \kappa \mid t(\langle f_{\alpha\beta} \mid \alpha \in a, \beta < \eta\rangle, \langle c_{\alpha\beta} \mid \alpha \in a, \beta < \eta\rangle, \langle f_{\alpha\beta} \mid \alpha \in \mathrm{dom}(a_\alpha) \setminus a, \beta < \eta\rangle,$$

$$\langle c_{\alpha\beta} \mid \alpha \in \mathrm{dom}(a_\alpha) \setminus a, \beta < \eta\rangle)(\nu) = 1 \wedge o^{\vec{U}}(\nu) = 0\}.$$

Consider also the following set

$$Z = \{\nu < \kappa \mid o^{\vec{U}}(\nu) > 0 \text{ or } o^{\vec{U}}(\nu) = 0 \text{ and there is no Cohens} * \text{Clubs generic } \vec{f}, \vec{c} \text{ such that}$$

$$t(\langle f_{\alpha\beta} \mid \alpha \in a, \beta < \eta\rangle, \langle c_{\alpha\beta} \mid \alpha \in a, \beta < \eta\rangle, \vec{f}, \vec{c})(\nu) = 1\}.$$

Clearly $A_\alpha \cap Z = \emptyset$, for every $\alpha < \kappa^{++}$.
Let B_α's be as above (even we take β^*_α always to be η). The choice of η insures that $A_\alpha \cap B_\alpha \in F^+$, but clearly not in H^+, since on ν's in $A_\alpha \cap B_\alpha$ we have $f_{\alpha\eta}(\nu) < f_{\alpha\beta_\alpha}(\nu)$.
Using κ^{++}–c.c. of $Col(\omega, < \kappa^{++})$ find $\xi_\alpha < \kappa^{++}$ such that the weakest condition forces that

$$\kappa \in i_{0\kappa^{++}}(Z) \text{ or } i_{0\kappa^{++}}(t)(\langle i_{0\kappa^{++}}(f_{\alpha\beta}) \restriction \xi_\alpha \mid \alpha \in a, \beta < \eta\rangle, \langle i_{0\kappa^{++}}(c_{\alpha\beta}) \restriction \xi_\alpha \mid \alpha \in a, \beta < \eta\rangle,$$

$$\langle i_{0\kappa^{++}}(f_{\alpha\beta}) \restriction \xi_\alpha \mid \alpha \in \mathrm{dom}(a_\alpha) \setminus a, \beta < \eta\rangle, \langle i_{0\kappa^{++}}(c_{\alpha\beta}) \restriction \xi_\alpha \mid \alpha \in \mathrm{dom}(a_\alpha) \setminus a, \beta < \eta\rangle))(\kappa) = 1.$$

Find $S \subseteq \kappa^{++}$ such that for every $\alpha < \alpha'$ in S we have $\xi_\alpha < \alpha'$. Let $\langle \alpha_n | n < \omega\rangle$ be an increasing sequence of elements of S (in $\tilde{V}$). The construction of generics for blocks of Cohen functions in 2.1.5 implies then, as in Lemma 2.2, that the set $Z \cup (\bigcup_{n<\omega}(A_{\alpha_n} \cap B_{\alpha_n}))$ contains a club. This is impossible, since A_α's are H–positive, disjoint to Z, each $A_{\alpha_n} \cap B_{\alpha_n}$ is in the ideal dual to H and H is countably complete.

4 A construction of a precipitous ideal.

Let show now how to construct a precipitous filter in $\tilde{V}$.
The basic idea will be to use κ^{++} as an additional generator. We continue the iteration from $M_{\kappa^{++}}$ using $i_{0\kappa^{++}}(\langle U_\beta \mid \beta < \kappa^{++}\rangle)$. Let $M^2_{\kappa^{++}}$ denotes the final model and $i^2_{0\kappa^{++}} : V \to M^2_{\kappa^{++}}$ the corresponding embedding.

Deal now with a two dimensional analog F_2 of F:

$$F_2 = \{X \subseteq \kappa^2 \mid 0_{Col(\omega,<i^2_{0\kappa^{++}}(\kappa))/G_1*G_2*G_3} \Vdash \langle \kappa, \kappa^{++}\rangle \in \underset{\sim}{i}^{\,2}_{0\kappa^{++}}(\underset{\sim}{X})\}.$$

The crucial difference between F and F_2 is that F_2 has anti-chains of size κ^{++}. Thus we have here $Col(\omega, \{\kappa^{++}\})$. Let $\underset{\sim}{H}$ be an F_2^+ name of a generic function from ω onto κ^{++}. Fix a maximal antichain of elements $\langle A_\xi \mid \xi < \kappa^{++}\rangle$ of F_2^+ which decide $\underset{\sim}{H}(0)$.

Now we turn to a recursive process of extending F_2 similar to those used in [6] and [2] . Let $\langle X_\alpha \mid \alpha < \kappa^{++}\rangle$ be an enumeration of all F_2-positive subsets of κ^2 (in $V[G_0, G_1]$).
Start with $n = 0$. Define a sequence of ordinals $\langle \xi_{\langle\alpha\rangle} \mid \alpha < \kappa^{++}\rangle$ and filters $\langle F_{\langle\alpha\rangle} \mid \alpha < \kappa^{++}\rangle$ by recursion as follows. Let $\alpha < \kappa^{++}$.
If there is $\xi < \kappa^{++}$ such that $\xi \neq \xi_{\langle\beta\rangle}$, for each $\beta < \alpha$ and $X_\alpha \cap A_\xi \in F_2^+$, then let $\xi_{\langle\alpha\rangle}$ be the least such ξ. Extend F_2 to $F_2 + X_\alpha \cap A_{\xi_{\langle\alpha\rangle}}$. Then pick β^0_α to be the least $\beta < \kappa^+$ such that for each $k < \omega, \gamma_1, ..., \gamma_k \in \kappa^+ \setminus \{\beta\}$ and $t \in {}^{\kappa^k}\kappa \cap V$ the set

$$\{\langle \nu_0, \nu_1\rangle \mid f_{\xi_{\langle\alpha\rangle}\beta}(\nu_0) < t(f_{\xi_{\langle\alpha\rangle}\gamma_1}(\nu_0), ..., f_{\xi_{\langle\alpha\rangle}\gamma_k}(\nu_0))\} \in (F_2 + X_\alpha \cap A_{\xi_{\langle\alpha\rangle}})^+.$$

Pick β^1_α to be the least $\beta < \kappa^+$ such that for each $k < \omega, \gamma_1, ..., \gamma_k \in \kappa^+ \setminus \{\beta\}$ and $t \in {}^{\kappa^k}\kappa \cap V$ the set

$$\{\langle \nu_0, \nu_1\rangle \mid f_{\xi_{\langle\alpha\rangle}\beta}(\nu_1) < t(f_{\xi_{\langle\alpha\rangle}\gamma_1}(\nu_1), ..., f_{\xi_{\langle\alpha\rangle}\gamma_k}(\nu_1))\} \in (F_2 + X_\alpha \cap A_{\xi_{\langle\alpha\rangle}})^+.$$

Note that always there are such β^0_α, β^1_α, since a single condition in $Col(\omega, < i^2_{0\kappa^{++}}(\kappa))/G_1 * G_2 * G_3$ decides which of the functions of the $\xi_{\langle\alpha\rangle}$–th block of Cohen functions $\langle f_{\xi_{\langle\alpha\rangle}\beta} \mid \beta < \kappa^+\rangle$ is the least. Now let $F_{\langle\alpha\rangle}$ be the $\aleph_1$-complete filter generated by $F_2 + X_\alpha \cap A_{\xi_{\langle\alpha\rangle}}$ together with all the sets $\{\langle \nu_0, \nu_1\rangle \mid f_{\xi_{\langle\alpha\rangle}\beta^0_\alpha}(\nu_0) < t(f_{\xi_{\langle\alpha\rangle}\gamma_1}(\nu_0), ..., f_{\xi_{\langle\alpha\rangle}\gamma_k}(\nu_0))\}$, $\{\langle \nu_0, \nu_1\rangle \mid f_{\xi_{\langle\alpha\rangle}\beta^1_\alpha}(\nu_1) < t(f_{\xi_{\langle\alpha\rangle}\gamma_1}(\nu_1), ..., f_{\xi_{\langle\alpha\rangle}\gamma_k}(\nu_1))\}$

Intuitively, $f_{\xi_{\langle\alpha\rangle}\beta^0_\alpha}$ is the function corresponding to $\kappa_{\xi_{\langle\alpha\rangle}}$ below $F_{\langle\alpha\rangle}$.[6]
If there is no ξ as above then we leave $\xi_{\langle\alpha\rangle}$ and $F_{\langle\alpha\rangle}$ undefined.

Note that it is impossible to have some $f_{\xi_{\langle\alpha\rangle}\beta}$ that will correspond to κ^{++}. Suppose for a moment that $f_{\alpha^*\beta^*}$ is such a function, for some $\alpha^* < \kappa^{++}$ and $\beta^* < \kappa^+$. Then for κ^{++} many α's and $\beta < \kappa^+$ we will have the set $\{\langle\nu_0,\nu_1\rangle \in \kappa^2 \mid f_{0\beta}(\nu_1) \geq f_{\alpha 0}(\nu_0)\}$ in F_2^+, but now $\{\langle\nu_0,\nu_1\rangle \in \kappa^2 \mid f_{0\beta}(\nu_1) \geq f_{\alpha 0}(\nu_0)\} = \{\langle\nu_0,\nu_1\rangle \in \kappa^2 \mid f_{0\beta}(f_{\alpha^*\beta^*}(\nu_0)) \geq f_{\alpha 0}(\nu_0)\}$ and the complement of the projection of the last set to the first coordinate contains a club for any $\alpha \geq \alpha^* + 1$.

Note also that for each $\mu < \kappa^{++}$, A_μ appears in the list $\langle X_\alpha \mid \alpha < \kappa^{++}\rangle$. Hence, $\{\xi_{\langle\alpha\rangle} \mid \alpha < \kappa^{++}, \xi_{\langle\alpha\rangle} \text{ is defined }\} = \kappa^{++}$. In particular each $\kappa_\mu(\mu > 1)$ has a chance to get a corresponding function.

Set $F(0) = \bigcap\{F_{\langle\alpha\rangle} \mid F_{\langle\alpha\rangle} \text{ is defined }\}$. Denote the corresponding dual ideals by $I_{\langle\alpha\rangle}$ and $I(0)$.

The following lemma follows from the construction (or see [6]):

Lemma 4.1 *For each $X \in F_2^+$, either $X \in F_{\langle\alpha\rangle}$, for some $\alpha < \kappa^{++}$, or the set*

$$\{\xi < \kappa^{++} \mid X \cap A_\xi \in F_2^+\}$$

has cardinality at most κ^+.

Next we deal with $n = 1$.
Let $\alpha < \kappa^{++}$ and $F_{\langle\alpha\rangle}$ be defined. We split (mod$(F_{\langle\alpha\rangle})$ $X_\alpha \cap A_{\xi_\alpha}$ into κ^{++}-many sets which decide $\underset{\sim}{H}(n_\alpha)$, where n_α is the least possible that allows κ^{++}-many possible values. Note that such n_α exists since otherwise $F^+_{\langle\alpha\rangle}$ will force that $\underset{\sim}{H}$ is bounded in κ^{++}, but the filter $F_{\langle\alpha\rangle}$ is obtained from F_2 basically by deciding the function which corresponds to $\kappa_{\xi_{\langle\alpha\rangle}}$.
Let $\langle A_{\alpha\mu} \mid \mu < \kappa^{++}\rangle$ be a maximal antichain below $X_\alpha \cap A_{\xi_\alpha}$ in $F^+_{\langle\alpha\rangle}$ consisting of sets which decide $\underset{\sim}{H}(n_\alpha)$.
Repeat the procedure above and define $\xi_{\langle\alpha\gamma\rangle}, F_{\langle\alpha\gamma\rangle}$, for $\gamma < \kappa^{++}$.
Thus, if there is $\xi < \kappa^{++}$ such that $\xi \neq \xi_{\langle\alpha\beta\rangle}$, for each $\beta < \gamma$ and $X_\gamma \cap A_{\alpha\xi} \in F^+_{\langle\alpha\rangle}$, then let $\xi_{\langle\alpha\gamma\rangle}$ be the least such ξ. Extend $F_{\langle\alpha\rangle}$ to $F_{\langle\alpha\rangle} + X_\alpha \cap A_{\alpha\xi_{\langle\alpha\gamma\rangle}}$. Then pick $\beta^0_{\langle\alpha\gamma\rangle}$ to be the least $\beta < \kappa^+$ such that for each $k < \omega, \delta_1, ..., \delta_k \in \kappa^+ \setminus \{\beta\}$ and $t \in {}^{\kappa^k}\kappa \cap V$ the set

$$\{\langle\nu_0,\nu_1\rangle \mid f_{\xi_{\langle\alpha\gamma\rangle}\beta}(\nu_0) < t(f_{\xi_{\langle\alpha\gamma\rangle}\delta_1}(\nu_0), ..., f_{\xi_{\langle\alpha\gamma\rangle}\delta_k}(\nu_0))\} \in (F_{\langle\alpha\rangle} + X_\alpha \cap A_{\alpha\xi_{\langle\alpha\gamma\rangle}})^+.$$

[6] Actually, it corresponds to some $\kappa_\gamma \geq \kappa_{\xi_{\langle\alpha\rangle}}$, by the construction 2.1.5, since we jumped over κ_τ's with $o^{\tilde{M}_{\kappa^{++}}}(\kappa_\tau) = 0$. Note that such κ_τ's will be still represented. Thus, for example, if $f_{0\beta}$ represents κ_ω, then the function $\nu \mapsto$ the n-th element of the Prikry sequence of $f_{0\beta}(\nu)$ will represent κ_n, for every $n < \omega$.

Pick $\beta^1_{\langle\alpha\gamma\rangle}$ to be the least $\beta < \kappa^+$ such that for each $k < \omega, \delta_1, ..., \delta_k \in \kappa^+\setminus\{\beta\}$ and $t \in {}^{\kappa^k}\kappa \cap V$ the set

$$\{\langle\nu_0, \nu_1\rangle \mid f_{\xi_{\langle\alpha\gamma\rangle}\beta}(\nu_1) < t(f_{\xi_{\langle\alpha\gamma\rangle}\delta_1}(\nu_1), ..., f_{\xi_{\langle\alpha\gamma\rangle}\delta_k}(\nu_1))\} \in (F_{\langle\alpha\rangle} + X_\alpha \cap A_{\alpha\xi_{\langle\alpha\gamma\rangle}})^+.$$

Now let $F_{\langle\alpha\gamma\rangle}$ be the $\aleph_1$-complete filter generated by $F_{\langle\alpha\rangle} + X_\alpha \cap A_{\alpha\xi_{\langle\alpha\gamma\rangle}}$ together with all the sets $\{\langle\nu_0, \nu_1\rangle \mid f_{\xi_{\langle\alpha\gamma\rangle}\beta^0_{\langle\alpha\gamma\rangle}}(\nu_0) < t(f_{\xi_{\langle\alpha\gamma\rangle}\delta_1}(\nu_0), ..., f_{\xi_{\langle\alpha\gamma\rangle}\delta_k}(\nu_0))\}$, $\{\langle\nu_0, \nu_1\rangle \mid f_{\xi_{\langle\alpha\gamma\rangle}\beta^1_{\langle\alpha\gamma\rangle}}(\nu_1) < t(f_{\xi_{\langle\alpha\gamma\rangle}\delta_1}(\nu_1), ..., f_{\xi_{\langle\alpha\gamma\rangle}\delta_k}(\nu_1))\}$.
If there is no ξ as above then we leave $\xi_{\langle\alpha\gamma\rangle}$ and $F_{\langle\alpha\gamma\rangle}$ undefined.
Note that for each $\mu < \kappa^{++}$, $A_{\alpha\mu}$ appears in the list $\langle X_\tau \mid \tau < \kappa^{++}\rangle$. Hence, $\{\xi_{\langle\alpha\gamma\rangle} \mid \gamma < \kappa^{++}, \xi_{\langle\alpha\gamma\rangle}$ is defined $\} = \kappa^{++}$. In particular each $\kappa_\mu(\mu > 1)$ has a chance to get a corresponding function.

Set $F(1) = \bigcap\{F_{\langle\alpha\gamma\rangle} \mid F_{\langle\alpha\gamma\rangle}$ is defined $\}$. Let $I(1)$ be the dual ideal.

The following analog of 4.1 follows from the construction:

Lemma 4.2 *Let $\alpha < \kappa^{++}$ and $F_{\langle\alpha\rangle}$ be defined. For each $X \in F^+_{\langle\alpha\rangle}$, either $X \in F_{\langle\alpha\gamma\rangle}$, for some $\gamma < \kappa^{++}$, or the set*

$$\{\xi < \kappa^{++} \mid X \cap A_{\alpha\xi} \in F^+_{\langle\alpha\rangle}\}$$

has cardinality at most κ^+.

Continue further and define in a similar fashion $F_\sigma, I_\sigma, F(n), I(n), \sigma \in {}^{\omega>}\kappa^{++}, \langle A_{\sigma^\frown\xi} \mid \xi < \kappa^{++}\rangle, n < \omega$.

We will have the following:

Lemma 4.3 *Let $\sigma \in {}^{\omega>}\kappa^{++}$ and F_σ be defined. For each $X \in F^+_\sigma$, either $X \in F_{\sigma^\frown\gamma}$, for some $\gamma < \kappa^{++}$, or the set*

$$\{\xi < \kappa^{++} \mid X \cap A_{\sigma^\frown\xi} \in F^+_\sigma\}$$

has cardinality at most κ^+.

Lemma 4.4 *Let $\sigma \in {}^{\omega>}\kappa^{++}$ and F_σ be defined. Then $F_\sigma \subseteq F(n)^+$, for every $n < \omega$.*

Proof. The lemma is trivial for every $n \leq |\sigma|$ and follows by the construction of $F(n)$'s for $n > |\sigma|$ (see [6] for similar arguments).
□

Finally set

$$F(\omega) = \text{ the closure under } \omega \text{ intersections of } \bigcup_{n<\omega} F_n$$

and

$$I(\omega) = \text{ the closure under } \omega \text{ unions of } \bigcup_{n<\omega} I_n.$$

The next two lemmas follow easily from the definitions.

Lemma 4.5 *$F_2 \subseteq F(0) \subseteq ... \subseteq F(n) \subseteq ... \subseteq F(\omega)$ and $I_2 \subseteq I(0) \subseteq ... \subseteq I(n) \subseteq ... \subseteq I(\omega)$.*

Lemma 4.6

$$F(\omega) = \{X \subseteq \kappa^2 \mid \exists \langle X_n \mid n < \omega\rangle \forall n < \omega \quad X_n \in F(n) \text{ and } X = \bigcap_{n<\omega} X_n\}$$

and

$$I(\omega) = \{X \subseteq \kappa^2 \mid \exists \langle X_n \mid n < \omega\rangle \forall n < \omega \quad X_n \in I(n) \text{ and } X = \bigcup_{n<\omega} X_n\}.$$

Lemma 4.7 *$I(\omega)$ is a proper κ-complete ideal over κ^2.*

Proof. Let $\langle X_n \mid n < \omega\rangle$ be a sequence such that $X_n \in I(n)$, for every $n < \omega$ and $X = \bigcup_{n<\omega} X_n$. Assume that each X_n is F_2-positive. Consider for every $n < \omega$ the set

$$Z_n = \{\xi < \kappa^{++} \mid X_n \cap A_\xi \in F_2^+\}.$$

Then, by Lemmas 4.1,4.4 $|Z_n| \leq \kappa^+$. Hence $|\bigcup_{n<\omega} Z_n| \leq \kappa^+$. Note that

$$Z := \{\xi < \kappa^{++} \mid X \cap A_\xi \in F_2^+\} = \bigcup_{n<\omega} Z_n$$

and so Z has cardinality at most κ^+ as well.
Pick now any $\xi \in \kappa^{++} \setminus Z$. Then $X \cap A_\xi \notin F_2^+$ which implies that $I(\omega)$ is a proper ideal, since, in particular, X never can be κ^2.
□

Lemma 4.8 *$X \in F(\omega)^+$ iff there is $\sigma \in {}^{\omega>}\kappa^{++}$ such that $X \in F_\sigma$.*

Proof. ($\Rightarrow$) Let $X \in F(\omega)^+$. Suppose that $X \notin F_\sigma$, for any $\sigma \in {}^{\omega>}\kappa^{++}$. Set

$$Z_0 = \{\xi < \kappa^{++} \mid X \cap A_\xi \in F_2^+\}.$$

By Lemmas 4.1,4.4, $|Z_0| \leq \kappa^+$. Then for every $\xi \in Z_0$, set

$$Z_{1\xi} = \{\rho < \kappa^{++} \mid X \cap A_\xi \cap A_{\xi\rho} \in F_{\langle\xi\rangle}^+\}$$

and

$$Z_1 = \bigcup_{\xi \in Z_0} Z_{1\xi}.$$

Then $|Z_1| \leq \kappa^+$, by Lemmas 4.2,4.4.
Similarly define Z_n, for each $n < \omega$.
There is $\eta_0 < \kappa^{++}$ such that

$$X \Vdash_{F_2^+} \underset{\sim}{H}(0) < \eta_0,$$

since $|Z_0| \leq \kappa^+$. Similar for each $n < \omega$ there will be $\eta_n < \kappa^{++}$ such that

$$X \Vdash_{F_2^+} \underset{\sim}{H}(n) < \eta_n.$$

But then

$$X \Vdash_{F_2^+} \text{rng}(\underset{\sim}{H}) \text{ is bounded in } \kappa^{++}.$$

Which is impossible by the choice of $\underset{\sim}{H}$. Contradiction.
($\Leftarrow$) The argument repeats those of Lemma 4.7 with F_2 replaced by F_σ.
Let $X \in F_\sigma$, for some $\sigma \in {}^{\omega>}\kappa^{++}$.
Suppose that $X \in I(\omega)$. Let $\langle X_n \mid n < \omega\rangle$ be a sequence such that $X_n \in I(n)$, for every $n < \omega$ and $X \subseteq \bigcup_{n<\omega} X_n$. Assume that each X_n is F_σ-positive. Consider for every $n < \omega$ the set

$$Z_n = \{\xi < \kappa^{++} \mid X_n \cap A_{\sigma^\frown\xi} \in F_\sigma^+\}.$$

Then, by Lemmas 4.3,4.4, $|Z_n| \leq \kappa^+$. Hence $|\bigcup_{n<\omega} Z_n| \leq \kappa^+$. Note that

$$Z := \{\xi < \kappa^{++} \mid X \cap A_{\sigma^\frown\xi} \in F_\sigma^+\} = \bigcup_{n<\omega} Z_n$$

and so Z has cardinality at most κ^+ as well.
Pick now any $\xi \in \kappa^{++} \setminus Z$. Then $X \cap A_{\sigma^\frown\xi} \notin F_\sigma^+$, but this is impossible since $X \in F_\sigma$ and $A_{\sigma^\frown\xi} \in F_\sigma^+$. Contradiction.
$\square$

Lemma 4.9 *$F(\omega)$ is a precipitous filter over κ^2.*

Proof. It is enough to show that for each $X \in F(\omega)^+$ and $\eta < \kappa^{++}$ there is $Y \subseteq X, Y \in F(\omega)^+$ deciding which function from $\{f_{\eta\beta} \mid \beta < \kappa^+\}$ will be least one (i.e. basically correspond to κ_η). By Lemma 4.8 there is $\sigma \in {}^{\omega>}\kappa^{++}$ such that $X \in F_\sigma$. Find $\gamma < \kappa^{++}$ such that $\xi_{\sigma^\frown\gamma} = \eta$. Set $Y = X \cap A_{\sigma\xi_{\sigma^\frown\gamma}}$. It will be as desired.
$\square$

5 Open problems

In conclusion let us state some problems on the subject that remain open.

Question 1. Is the assumption $o(\kappa) = \kappa^{++}$ needed for a model with a precipitous ideal on $\aleph_1$ but without a normal one?

We think that it is likely to be possible to show that if $\aleph_1$ is ∞–semi precipitous with a witnessed forcing satisfying $\aleph_3$–c.c. and with image of $\aleph_1$ under the corresponding generic embedding is at least $\aleph_3$, then $o(\kappa) = \kappa^{++}$ in an inner model. But probably there is no need to go via a construction of such ∞–semi precipitous.

Question 2. Is it possible to have a GCH model with a precipitous ideal on $\aleph_1$ but without a normal one?

By [7] large cardinals not far from $o(\kappa) = \kappa^{++}$ are needed for such a model.

Question 3. Is it possible to generalize the present result to cardinals bigger than $\aleph_1$? Simplest case: Is there a model with a precipitous ideal on $\aleph_2$ but without a normal one?

The next question is well known with partial answers given by Schimmerling, Velickovic [13], Woodin [14](8.1 Condensation Principles) and recently by Wu.

Question 4. Is it consistent that there is a supercompact cardinal and $\aleph_1$ does not carry a precipitous ideal?

The construction above can be carried out below a supercompact cardinal and so it provides a model with a supercompact and no precipitous filters on $\aleph_1$ which extend $Cub_{\aleph_1}$ restricted to a stationary set. It is natural so ask the following question:

Question 5. Is it consistent that there is a supercompact cardinal and $\aleph_1$ does not carry precipitous filters that are Q–points, i.e. isomorphic to filters which extend $Cub_{\aleph_1}$?

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