Gap 3: Lectures March - May 2008

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1 The Preparation Forcing

We assume GCH.
A condition in the preparation forcing $\mathcal{P}'$, which we define below, will consists basically of an elementary chain of models of cardinality κ^{++} and a directed system elementary submodels of cardinality κ^+. Inside this directed system a crucial role will be played by a certain elementary chain which will be called *central line* . Let us give first a definition of both elementary chains.

Definition 1.1 The set $\mathcal{P}''$ consists of elements of the form

$$\langle B^{1\kappa^+}, A^{1\kappa^{++}} \rangle$$

so that the following hold:

1. $A^{1\kappa^{++}}$ is a continuous closed chain of length less than κ^{+3} of elementary submodels of $\langle H(\kappa^{+3}), \in, <, \subseteq, \kappa \rangle$ each of cardinality κ^{++}.

2. For each $X \in A^{1\kappa^{++}}$, we have $X \cap \kappa^{+3} \in On$. So, $X \supseteq \kappa^{++}$. Further we shall frequently identify such model X with the ordinal $X \cap \kappa^{+3}$ and also view $A^{1\kappa^{++}}$ as a closed set of ordinals.

3. If X is a non-limit element of the chain $A^{1\kappa^{++}}$ then

   (a) $A^{1\kappa^{++}} \restriction X := \{Y \mid Y \subset X, Y \in A^{1\kappa^{++}}\} \in X$,

(b) ${}^{\kappa^+}X \subseteq X$.

4. $B^{1\kappa^+}$ is a continuous closed chain of length less than κ^{++} of elementary submodels of $\langle H(\kappa^{+3}), \in, <, \subseteq, \kappa\rangle$, each of cardinality κ^+. $B^{1\kappa^+}$ has the last element which we denote by $\max(B^{1\kappa^+})$.

5. For each $X \in B^{1\kappa^+}$, we have $X \cap \kappa^{++} \in On$. Hence $X \supseteq \kappa^+$.

6. If X is a non-limit element of the chain $B^{1\kappa^+}$ then

   (a) $B^{1\kappa^+} \restriction X := \langle Y \mid Y \subset X, Y \in B^{1\kappa^+}\rangle \in X$,

   (b) ${}^{\kappa}X \subseteq X$,

   (c) If $\delta < \sup(X)$ for some $\delta \in A^{1\kappa^{++}}$ (we identify here an element of $A^{1\kappa^{++}}$ with an ordinal), then $\min(X \setminus \delta) \in A^{1\kappa^{++}}$.

The following technical notion will be needed in order to define $\mathcal{P}'$ (and will be used further as well).

Definition 1.2 Suppose that $\langle B^{1\kappa^+}, A^{1\kappa^{++}}\rangle \in \mathcal{P}''$, $F \in B^{1\kappa^+}$ and $F_0, F_1 \in F$. We say that the triple F_0, F_1, F is of Δ-*system type* iff

1. F_0 is the immediate predecessor of F in the chain $B^{1\kappa^+}$,

2. $F_1 \prec F$,

3. ?(it looks like it is possible also without this) ${}^{\kappa}F_1 \subseteq F_1$,

4. ?$A^{1\kappa^{++}} \restriction \sup(F_1) \in F_1$. Replacement:

5. If $\delta < \sup(F_1 \cap On)$ for some $\delta \in A^{1\kappa^{++}}$, then $\min((F_1 \cap On) \setminus \delta) \in A^{1\kappa^{++}}$.

6. There are $\alpha_0, \alpha_1 \in A^{1\kappa^{++}}$ such that

   (a) $\mathrm{cof}(\alpha_0) = \mathrm{cof}(\alpha_1) = \kappa^{++}$,

   (b) $\alpha_0 \in F_0$ and $\alpha_1 \in F_1$,

(c) $F_0 \cap F_1 \cap On = F_0 \cap \alpha_0 = F_1 \cap \alpha_1$,

(d) either $\alpha_0 > \sup(F_1 \cap On)$ or $\alpha_1 > \sup(F_0 \cap On)$.

Intuitively, this means that F_0, F_1 behave as in a Δ-system with the common part below $\min \alpha_0, \alpha_1$.
Further let us call α_0, α_1 the witnessing ordinals for F_0, F_1, F.

The next condition will require more similarity:

7. (isomorphism condition)
the structures

$$\langle F_0, \in, <, \subseteq, \kappa, A^{1\kappa^{++}} \cap F_0, f_{F_0} \rangle$$

and

$$\langle F_1, \in, <, \subseteq, \kappa, A^{1\kappa^{++}} \cap F_1,\ f_{F_1} \rangle$$

are isomorphic over $F_0 \cap F_1$, i.e. the isomorphism $\pi_{F_0 F_1}$ between them is the identity on $F_0 \cap F_1$, where ? (it seems unnecessary for gap 3 to have this f_{F_i}) $f_{F_0} : \kappa^+ \longleftrightarrow F_0$, $f_{F_1} : \kappa^+ \longleftrightarrow F_1$ are some fixed in advance bijections.

Note that, in particular, we will have that $\mathrm{otp}(F_0) = \mathrm{otp}(F_1)$ and $F_0 \cap \kappa^{++} = F_1 \cap \kappa^{++}$.

Definition 1.3 The set $\mathcal{P}'$ consists of elements of the form

$$\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle$$

so that the following hold:

1. $A^{0\kappa^+} \in A^{1\kappa^+}$,

2. every $X \in A^{1\kappa^+}$ is either equal to $A^{0\kappa^+}$ or belongs to it,

3. $C^{\kappa^+} : A^{1\kappa^+} \to P(A^{1\kappa^+})$,

4. for every $X \in A^{1\kappa^+}$, $\langle C^{\kappa^+}(X), A^{1\kappa^{++}}\rangle \in \mathcal{P}''$ and X is the maximal model of $C^{\kappa^+}(X)$. In particular, each $C^{\kappa^+}(X)$ is an increasing continuous chain of models of cardinality κ^+.

5. (Coherence) If $X, Y \in A^{1\kappa^+}$ and $X \in C^{\kappa^+}(Y)$, then $C^{\kappa^+}(X)$ is an initial segment of $C^{\kappa^+}(Y)$ with X being the largest element of it.

   We call $C^{\kappa^+}(A^{0\kappa^+})$ *central line* of $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$. The following conditions describe a special way in which $A^{1\kappa^+}$ is generated from the central line.

6. Let $B \in A^{1\kappa^+}$. Then $B \in C^{\kappa^+}(A^{0\kappa^+})$ (i.e., it is on the central line) or there are $n < \omega$ and sequences $\langle A_1, ..., A_n\rangle$, $\langle B_1, ..., B_n\rangle$ of elements of $A^{1\kappa^+}$ such that

   (a) $A_1 \in C^{\kappa^+}(A^{0\kappa^+})$ is the least model of the central line $C^{\kappa^+}(A^{0\kappa^+})$ that contains B.

   (b) A_1 is a successor model in $C^{\kappa^+}(A^{0\kappa^+})$. Let A_1^- denotes its immediate predecessor in $C^{\kappa^+}(A^{0\kappa^+})$.

   (c) The triple A_1^-, B_1, A_1 is of a Δ-system type with respect to $A^{1\kappa^{++}}$.

   (d) For each $m, 1 < m \le n$,

      i. $A_m \in C^{\kappa^+}(B_{m-1})$ (i.e. it is on the central line of B_{m-1}) is the least model in $C^{\kappa^+}(B_{m-1})$ that contains B.

      ii. A_m is a successor model in $C^{\kappa^+}(B_{m-1})$. Let A_m^- denotes its immediate predecessor in $C^{\kappa^+}(B_{m-1})$.

      iii. The triple A_m^-, B_m, A_m is of a Δ-system type with respect to $A^{1\kappa^{++}}$.

   (e) $B \in C^{\kappa^+}(B_n)$.

   We refer to the sequence $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n\rangle$ as the *walk* from $A^{0\kappa^+}$ (or from the central line) to B. Denote it by $wk(A^{0\kappa^{++}}, B)$.

Let us call n *distance of B from the central line*, denote it by $dcl(B)$. If it is on the central line, then set $dcl(B) = 0$.

The next condition strengthens a bit the isomorphism condition (7) of Definition 1.2.

7. (isomorphism condition) Let $F_0, F_1, F \in A^{1\kappa^+}$ be of a Δ-system type and $X \in A^{1\kappa^+}$. Then $X \in F_0$ iff $\pi_{F_0F_1}[X] \in F_1 \cap A^{1\kappa^+}$. This means that the structures of 1.2(7) remain isomorphic even if we add $F_0 \cap A^{1\kappa^+}$ to the first and $F_1 \cap A^{1\kappa^+}$ to the second.

8. (uniqueness) Let $F_0, F_1, F_1', F \in A^{1\kappa^+}$. If both triples F_0, F_1, F and F_0, F_1', F are of a Δ-system type, then $F_1 = F_1'$.

   Note that both conditions 7, 8 can be stated equivalently only in the case when F is on the central line.

Let us define also a walk to an ordinal.

Definition 1.4 Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $\alpha \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. The sequence $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, A_{n1}\rangle$ of elements of $A^{1\kappa^+}$ is called a *walk* from $A^{0\kappa^+}$ to α iff

1. $A_1 \in C^{\kappa^+}(A^{0\kappa^+})$ is the least model of $C^{\kappa^+}(A^{0\kappa^+})$ with $\alpha \in A_1$,

2. either

   - A_1 is the least model of $C^{\kappa^+}(A^{0\kappa^+})$ and then $A_n^- = A_1$,i.e. the walk consists of A_1 alone,

     or

   - A_1^- exists, it is the immediate predecessor of A_1 on $C^{\kappa^+}(A^{0\kappa^+})$. If A_1^- is the unique immediate predecessor of A_1, or there is an other one but α does belong to it, then the walk consists of $\langle A_1, A_1^-\rangle$. Otherwise, A_1^-, B_1, A_1 are of Δ-system type, $\alpha \in B_1$ and the walk continues.

3. For each $m, 1 < m \leq n$,

   (a) $A_m \in C^{\kappa^+}(B_{m-1})$ (i.e. it is on the central line of B_{m-1}) is the least model in $C^{\kappa^+}(B_{m-1})$ with $\alpha \in A_m$, either

   - A_m is the least model of $C^{\kappa^+}(B_{m-1})$ and then $B_n^- = A_m$, or
   - A_m^- exists, it is the immediate predecessor of A_m on $C^{\kappa^+}(B_{m-1})$. If A_m^- is the unique immediate predecessor of A_m, or there is an other one but α does belong to it, then $A_n^- = A_m^-$. Otherwise, A_m^-, B_m, A_m are of Δ-system type, $\alpha \in B_m$ and the walk continues.

4. $\alpha \in A_n$ and either

   - A_n is the least model of $C^{\kappa^+}(B_{n-1})$ and then $A_n^- = A_{n1} = A_n$, i.e. the walk terminates at A_n;
   or
   - there exists the immediate predecessor of A_n in $C^{\kappa^+}(B_{m-1})$. Then A_n^- is this immediate predecessor of A_n and there is no $Z \in A^{1\kappa^+}$ such that A_n^-, Z, A_n is of a Δ-system type. In this case $A_{n1} = A_n^-$ and the walk terminates at A_n^-;
   or
   - there exists the immediate predecessor of A_n in $C^{\kappa^+}(B_{m-1})$. Then A_n^- is this immediate predecessor of A_n and there is $Z \in A^{1\kappa^+}$ such that A_n^-, Z, A_n is of a Δ-system type, witnessed by $\xi_0 \in A_n^- \cap A^{1\kappa^{++}}, \xi_1 \in Z \cap A^{1\kappa^{++}}$. Then $\alpha \notin Z$. If $\alpha \notin [\xi_1, \sup(Z)]$, then $A_{n1} = A_n^-$ and the walk to α terminates at A_n^-. If $\alpha \in [\xi_1, \sup(Z)]$, then $A_{n1} = Z$.

Note that walks to ordinals terminate by the last model A_n to which the ordinal belongs followed by its immediate predecessor in $C^{\kappa^+}(A_n)$, whenever such predecessor exists.

Definition 1.5 (Complexity of walks)
Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$.

- Suppose that $A, B \in A^{1\kappa^+}$. We say that the walk from $A^{0\kappa^+}$ to A is *simpler* than the walk from $A^{0\kappa^+}$ to B iff

  1. $A \subset B$, or
  2. $A \not\subset B, B \not\subset A, A \neq B$ and if $F \in A^{1\kappa^+}$ is the last common point of both walks, then $A \subseteq F_0$, where F_0 is the immediate predecessor of F in $C^{\kappa^+}(F)$. Note that necessarily, there is $F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F is a triple of a Δ-system type and $B \subseteq F_1$.

- Suppose that $A \in A^{1\kappa^+}$ and $\alpha \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. We say that the walk from $A^{0\kappa^+}$ to A is *simpler* than the walk from $A^{0\kappa^+}$ to α iff

  1. A is one of the models of the walk to α,
     or
  2. if F is the last common model of the walks, then $A \in C^{\kappa^+}(F)$, or $A \notin C^{\kappa^+}(F)$ and $A \subseteq F_0$, where F_0 is the immediate predecessor of F in $C^{\kappa^+}(F)$. Note, if the second possibility occurs, then, necessarily, there is $F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F is a triple of a Δ-system type and $\alpha \in F_1$.

- Suppose that $\alpha, \beta \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. We say that the walk from $A^{0\kappa^+}$ to α is *simpler* than the walk from $A^{0\kappa^+}$ to β iff $\alpha \neq \beta$, there is $F \in A^{1\kappa^+}$ which is the last common point of both walks and

  1. there are $D, E \in C^{\kappa^+}(F)$ such that $\alpha \in D \in E$ and $\beta \in E \setminus D$,
     or
  2. there are $F_0, F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F are of a Δ-system type, $F_0 \in C^{\kappa^+}(F)$, $\alpha \in F_0$ and $\beta \in F_1$,

3. there are $F_0, F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F are of a Δ-system type, $F_0 \in C^{\kappa^+}(F)$,ξ_0, ξ_1 the witnessing ordinals, and $\beta \in F \setminus (F_0 \cup F_1)$, $\xi_1 \leq \beta \leq \sup(F_1)$ and $\alpha \in F_1$,
   or

4. there are $F_0, F_1 \in A^{1\kappa^+}$ such that F_0, F_1, F are of a Δ-system type, $F_0 \in C^{\kappa^+}(F)$,ξ_0, ξ_1 the witnessing ordinals, and $\alpha \in F \setminus (F_0 \cup F_1)$, $\beta \in F_1$ and $\alpha < \xi_1$ or $\alpha > \sup(F_1)$.

The above defines a well-founded relation. We will use further the walks complexity in inductive arguments.

Lemma 1.6 *Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Then*

1. *$\langle\langle B, A^{1\kappa^+} \cap (B \cup \{B\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (B \cup \{B\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$.*

2. *If $B' \in A^{1\kappa^+}$ and $B' \subsetneq B$, then $B' \in B$.*

Proof. We prove both statements simultaneously by an induction on $dcl(B)$ -the distance from the central line. If B is on the central line, then it is clear. Suppose that B is not on the central line. Consider the walk $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n\rangle$ from $A^{0\kappa^+}$ to B. We have

$$\langle\langle A_1^-, A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'.$$

Recall that A_1^-, B_1, A_1 are of the Δ-system type. Hence we have the isomorphism $\pi_{A_1^-, B_1}$ between A_1^- and B_1 which preserves all the relevant structure. In particular, it will move the walk from A_1^- to a model in $A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\})$ to the walk from B_1 to the corresponding under $\pi_{A_1^-, B_1}$ model of $A^{1\kappa^+} \cap (B_1 \cup \{B_1\})$. This easily implies that

$$\langle\langle B_1, A^{1\kappa^+} \cap (B_1 \cup \{B_1\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (B_1 \cup \{B_1\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'.$$

Suppose now that we have some $B' \in A^{1\kappa^+}, B' \subsetneq B_1$. If $B' \not\subseteq A_1^-$, then the walk from $A^{0\kappa^+}$ to B' goes via B_1, and hence $B' \in B_1$. Suppose that

$B' \subseteq A_1^-$. It is impossible to have $B' = A_1^-$, since then

$$A_1^- \cap B_1 \supseteq B' = A_1^-,$$

which is clearly not the case. So, $B' \subsetneq A_1^-$. Then the walk from $A^{0\kappa^+}$ to B' goes via A_1^-, and hence $B' \in A_1^-$. Then $\pi_{A_1^-, B_1}(B') \in B_1$, but

$$\pi_{A_1^-, B_1}(B') = \pi_{A_1^-, B_1}{}``B' = B'.$$

So we are done.
Hence, $A^{1\kappa^+} \cap (B_1 \cup \{B_1\}) = A^{1\kappa^+} \cap \mathcal{P}(B_1)$.

Now we deal with B and $\langle\langle B_1, A^{1\kappa^+} \cap (B_1 \cup \{B_1\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (B_1 \cup \{B_1\})\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$. The walk distance from B_1 to B is shorter than those from $A^{0\kappa^+}$ to B. So the induction hypothesis applies.
$\square$

The next lemma is trivial.

Lemma 1.7 *Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $Z \in A^{1\kappa^{++}}$ is so that $Z \cap \kappa^{+3} \geq \sup(A^{0\kappa^+})$. Then $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, \{Y \in A^{1\kappa^{++}} \mid Y \subseteq Z\}\rangle \in \mathcal{P}'$.*

Let us introduce the following notation:

Definition 1.8 Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Then set

$$p \restriction B := \langle\langle B, A^{1\kappa^+} \cap (B \cup \{B\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (B \cup \{B\})\rangle, A^{1\kappa^{++}}\rangle.$$

We call $p \restriction B$ the restriction of p to B.
Similar, if $Z \in A^{1\kappa^{++}}$, then set

$$p \restriction Z := \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, \{Y \in A^{1\kappa^{++}} \mid Y \subseteq Z\}\rangle$$

. Also, let $p \restriction (B, Z) := (p \restriction B) \restriction Z$.

By the previous lemmas, $p \restriction (B, Z) \in \mathcal{P}'$.

The next lemma follows easily from the definitions.

Lemma 1.9 *Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$, $A \in A^{1\kappa^+}$ *and* $\delta \in A^{1\kappa^{++}}$. *If* $\delta < \sup(A)$, *then* $\min(A \setminus \delta) \in A^{1\kappa^{++}}$.

Proof. By 1.3(4), $\langle C^{\kappa^+}(A), A^{1\kappa^{++}}\rangle \in \mathcal{P}''$. So, it satisfies 1.1(6(d)), (or ?**??**) and we are done, if A is a successor model of $C^{\kappa^+}(A)$. Suppose A is a limit model of $C^{\kappa^+}(A)$. Let $\langle A_i \mid i < \eta\rangle$ be an increasing sequence of successor models of $C^{\kappa^+}(A)$ with $\bigcup_{i<\eta} A_i = A$. Now, $\delta < \sup(A)$, so starting with some $i^* < \eta$, we have $\delta < \sup(A_i)$. Just note that $i < j$ implies $A_i \in A_j$, hence $\langle \sup(A_i) \mid i < \eta\rangle$ is an increasing sequence of ordinals with limit $\sup(A)$. Set $\alpha_i = \min(A_i \setminus \delta)$, for each $i, i^* \leq i < \eta$. By 1.1(6(d)), $\alpha_i \in A^{1\kappa^{++}}$. Clearly, $i \geq j$ implies $\alpha_i \leq \alpha_j$. Hence, the sequence $\langle \alpha_i \mid i^* \leq i < \eta\rangle$ is eventually constant. Let α^* be this constant value. Then $\min(A \setminus \delta) = \alpha^*$ and we are done.
$\square$

Definition 1.10 Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $A, B \in A^{1\kappa^+}$. We say that A *satisfies the intersection property with respect to* B or shortly $ip(A, B)$ iff either

1. $A \supseteq B$, or
2. $B \supseteq A$, or
3. $A \not\supseteq B, B \not\supseteq A$, and then there are $A' \in A^{1\kappa^+} \cap (A \cup \{A\})$ and $\eta \in A^{1\kappa^{++}} \cap A')$ such that
$$A \cap B = A' \cap \eta,$$
or just
$$A \cap B = A'.$$

Let $ipb(A, B)$ denotes that both $ip(A, B)$ and $ip(B, A)$ hold.

Lemma 1.11 *(The intersection lemma) Let* $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}$ *and* $X, Y \in A^{1\kappa^+}$. *Then* $ipb(X, Y)$.

Proof. Assume that $X \not\supseteq Y, Y \not\supseteq X$.
Consider the walks $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n\rangle$ from $A^{0\kappa^+}$ to X and $\langle D_1, D_1^-, E_1, ..., D_{m-1}, D_{m-1}^-, E_{m-1}, D_m, D_m^-, E_m\rangle$ from $A^{0\kappa^+}$ to Y. Let $B_k = E_k$ be the last place up to which the walks coincide. Then we have both A_{k+1}, D_{k+1} in $C^{\kappa^+}(B_k)$ but at different places.
Suppose first that A_{k+1} is above D_{k+1}. Then $A_{k+1}^- = D_{k+1}$ or $A_{k+1} \supset D_{k+1}$, and then $D_{k+1} \in A_{k+1}^-$. Now, $A_{k+1}^-, B_{k+1}, A_{k+1}$ are of a Δ-system type. Hence by Definition 1.2(6), there are ordinals $\alpha_0, \alpha_1 \in A^{1\kappa^{++}} \cap A_{k+1}, \alpha_0 \in A_{k+1}^-$ and $\alpha_1 \in B_{k+1}$ such that

$$A_{k+1}^- \cap B_{k+1} = A_{k+1}^- \cap \alpha_0 = B_{k+1} \cap \alpha_1.$$

Recall that $X \subseteq B_{k+1}$ and $Y \subseteq A_{k+1}^-$. Hence,

$$X \cap Y = (X \cap B_{k+1}) \cap (Y \cap A_{k+1}^-) = (X \cap \alpha_1) \cap (Y \cap \alpha_0).$$

Let us use (7) of 1.3. Then

$$X' = \pi_{B_{k+1}, A_{k+1}^-}[X] \in A_{k+1} \cap A^{1\kappa^+}.$$

Also,

$$X \cap \alpha_1 = X' \cap \alpha_0,$$

since the isomorphism π_{B_{k+1}, A_{k+1}^-} is the identity over $B_{k+1} \cap A_{k+1}^-$. Hence,

$$X \cap Y = X \cap \alpha_1 \cap Y = X' \cap \alpha_0 \cap Y.$$

Consider

$$p := \langle\langle A_{k+1}^-, A^{1\kappa^+} \cap (A_{k+1}^- \cup \{A_{k+1}^-\}), C^{\kappa^+} \restriction A^{1\kappa^+} \cap (A_{k+1}^- \cup \{A_{k+1}^-\})\rangle, A^{1\kappa^{++}}\rangle.$$

By Lemma 1.6, it is in $\mathcal{P}'$. We can apply the inductive hypothesis to p,X' and Y, since the walk from A_{k+1}^- to X' shorter than those from $A^{0\kappa^+}$ to X (it is just a copy under π_{B_{k+1}, A_{k+1}^-} of the final segment $\langle B_{k+1}, ..., A_n, A_n^-, B_n\rangle$

of the original walk to X from $A^{0\kappa^+}$). Hence there are $Y' \in A^{1\kappa^+} \cap (Y \cup \{Y\})$ and $\eta \in A^{1\kappa^{++}} \cap A$ such that

$$X' \cap Y = Y' \cap \eta.$$

Then

$$X \cap Y = Y \cap Y' \cap \eta \cap \alpha_0.$$

If $\alpha_0 \in Y$, then we are done. Suppose otherwise. If $\alpha_0 \geq \sup(Y)$, then we can just remove it from the intersection above. If $\alpha_0 < \sup(Y)$, then replace it by $\min(Y \cap \alpha_0)$, which is in $A^{1\kappa^{++}}$ by Lemma 1.9.
This shows $ip(Y, X)$. Finally, using $\pi_{A^-_{k+1}, B_{k+1}}$ and moving Y to B_{k+1}, the same argument shows $ip(X, Y)$.
$\square$

It is easy to deduce the following generalization using an induction:

Lemma 1.12 *Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}$ and $A_1, ..., A_n \in A^{1\kappa^+}$, for some $n < \omega$. Then there are $A' \in A^{1\kappa^+} \cap (A_1 \cup \{A_1\})$ and $\eta \in A^{1\kappa^{++}} \cap A'$ such that $A_1 \cap ... \cap A_n = A' \cap \eta$ or just $A_1 \cap ... \cap A_n = A'$.*

We need to allow a possibility to change the component C^{κ^+} in elements of $\mathcal{P}'$ and replace one central line by another. It is essential for the definition of an order on $\mathcal{P}'$ given below.

Definition 1.13 Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Define $swt(p, B)$ (here swt stands for switch) to be

$$\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, D^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle ,$$

where D^{κ^+} is obtained from C^{κ^+} as follows:
$D^{\kappa^+} = C^{\kappa^+}$ unless B has exactly two immediate predecessors in $A^{1\kappa^+}$. If $B_0 \neq B_1$ are such predecessors of B and, say $B_0 \in C^{\kappa^+}(B)$, then we set $D^{\kappa^+}(B) = C^{\kappa^+}(B_1)^\frown B$. Extend D^{κ^+} on the rest in the obvious fashion just replacing $C^{\kappa^+}(B_0)$ by $C^{\kappa^+}(B_1)$ for models including B and then moving over

isomorphic models.
Intuitively, we switched here from B_0 to B_1.

Note that $swt(swt(p,B),B)=p$.
Define $q=swt(p,B_1,\dots,B_n)$ by applying the operation swt n-times: $p_{i+1}=swt(p_i,B_i)$, for each $1\le i\le n$, where $p_1=p$ and $q=p_{n+1}$.
The following simple observation will be useful further.

Lemma 1.14 *Let $p=\langle\langle A^{0\kappa^+}(p),A^{1\kappa^+}(p),C^{\kappa^+}(p)\rangle,A^{1\kappa^{++}}(p)\rangle\in\mathcal{P}'$ and $B\in A^{1\kappa^+}(p)$. Then there are $E_1,...,E_m\in A^{1\kappa^+}(p)$ such that $B\in C^{\kappa^+}(q)(A^{0\kappa^+}(p))$, where*

$$q=\langle\langle A^{0\kappa^+}(p),A^{1\kappa^+}(p),C^{\kappa^+}(q)\rangle,A^{1\kappa^{++}}(p)\rangle=swt(p,E_1,...,E_m).$$

Proof. If $B\in C^{\kappa^+}(p)(A^{0\kappa^+}(p)$, then let $q=p$. Otherwise, Consider the walk $\langle A_1,A_1^-,B_1,...,A_{n-1},A_{n-1}^-,B_{n-1},A_n,A_n^-,B_n\rangle$ from $A^{0\kappa^+}$ to B. Then

$$q=\langle\langle A^{0\kappa^+}(p),A^{1\kappa^+}(p),C^{\kappa^+}(q)\rangle,A^{1\kappa^{++}}(p)\rangle=swt(p,A_1^-,B_1,A_2^-,B_2,...,A_n^-,B_n)$$

will be as desired.
$\square$

Definition 1.15 Let $r,q\in\mathcal{P}'$. Then $r\ge q$ (r is stronger than q) iff there is $p=swt(r,B_1,\dots,B_n)$ for some $B_1,\dots,B_n$ appearing in r so that the following hold, where

$$p=\langle\langle A^{0\kappa^+},A^{1\kappa^+},C^{\kappa^+}\rangle,A^{1\kappa^{++}}\rangle$$
$$q=\langle\langle B^{0\kappa^+},B^{1\kappa^+},D^{\kappa^+}\rangle,B^{1\kappa^{++}}\rangle$$

(1) $A^{1\kappa^{++}}\cap(\max(B^{1\kappa^{++}})+1)=B^{1\kappa^{++}}$

(2) $B^{0\kappa^+}\in C^{\kappa^+}(A^{0\kappa^+})$ and $D^{\kappa^+}(B^{0\kappa^+})$ is an initial segment of $C^{\kappa^+}(A^{0\kappa^+})$

(3) $q=p\restriction(B^{0\kappa^+},B^{0\kappa^{++}})$(as it was defined in 1.8).

Remarks (1) Note that if $t = swt(p, B_0, \ldots, B_n)$ is defined, then $t \geq p$ and $p = swt(swt(p, B_0, \ldots, B_n), B_n, B_{n-1}, \ldots, B_0) = swt(t, B_n, \ldots, B_0) \geq t$. Hence the switching produces equivalent conditions.

(2) We need to allow $swt(p, B)$ for the Δ-system argument. Since in this argument two conditions are combined into one and so C^{κ^+} should pick one of them only. Also it is needed for proving a strategic closure of the forcing.

(3) The use of finite sequences $B_0, \ldots, B_n$ is needed in order to insure transitivity of the order $\leq$ on $\mathcal{P}'$.

Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle,\ A^{1\kappa^{++}}\rangle \in \mathcal{P}'$. Set $p \backslash \kappa^{++} = A^{1\kappa^{++}}$. Define $\mathcal{P}'_{\geq\kappa^{++}}$ to be the set of all $p\backslash\kappa^{++}$ for $p \in \mathcal{P}'$.

The next lemma is obvious.

Lemma 1.16 *$\langle \mathcal{P}'_{\geq\kappa^{++}}, \leq \rangle$ is κ^{+3}-closed.*

Set $p \restriction \kappa^{++} = \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle$ where $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle,\ A^{1\kappa^{++}}\rangle \in \mathcal{P}'$.

Let $G(\mathcal{P}'_{\geq\kappa^{++}})$ be a generic subset of $\mathcal{P}'_{\geq\kappa^{++}}$. Define $\mathcal{P}'_{<\kappa^{++}}$ to be the set of all $p \restriction \kappa^{++}$ for $p \in \mathcal{P}'$ with $p\backslash\kappa^{++} \in G(\mathcal{P}'_{\geq\kappa^{++}})$.

Let $p \in \mathcal{P}'$ and $q \in \mathcal{P}'_{\geq\kappa^{++}}$. Then $q^\frown p$ denotes the set obtained from p by adding q to the last component of p, i.e. to $A^{1\kappa^{++}}$.

The following lemma is trivial.

Lemma 1.17 *Let $p \in \mathcal{P}'$, $q \in \mathcal{P}'_{\geq\kappa^{++}}$ and $q \geq_{\mathcal{P}'_{\geq\kappa^{++}}} p\backslash\kappa^{++}$. Then $q^\frown p \in \mathcal{P}'$ and $q^\frown p \geq p$.*

It follows now that $\mathcal{P}'$ projects to $\mathcal{P}'_{<\kappa^{++}}$.

Let us turn to the chain condition.

Lemma 1.18 *The forcing $\mathcal{P}'_{<\kappa^{++}}$ satisfies κ^{+3}-c.c. in $V^{\mathcal{P}'_{\geq\kappa^{++}}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \Vdash_{\mathcal{P}'_{\geq\kappa^{++}}} (\langle \underset{\sim}{p}_\alpha = \langle \underset{\sim}{A}^{0\kappa^+}_\alpha,\ \underset{\sim}{A}^{1\kappa^+}_\alpha,\ \underset{\sim}{C}^{\kappa^+}_\alpha \rangle \mid \alpha < \kappa^{+3}\rangle \text{ is an antichain in } \underset{\sim}{\mathcal{P}}'_{<\kappa^{++}})$$

Without loss of generality we can assume that each $A^{0\kappa^+}_\alpha$ is forced to be a successor model, otherwise just extend conditions by adding one additional model on the top. Define by induction, using Lemma 1.16, an increasing sequence $\langle q_\alpha \mid \alpha < \kappa^{+3}\rangle$ of elements of $\mathcal{P}'_{\geq\kappa^{++}}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{+3}\rangle$, $p_\alpha = \langle A^{0\kappa^+}_\alpha, A^{1\kappa^+}_\alpha, C^{\kappa^+}_\alpha\rangle$ so that for every $\alpha < \kappa^{+3}$

$$q_\alpha \Vdash_{\mathcal{P}'_{\geq\kappa^3}} \langle \underset{\sim}{A}^{0\kappa^+}_\alpha,\ \underset{\sim}{A}^{1\kappa^+}_\alpha,\ \underset{\sim}{C}^{\kappa^+}_\alpha\rangle = \check{p}_\alpha\ .$$

For a limit $\alpha < \kappa^{+3}$ let

$$\overline{q}_\alpha = \bigcup_{\beta<\alpha} q_\beta \cup \{\sup \bigcup_{\beta<\alpha} q_\beta\}$$

and q_α be its extension deciding $\underset{\sim}{p}_\alpha$. Also assume that $\max q_\alpha \geq \sup(A^{0\kappa^+}_\alpha \cap \kappa^{+3})$.

We form a Δ-system. By shrinking if necessary assume that for some stationary $S \subseteq \kappa^{+3}$ and $\delta < \kappa^{+3}$ we have the following for every $\alpha < \beta$ in S:

(a) $A^{0\kappa^+}_\alpha \cap \alpha = A^{0\kappa^+}_\beta \cap \beta \subseteq \delta$

(b) $A^{0\kappa^+}_\alpha \backslash \alpha \neq \emptyset$

(c) $\sup A^{0\kappa^+}_\alpha < \beta$

(d) $\sup \overline{q}_\alpha = \alpha + 1$

(e)
$$\langle A^{0\kappa^+}_\alpha, \in, \leq, \subseteq, \kappa, C^{\kappa^+}_\alpha, f_{A^{0\kappa^+}_\alpha},\ A^{1\kappa^+}_\alpha, q_\alpha \cap A^{0\kappa^+}_\alpha\rangle$$
$$\langle A^{0\kappa^+}_\beta, \in, \leq, \subseteq, \kappa, C^{\kappa^+}_\beta, f_{A^{0\kappa^+}_\beta},\ A^{1\kappa^+}_\beta, q_\beta \cap A^{0\kappa^+}_\beta\rangle$$

are isomorphic over δ, i.e. by isomorphism fixing every ordinal below δ, where

$$f_{A^{0\kappa^+}_\alpha} : \kappa^+ \longleftrightarrow A^{0\kappa^+}_\alpha$$

and

$$f_{A^{0\kappa^+}_\beta} : \kappa^+ \longleftrightarrow A^{0\kappa^+}_\beta$$

are the fixed enumerations.

We claim that for $\alpha < \beta$ in S it is possible to extend q_β to a condition forcing compatibility of p_α and p_β. Proceed as follows. Pick A to be an elementary submodel of cardinality κ^+ so that

(i) $A^{1\kappa^+}_\alpha, A^{1\kappa^+}_\beta \in A$

(ii) $C^{\kappa^+}_\alpha, C^{\kappa^+}_\beta \in A$

(iii) $q_\beta \in A$.

Extend q_β to $q = q_\beta \cup \sup(A \cap \kappa^{+3})$. Set $p = \langle A, A^{1\kappa^+}, C^{\kappa^+}\rangle$, where $A^{1\kappa^+} := A^{1\kappa^+}_\alpha \cup A^{1\kappa^+}_\beta \cup \{A\}$, $C^{\kappa^+} := C^{\kappa+}_\alpha \cup C^{\kappa^+}_\beta \cup \langle A, C^{\kappa^+}_\beta(A^{0\kappa^+}_\beta)^\frown A\rangle\rangle$.
Clearly, $\langle C^{\kappa^+}(A), q\rangle \in \mathcal{P}''$.
The triple $A^{0\kappa^+}_\beta, A^{0\kappa^+}_\alpha, A$ is of a Δ-system type relatively to q, by (e) above. It follows that $\langle p, q\rangle \in \mathcal{P}'$. Thus the condition (6) of Definition 1.3 holds since each of $\langle p_\alpha, q\rangle, \langle p_\beta, q\rangle$ satisfies it. The condition (7) of Definition 1.3 follows from (e) above and since both $\langle p_\alpha, q\rangle, \langle p_\beta, q\rangle$ satisfy it.

$\square$

Lemma 1.19 *$\mathcal{P}'$ is κ^{++}-strategically closed.*

Proof. We define a winning strategy for the player playing at even stages. Thus suppose $\langle p_j \mid j < i\rangle$, $p_j = \langle\langle A^{0\kappa^+}_j, A^{1\kappa^+}_j, C^{\kappa^+}_j\rangle, A^{1\kappa^{++}}_j\rangle$ is a play according to this strategy up to an even stage $i < \kappa^{++}$. Set first

$$B^{0\kappa^+}_i = \bigcup_{j<i} A^{0\kappa^+}_j, B^{1\kappa^+}_i = \bigcup_{j<i} A^{1\kappa^+}_j \cup \{B^{0\kappa^+}_i\},$$

$$D^{\kappa^+}_i = \bigcup_{j<i} C^{\kappa^+}_j \cup \{\langle B^{0\kappa^+}_i, \{B^{0\kappa^+}_i\} \cup \{C^{\kappa^+}_j(A^{0\kappa^+}_j) \mid \ j \text{ is even}\}\rangle\}$$

and

$$B_i^{1\kappa^{++}} = \bigcup_{j<i} B_j^{1\kappa^{++}} \cup \{\sup \bigcup_{j<i} B_j^{1\kappa^{++}}\}.$$

Then pick $A_i^{0\kappa^+}$ to be a model of cardinality κ^+ such that

(a) ${}^{\kappa}A_i^{0\kappa^+} \subseteq A_i^{0\kappa^+}$

(b) $B_i^{0\kappa^+}, B_i^{1\kappa^+}, D_i^{\kappa^+}, B_i^{1\kappa^{++}} \in A_i^{0\kappa^+}$.

Set $A_i^{1\kappa^+} = B_i^{1\kappa^+} \cup \{A_i^{0\kappa^+}\}$, $C_i^{\kappa^+} = D_i^{\kappa^+} \cup \{\langle A_i^{0\kappa^+}, D_i^{\kappa^+}(B_i^{0\kappa^+}) \cup \{A_i^{0\kappa^+}\}\rangle\}$ and $A_i^{1\kappa^{++}} = B_i^{1\kappa^{++}} \cup \{\sup(A_i^{0\kappa^+} \cap \kappa^{+3}\}$. As an inductive assumption we assume that at each even stage $j < i$, p_j was defined in the same fashion. Then $p_i = \langle\langle A_i^{0\kappa^+}, A_i^{1\kappa^+}, C_i^{\kappa^+}\rangle, A_i^{1\kappa^{++}}\rangle$ will be a condition in $\mathcal{P}'$ stronger than each p_j for $j < i$. The switching may be required here once moving from an odd stage to its immediate successor even stage.

$\square$

2 Suitable structures and assignment functions

In the gap 2 case assignment functions a_n (those connecting the level κ with level $\kappa_n, n < \omega$) were order preserving. In other words a_n is an isomorphism between structures in the language containing only the predicate for the order relation. Here, in the gap 3 case (and beyond), a_n's will be isomorphisms between structures in more complicated languages.

Let us start with two definitions which will specify relevant structures.

Definition 2.1 A three sorted structure $\langle\langle X, Y, Z\rangle, C, \in, \subseteq\rangle$ is called *suitable structure* iff

1. X has a maximal under inclusion element. Denote it by $\max(X)$.

2. $Y \subseteq \max(X)$,

3. C is a binary relation X,

4. $\langle\langle \max(X), X, C\rangle, Y\rangle \in \mathcal{P}'$, where for every $A \in X$ we identify $C(A)$ with the set $\{B \in X \mid \langle A, B\rangle \in C\}$.

5. $Z = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X \cup Y\}$.

Note that by Lemma 1.11, an intersection $t_1 \cap ... \cap t_n$ above is really simple, thus it is equal to an element of X or of Y or to $s \cap \alpha$, where $s \in X$ and $\alpha \in Y$.

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.

Definition 2.2 A suitable structure $\langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ is called *suitable generic structure* iff there is $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ such that

1. $\langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ is a substructure (not necessarily elementary) of $\langle\langle A^{1\kappa^+}, A^{1\kappa^{++}}, \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in A^{1\kappa^+} \cup A^{1\kappa^{++}}\}\rangle,$ $C^{\kappa^+}, \in, \subseteq\ \rangle$,

2. $\max(X) \in C^{\kappa^+}(A^{0\kappa^+})$,

3. $\langle\langle \max(X), X, C\rangle, Y\rangle$ and $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ agree about the walks to members of X and to ordinals in $\max(X) \cap Y$. In other words we require that all the elements of walks in $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ to elements of X and to ordinals in $\max(X) \cap Y$ are in X.

Note that, as a condition in $\mathcal{P}'$, $\langle\langle \max(X), X, C\rangle, Y\rangle$ need not be weaker than $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$, and hence it need not be in $G(\mathcal{P}')$. Thus, for example, $A^{1\kappa^{++}}$ need not be an end extension of Y.
Note also, that any stronger condition $\langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ with $C^{\kappa^+}(A^{0\kappa^+})$ being an initial segment of $D^{\kappa^+}(B^{0\kappa^+})$ will witness that $\langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ is a suitable generic structure.

Lemma 2.3 *Let $\langle\langle X,Y,Z\rangle,C,\in,\subseteq\ \rangle$ be a suitable generic structure as witnessed by $\langle\langle A^{0\kappa^+},A^{1\kappa^+},C^{\kappa^+}\rangle,A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$. Suppose that $F_0,F_1,F \in A^{1\kappa^+}$, $F_0,F \in C^{\kappa^+}(A^{0\kappa^+})$ is a triple of a Δ-system type with α_0,α_1 as in Definition 1.2, and $\alpha_1 \in Y$. Then $F_0,F_1 \in X \cap \max(X), F \in X, \alpha_0 \in \max(X)\cap Y$.*

Proof. The walk to α_1 from $\max(X)$ (or the same from $A^{0\kappa^+}$) passes through F and turns to F_1. Hence, by 2.2(3), $F_0,F_1,F \in X$. Recall that by 2.1(3) we have $\langle\langle \max(X),X,C\rangle,Y\rangle \in \mathcal{P}'$. Hence F_0,F_1,F are of a Δ-system type in $\langle\langle \max(X),X,C\rangle,Y\rangle$. Then there are $\alpha_0' \in F_0\cap Y, \alpha_1' \in F_1 \cap Y$ such that

$$F_0\cap F_1 = F_0\cap\alpha_0' = F_1\cap\alpha_1'.$$

But, also

$$F_0\cap F_1 = F_1\cap\alpha_1$$

and $\alpha_1,\alpha_1' \in Y \subseteq A^{1\kappa^{++}}$.
Hence, $\alpha_1=\alpha_1'$. Finally, $\alpha_0'=\pi_{F_1,F_0}(\alpha_1)=\alpha_0$. Hence, $\alpha_0\in\max(X)\cap Y$.
$\square$

Lemma 2.4 *Let $p=\langle\langle X,Y,Z\rangle,C,\in,\subseteq\ \rangle$ and $p'=\langle\langle X',Y',Z'\rangle,C',\in,\subseteq\ \rangle$ be isomorphic suitable structures (even over different cardinals) and a an isomorphism between them. Suppose that F_0,F_1,F is a triple in X of a Δ-system type and $\alpha_0\in F_0\cap Y, \alpha_1\in F_1\cap Y$ are witnessing this ordinals. Then $a(F_0),a(F_1),a(F)$ is a triple in X' of a Δ-system type witnessed by $a(\alpha_0)$ and $a(\alpha_1)$.*

Proof. Obviously, α_0 and α_1 are uniquely determined by F_0 and F_1.
Denote $a(F_0)$ by F_0',$a(F_1)$ by F_1',$a(F)$ by F', $a(\alpha_0)$ by α_0' and $a(\alpha_1)$ by α_1'. Now, $F_0',F_1'\in F'$, moreover F_0' is the immediate predecessor of F' in $C(F')$ and F_1' is an additional predecessor of F' under the inclusion relation, since a is an isomorphism between p and p'. Note that by 2.1(3) this implies that

F_0', F_1', F' is a Δ-system type triple in p'.
Let $\alpha_0'' \in F_0' \cap a(Y)$ and $\alpha_1'' \in F_1' \cap a(Y)$ be such that

$$F_0' \cap F_1' = F_0' \cap \alpha_0'' = F_1' \cap \alpha_1''.$$

Also $\alpha_0' \in F_0' \cap a(Y)$ and $\alpha_1' \in F_1' \cap a(Y)$, since a respects $\in$-relation. But then, necessarily, $\alpha_0' = \alpha_0'', \alpha_1' = \alpha_1''$.
$\square$

Lemma 2.5 *Let $p = \langle\langle X, Y, Z\rangle, C, \in, \subseteq \rangle$ and $p' = \langle\langle X', Y', Z'\rangle, C', \in, \subseteq \rangle$ be isomorphic suitable structures (even over different cardinals) and a an isomorphism between them. Then a respects walks, i.e. for every $A \in X$ and $B \in (X \cup Y) \cap A$, a maps the walk between A and B in p onto the walk between $a(A)$ and $a(B)$.*

Proof. Induction on walks length. Thus, if B in $C(A)$ or if $B \in Y$ and the walk to it from A involves only $C(A)$, then the isomorphism a guaranties the same for the images. Suppose that the walk proceeds with splitting. Let F_0, F_1, F be the first split on the way to B, i.e. $F \in C(A)$, the triple F_0, F_1, F is of a Δ-system type, $B \not\subseteq F_0$ (or, if $B \in Y$, $B \notin F_0$) and $B \subseteq F_1$ (or $B \in F_1 \cup \{F_1\}$). By the previous lemma (Lemma 2.4), $a(F_0), a(F_1), a(F)$ is a triple in X' of a Δ-system type. a is isomorphism, hence $a(F) \in C(a(A)), a(F_0) \in C(a(F_0))$, $a(B) \not\subseteq a(F_0)$ (or, if $B \in Y$, $a(B) \notin a(F_0)$) and $a(B) \subseteq a(F_1)$ (or $a(B) \in a(F_1) \cup \{a(F_1)\}$).
But this means that the walk from $a(A)$ to $a(B)$ goes via $a(F_1)$. Now we can apply induction to the walk from F_1 to B, since it is shorter than the original one from A to B.
$\square$

Lemma 2.6 *Let $p = \langle\langle X, Y, Z\rangle, C, \in, \subseteq \rangle$ and $p' = \langle\langle X', Y', Z'\rangle, C', \in, \subseteq \rangle$ be isomorphic suitable structures (even over different cardinals), a an isomorphism between them and $F_0, F_1, F \in X$ a triple of a Δ-system type. Then*

a respects π_{F_0,F_1}, i.e. for every $A \in F_0 \cap (X \cup Y)$ we have $a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A))$.

Proof. Let $F_0, F_1, F \in X$ be a triple of a Δ-system type and $A \in F_0 \cap (X \cup Y)$. We prove the lemma by induction on the length of the walk from F_0 to A. Suppose first that $A \in C(F_0)$ (or in case $A \in Y$ the walk to A involves only $C(F_0)$). The isomorphism a moves $C(F_0)$ to $C(a(F_0))$ and $C(F_1)$ to $C(a(F_1))$. By Lemma 2.4, the triple $a(F_0), a(F_1), a(F)$ is of a Δ-system type. So, $\pi_{a(F_0),a(F_1)}$ moves $C(a(F_0))$ onto $C(a(F_1))$ respecting the inclusion relation. Then $\pi_{a(F_0),a(F_1)}(a(A))$ should an element of $C(a(F_1))$ at the same place as $a(A)$ in $C(a(F_0))$, which, in turn is at the same place as A in $C(F_0)$ and $\pi_{F_0,F_1}(A)$ in $C(F_1)$. Hence

$$a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A)).$$

Suppose no that $A \notin C(F_0)$. Let H_0, H_1, H be the first splitting on the way to A from F_0. The induction applies to H_1, A . Hence

$$a(\pi_{H_1,H_0}(A)) = \pi_{a(H_1),a(H_0)}(a(A)).$$

Let $A' = \pi_{H_1,H_0}(A)$. Apply the induction to F_0, A'. Then

$$a(\pi_{F_0,F_1}(A')) = \pi_{a(F_0),a(F_1)}(a(A')).$$

Again, apply induction to F_0, H_0 and F_0, H_1. So,

$$a(\pi_{F_0,F_1}(H_0)) = \pi_{a(F_0),a(F_1)}(a(H_0))$$

and

$$a(\pi_{F_0,F_1}(H_1)) = \pi_{a(F_0),a(F_1)}(a(H_1)).$$

Finally,

$$\pi_{F_0,F_1}(A) = \pi_{\pi_{F_0,F_1}(H_0),\pi_{F_0,F_1}(H_1)}(\pi_{F_0,F_1}(\pi_{H_1,H_0}(A'))).$$

Applying a, we obtain

$$a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A)).$$

$\square$

Note that the proofs of Lemmas 2.5, 2.6 rely only on Lemma 2.4 and ?(also this lemma does not use Z) do not use the component of suitable structures consisting of intersections. Let us isolate a weaker notion that still will capture all the essential parts.

Definition 2.7 A two sorted structure $\langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ is called *weak suitable structure* iff

1. X has a maximal under inclusion element. Denote it $\max(X)$,
2. $Y \subseteq \max(X)$,
3. C is a binary relation X,
4. $\langle\langle \max(X), X, C\rangle, Y\rangle \in \mathcal{P}'$, where for every $A \in X$ we identify $C(A)$ with the set $\{B \in X \mid \langle A, B\rangle \in C\}$.

The following analogs of Lemmas 2.5, 2.6 were actually proved above:

Lemma 2.8 *Let $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ and $p' = \langle\langle X', Y'\rangle, C', \in, \subseteq\ \rangle$ be isomorphic weak suitable structures (even over different cardinals) and a an isomorphism between them. Then a respects walks, i.e. for every $A \in X$ and $B \in (X \cup Y) \cap A$, a maps the walk between A and B in p onto the walk between $a(A)$ and $a(B)$.*

Lemma 2.9 *Let $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ and $p' = \langle\langle X', Y'\rangle, C', \in, \subseteq, \rangle$ be isomorphic weak suitable structures (even over different cardinals), a an isomorphism between them and $F_0, F_1, F \in X$ a triple of a Δ-system type. Then a respects π_{F_0,F_1}, i.e. for every $A \in F_0 \cap (X \cup Y)$ we have $a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A))$.*

Let $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ be a weak suitable structure. Consider $Z = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X \cup Y\}$. Then $\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ is a suitable structure. Let us call it *expansion of* p to a suitable structure.

Lemma 2.10 *Suppose that* $p = \langle\langle X, Y\rangle, C, \in, \subseteq\ \rangle$ *and* $p' = \langle\langle X', Y'\rangle, C', \in, \subseteq\ \rangle$ *are isomorphic weak suitable structures (even over different cardinals). Then their expansions are isomorphic as well.*

Proof. Let a be the isomorphism between p and p'. We show that it extends to an isomorphism between the expansions. Let $Z = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X \cup Y\}$ and $Z' = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X' \cup Y'\}$. Extend a to a function b in the obvious fashion: $b \restriction \mathrm{dom}(a) = a$ and $b(t_1 \cap ... \cap t_n) = a(t_1) \cap ... \cap a(t_n)$, for any $t_1, ..., t_n \in X \cup Y$. We need to check that such defined b is a function and an isomorphism.

Note first that for every $A, B \in X$, $A' \in (A \cup \{A\}) \cap X$ and $\alpha \in Y \cap A'$ such that $A \cap B = A' \cap \alpha$ we have $a(A) \cap a(B) = a(A') \cap a(\alpha)$. Use induction on the walks complexity from $\max(X)$ to A, B as in Lemma 1.11. The inductive step follows since a preserves Δ-system triples. Also, by Lemmas 2.8,2.9, a respects walks and images under Δ-system triples isomorphisms.
Similar, if instead of two sets we have finitely many $A_1, ..., A_n \in X$, $A' \in (A_1 \cup \{A_1\}) \cap X$ and $\alpha \in Y \cap A'$ such that $A_1 \cap ... \cap A_n = A' \cap \alpha$, then $a(A_1) \cap ... \cap a(A_n) = a(A') \cap a(\alpha)$. Also, the same holds if some (or actually one) of A_i's is in Y, i.e. is an ordinal.

Now, by Lemma 1.12, for every $A_1, ..., A_n \in X$ there are $A' \in (A_1 \cup \{A_1\}) \cap X$ and $\eta \in Y \cap A'$ such that $A_1 \cap ... \cap A_n = A' \cap \eta$, or just $A_1 \cap ... \cap A_n = A'$.

An alternative proof that works for higher gaps as well proceeds as follows. Suppose that

$$A_1 \cap ... \cap A_n = B_1 \cap ... \cap B_n,$$

for some $A_1, ..., A_n, B_1, ..., B_n \in X \cup Y$. We need to show that then

$$a(A_1) \cap ... \cap a(A_n) = a(B_1) \cap ... \cap a(B_n).$$

The proof is by induction on complexity of the walks to components of the intersections. Thus, suppose that A_1 has a maximal walk complexity among the components of the intersection. Consider the walks from $\max(X)$ to A_1 and to A_2. Go to the last point until which the walks coincide. Then, as in the proof of Lemma 1.11, we replace A_1 by $A_1' \in X$ and $\alpha_1 \in Y$ which are simpler than A_1 in the walk sense and such that

$$A_1 \cap A_2 = A_1' \cap \alpha_1 \cap A_2.$$

Now the induction applies.

$\square$

Fix $n < \omega$. We define an analog $\mathcal{P}_n'$ of $\mathcal{P}'$ on the level n just replacing κ by κ_n^{+n}. An assignment function a_n will be an isomorphism between a suitable generic structure of cardinality less than κ_n over κ and a suitable structure over κ_n^{+n}.

Define Q_{n0}.

Definition 2.11 Let Q_{n0} be the set of the triples $\langle a, A, f\rangle$ so that:

1. f is partial function from κ^{+3} to κ_n of cardinality at most κ

2. a is an isomorphism between a suitable generic structure $\langle\langle X, Y, Z\rangle, C, \in, \subseteq\ \rangle$ of cardinality less than κ_n and a suitable structure $\langle\langle X', Y', Z'\rangle, C', \in, \subseteq\ \rangle$ in $\mathcal{P}_n'$ so that

   (a) $\max(X')$ is above every $t \in X' \cup Y'$ in the order $\leq_{E_n}$ of the extender E_n, (or actually, the ordinal which codes $\max(X')$ in the fixed in advance nice codding of $[\kappa_n^{+n+3}]^{<\kappa_n}$. We need that each element of $[\kappa_n^{+n+3}]^{<\kappa_n}$ is coded by a stationary many ordinals below κ_n^{+n+3}).

   (b) if $t \in X' \cup Y'$ then for some $k, 2 < k < \omega$, ?$t \prec H(\chi^{+k})$, with χ big enough fixed in advance. (Alternatively, may be to work with subsets of κ_n^{+n+3} only and further require it

is a restriction of such model to κ_n^{+n+3}.) We deal with elementary submodels of $H(\chi^{+k})$, instead of those of $H(\kappa_n^{+n+3})$.
Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of a model $t \in X \cup Y$ will be an elementary submodel of $H(\chi^{+k})$.
The way to compare such models $t_1 \prec H(\chi^{+k_1}), t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{n,\max(X')}$,

4. for every ordinals α, β, γ which are elements of Y' or the ordinals coding models in X' we have

$$\begin{array}{l} \alpha \geq_{E_n} \beta \geq_{E_n} \gamma \quad \text{implies} \\ \pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho)) \end{array}$$

for every $\rho \in \pi``_{\max(X'),\alpha}(A)$.

Define a partial order on Q_{n0} as follows.

Definition 2.12 Let $\langle a, A, f\rangle$ and $\langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $a \supseteq b$,

2. $f \supseteq g$,

3. $\pi_{\max(\mathrm{rng}(a)),\max(\mathrm{rng}(b))}``A \subseteq B$,

4. $\mathrm{dom}(f) \cap Y^b = \mathrm{dom}(g) \cap Y^b$, where Y^b is the second component (i.e. the set of ordinals) of the suitable structure on which b is defined.
Note that here we do not require disjointness of the domain of g and of Y^b, but as it will follow from the further definition of non-direct extension, the value given by g will be those that eventually counts.

Definition 2.13 Q_{n1} consists of all partial functions $f : \kappa^{+3} \to \kappa_n$ with $|f| \le \kappa$. If $f, g \in Q_{n1}$, then set $f \ge_{n1} g$ iff $f \supseteq g$.

Definition 2.14 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\le_n^* = \le_{n0} \cup \le_{n1}$.
Let $p = \langle a, A, f \rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown \nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{rng}(a)), a(\alpha)}(\nu) \mid \alpha \in \mathrm{dom}(a) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $\mathrm{dom}(a)$ are used to produce non direct extensions, models disappear.

Now, if $p, q \in Q_n$, then we set $p \ge_n q$ iff either $p \ge_n^* q$ or $p \in Q_{n1}, q = \langle b, B, g \rangle \in Q_{n0}$ and for some $\nu \in B$, $p \ge_{n1} q^\frown \nu$.

Definition 2.15 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) for every $n < \omega$, $\quad p_n \in Q_n$,

(2) there is $\ell(p) < \omega$ such that

  (i) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,

  (ii) for every $n \ge \ell(p)$, we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$,

  (iii) there is $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in G(\mathcal{P}')$ which witnesses that $\mathrm{dom}(a_n(p))$ is a suitable generic structure (i.e. $\mathrm{dom}(a_n(p))$ and $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle$ satisfy 2.2), simultaneously for every $n, l(p) \le n < \omega$.

(3) for every $n \ge m \ge \ell(p)$, $\quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,

(4) ? for every n, $\ell(p) \le n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \kappa_m^{+m+3}$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \kappa_m^{+m+3}$. It is possible to define being k-good this way as well).

(5) ? For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.

Next lemma deals with extensions of elements of $\mathcal{P}$. The analogs for the gap 2 are trivial.

Lemma 2.16 *Let $p \in \mathcal{P}$ and $\langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$. Then*

1. *for every $\alpha \in B^{1\kappa^{++}}$ there is $q \geq^* p$ such that $\alpha \in \mathrm{dom}(a_n(q))$ for all but finitely many n's;*

2. *for every $A \in B^{1\kappa^+}$ there is $q \geq^* p$ such that $A \in \mathrm{dom}(a_n(q))$ for all but finitely many n's. Moreover, if $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \geq \langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle$ witnesses a generic suitability of p and $A \in C^{\kappa^+}(A^{0\kappa^+})$, then the addition of A does not require adding of ordinals and the only models that probably will be added together with A are its images under Δ-system type isomorphisms for triples in p.*

Proof. Pick some $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ stronger than $\langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle$ such that

1. $\alpha \in A^{1\kappa^{++}}$,

2. $A \in A^{1\kappa^+}$,

3. $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ witnesses that $\mathrm{dom}(a_n(p))$ is a suitable generic structure (i.e. $\mathrm{dom}(a_n(p))$ and $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$ satisfy 2.2), for every $n, l(p) \leq n < \omega$.

Note first that it is easy to add to p any $A \in C^{\kappa^+}(A^{0\kappa^+})$ such that the maximal models of p_n's belong to A. Just at each level $n \geq l(p)$ pick an elementary submodel of $H(\chi^{+\omega})$ of cardinality κ_n^{+n+1} which includes $\mathrm{rng}(a_n)$ as an element. Map A to such a model.

Hence it is enough to deal with α, A which are the members of the maximal model of p, just, if not, then we can add first $A^{0\kappa^+}$.

We proof the lemma simultaneously for α and A by induction on the walk distance or complexity.

Fix $n \geq l(p)$. Let $\mathrm{dom}(a_n(p)) = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\rangle$.

Suppose that the walk to α involves only the central line. The general case is treated similar.

Let $A_1 \in C^{\kappa^+}(\max(X))$ be the least model of $C^{\kappa^+}(\max(X))$ with $\alpha \in A_1$. We assume that $A_1 \in X$. Just otherwise use the induction to add it. This is possible, since the walk to A_1 is simpler than those to α.

Case 1. A_1 is the least model of $C^{\kappa^+}(\max(X))$.
The walk to α from $\max(X)$ (or from $A^{0\kappa^+}$) consists of A_1 alone. So, in order to add α we do not have to add models or other ordinals first.
Consider $\beta_1 = \min(A_1 \cap Y) \setminus \alpha)$ and $\gamma_1 = \max(A_1 \cap Y \cap \alpha)$ whenever defined. Suppose that both β_1 and γ_1 are defined. If one of them or both are undefined then the argument below will be only simpler.
Let us denote $a_n(\beta_1)$ by β_1^*, $a_n(\gamma_1)$ by γ_1^*, $a_n(X)$ by X^* and $a_n(A_1)$ by A_1^*. Let C^* be the function that corresponds to C in $\mathrm{rng}(a_n)$. Then $A_1^* \in C^*(X^*)$. Also, $\beta_1^*, \gamma_1^* \in A_1^* \cap a_n{}''Y$ and $\gamma_1^* < \beta_1^*$.
Assume that A_1^* and β_1^* are k-good, for some $k >> 2$. Pick now $M \in A_1^*$ such that

1. $M \in \beta_1^*$,
2. $|M| = \kappa_n^{+n+2}$,
3. M is $k-1$-good,
4. $\gamma_1^* \in M$.

Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.

Case 2. A_1 is not the least element of $C^{\kappa^+}(\max(X))$.
Then we will need to add also the immediate predecessor A_1^- of A_1 in $C^{\kappa^+}(\max(X))$. Do this using the induction.

Split the argument into three cases.

Case 2.1. $\alpha > \sup(A_1^-)$.

Then we proceed exactly as in Case 1 above only require in addition that $a_n(A_1^-) \in M$.

Case 2.2. $\alpha = \sup(A_1^-)$.

Set $B = a_n(A_1)$. Then, its immediate predecessor $B^- = a_n(A_1^-)$. Pick $k < \omega$ such that $B^- \prec H(\chi^{+k+1})$ and $B \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$. Then $H(\chi^{+k}) \in B^-$. Hence

$$B^- \models \forall \nu < \kappa_n^{+n+3} \forall t \in [H(\chi^{+k})]^{<\kappa^{+n+3}} \exists M \prec H(\chi^{+k}) \quad (M \supseteq \nu \cup t \text{ and } |M| < \kappa_n^{+n+3}).$$

Let $\delta = \sup(B^- \cap \kappa_n^{+n+3})$. Set M to be the Skolem hull of $\delta \cup (B^- \cap H(\chi^{+k}))$ in $H(\chi^{+k})$. Then $M \cap \kappa_n^{+n+3} = \delta$. Also, $M \in B$.

Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.

Case 2.3. $\alpha < \sup(A_1^-)$.

Consider $\alpha_1 = \min(A_1^- \setminus \alpha)$. We need to add α_1 before α and this can be done using the induction, since the walk to α_1 is simpler than those to α. So assume that α_1 is already in Y. Note that $\mathrm{cof}(\alpha_1) = \kappa^{++}$, since $A_1^- \supseteq \kappa^+$ and it is an elementary submodel of $H(\kappa^{+3})$.

We split the proof now into two cases.

Case 2.3.1. $\alpha = \sup(\alpha_1 \cap A_1^-)$.

This case is similar to Case 2.2 above. Set $B = a_n(A_1)$. Then, its immediate predecessor $B^- = a_n(A_1^-)$. Let $E = a_n(\alpha_1)$.

Pick $k < \omega$ such that $E \prec H(\chi^{+k+1})$, $B^- \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$ and $B \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$. Then $H(\chi^{+k}) \in E \cap B^-$.

$$E \cap B^- \models \forall \nu < \kappa_n^{+n+3} \forall t \in [H(\chi^{+k})]^{<\kappa^{+n+3}}$$

$$\exists M \prec H(\chi^{+k}) \quad (M \supseteq \nu \cup t \text{ and } |M| < \kappa_n^{+n+3}).$$

Let $\delta = \sup(E \cap B^- \cap \kappa_n^{+n+3})$. Set M to be the Skolem hull of $\delta \cup (E \cap B^- \cap H(\chi^{+k}))$ in $H(\chi^{+k})$. Then $M \cap \kappa_n^{+n+3} = \delta$. Also, $M \in B$.

Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.

Case 2.3.2. $\alpha > \sup(\alpha_1 \cap A_1^-)$.

Consider $\beta_1 = \min((A_1 \cap Y) \setminus \alpha)$ and $\gamma_1 = \max(A_1 \cap Y \cap \alpha)$ whenever defined. Suppose that both β_1 and γ_1 are defined. If one of them or both are undefined then the argument below will be only simpler.

Let us denote $a_n(\beta_1)$ by β_1^*, $a_n(\gamma_1)$ by γ_1^*, $a_n(X)$ by X^* and $a_n(A_1)$ by A_1^*. Let C^* be the function that corresponds to C in $\mathrm{rng}(a_n)$. Then $A_1^* \in C^*(X^*)$ and $a_n(A_1^-)$ is the immediate predecessor of A_1^* in $C^*(A_1^*)$. Also, $\beta^*, \gamma^* \in A_1^* \cap a_n{}''Y$ and $\gamma^* < \beta^*$.

Assume that A_1^* and β_1^* are k-good, for some $k >> 2$. Pick now $M \in A_1^*$ such that

1. $M \in \beta_1^*$,
2. $|M| = \kappa_n^{+n+2}$,
3. M is $k-1$-good,
4. $\gamma_1^*, a_n(A_1^-) \cap a_n(\alpha_1) \in M$.

Now, extend a_n by mapping α to M and all the images of it under Δ-system types triples isomorphisms to those of M.

Set

$$Y_1 = Y \cup \{\alpha' \mid \alpha' \text{ is the image of}$$

$$\alpha \text{ under } \Delta - \text{ system types triples (of } X) \text{ isomorphisms } \}.$$

Claim 2.16.1 *Y_1 is a closed set.*

Proof. We just prove that every limit point of Y_1 is a limit point of Y, and hence, is in Y. It is enough to deal limits of ω-sequences, since if every limit of an ω-sequence from Y_1 is in Y, then any limit will be in Y, because Y is closed.

Such images are generated as follows. Pick the smallest triple $F_0^1, F_1^1, F^1 \in X$

of a Δ-system type with $F_0^1, F^1 \in C(\max(X))$ and $F_0^1 \subseteq A$. We add $\alpha^1 = \pi_{F_0^1, F_1^1}(\alpha)$ to Y. Note that it is possible to have $\alpha = \alpha^1$. Let $\xi_0^1 \in F_0^1 \cap Y, \xi_1^1 \in F_1^1 \cap Y$ be as in Definition 1.2(6d). Then $\alpha^1 > \alpha$ implies $\xi_0^1 \leq \alpha < \xi_1^1 \leq \alpha^1$.
Then pick the smallest triple $F_0^2, F_1^2, F^2 \in X$ of a Δ-system type with $F_0^2, F^2 \in C(\max(X))$ and $F_0^2 \subseteq F^1$. We add $\alpha^{20} = \pi_{F_0^2, F_1^2}(\alpha)$ and $\alpha^{21} = \pi_{F_0^2, F_1^2}(\alpha^1)$ to Y. Again it is possible to have $\alpha^{2i} \in \{\alpha, \alpha^1\}$, where $i < 2$. Let $\xi_0^2 \in F_0^2 \cap Y, \xi_1^2 \in F_1^2 \cap Y$ be as in Definition 1.2(6d). Again, if one of the new α^{2i}'s is above its pre-image, then the corresponding ξ_i^2 will be above $\sup(F_0^2)$, and so, above both α, α^1.
Continue further all the way up to $\max(X)$. This way all the images of α are generated. Note that we move up over the central line of X.
At each stage j in the process the same effect observed above will take place- if one of α^{ji}'s is above its pre-image, then the corresponding ξ_i^j will be above $\sup(F_0^j)$, and so, above all the images $\alpha^{j'i'}$ of α generated at stages $j' < j$. But all such ξ_i^j are in Y. Hence, their limit, which is the same as those of increasing sequence of α^{ji}'s, is in Y as well.
$\square$ of the claim.

Turn now to the adding of a model.
Assume first that a model A is on the central line. Let us observe that no collision with ordinals in Y can occur. Thus if some $\alpha \in Y, \alpha \notin A$ and $\sup(A) > \alpha$ (if $\alpha = \sup(A)$, then by the walk closure we must have $A \in X$), then the same should hold with images, i.e. the image A^* of A must have supremum above $\alpha^* := a_n(\alpha)$ and $\alpha^* \notin A^*$. There may be infinitely many such α's and then, in general, it will be impossible to find A^*. In present situation, we have the advantage - X is closed under walks to ordinals of Y. This means, in particular, that there is $B_\alpha \in C(\max(X))$ such that $\alpha \in B_\alpha$ and B_α is the least model of $C(\max(X))$, or B_α has the immediate predecessor B_α^- in $C(\max(X))$ and $\alpha \notin B_\alpha^-$. In our case the first possibility is just impossible. Thus, we assumed that $A \in C^{\kappa^+}(A^{0\kappa^+})$, $\alpha \in B_\alpha \setminus A$. So,

B_α is not the least element of $C^{\kappa^+}(A^{0\kappa^+})$, which by 2.2(3) implies that B_α is not the least element of $C(\max(X))$ as well.
Hence, B_α^- exists and $A \subseteq B_\alpha^-$.
Consider now a set

$$T = \{B_\alpha^- \mid \alpha \in Y, \alpha \notin A, \sup(A) > \alpha\}.$$

T is a subset of the closed chain $C(\max(X))$. Let E be the least element of T under the inclusion. Then $A \subset E$, since $T \subseteq C^{\kappa^+}(A^{0\kappa^+})$ and so, both E and A are inside the chain $C^{\kappa^+}(A^{0\kappa^+})$, but E is of the form B_α^-, for some $\alpha \in B_\alpha \setminus A$, and $B_\alpha^- \in X, A \notin X$.
Now it is easy to add A in a fashion similar to adding an ordinal above.
First we pick the least $D \in C(E)$ which contains A. Let F be the last model of $C(E)$ inside D. Note that D can be a limit model of $C^{\kappa^+}(A^{0\kappa^+})$ and so D^- may not exist. Even if D^- exists, still it cannot be in X, since otherwise $A = D^-$ will be in X.
Set $\beta = \min((D \cap Y) \setminus \sup(A))$ whenever defined. Suppose that β is defined. If it is undefined then the argument below will be only simpler. Note that necessarily $\beta > \sup(A)$. Otherwise, $\sup(A) = \beta$ and it is in Y. Then the largest model W of $C^{\kappa^+}(A^{0\kappa^+})$ with $\sup(A) \notin W$ must be in X (walks closure to ordinals). But then $W = A$, since $W \neq A$ will imply $W \in A$ or $A \in W$, both possibilities are clearly impossible.
Note that every $\gamma \in D \cap Y \cap \beta$ is in F. Otherwise, let some $\gamma \in D \cap Y \cap \beta$ be not in F. The walk to γ goes via D but does continue further on $C^{\kappa^+}(D)$. Hence, D must be a successor model of $C^{\kappa^+}(A^{0\kappa^+})$ and D^- must be in X, which is impossible, as was observed above.

Let us denote $a_n(\beta)$ by β^*, $a_n(D)$ by D^*, $a_n(X)$ by X^* and $a_n(F)$ by F^*. Let C^* be the function that corresponds to C in $\mathrm{rng}(a_n)$. Then $D^*, F^* \in C^*(X^*)$ and $\beta^* \in D^* \cap a_n{}''Y$.
Assume that D^* and β^* are k-good, for some $k >> 2$. Pick now $M \in D^*$ such that

1. $M \in \beta^*$,

2. $|M| = \kappa_n^{+n+1}$,

3. M is $k-1$-good,

4. $F^* \in M$.

Now, extend a_n by mapping A to M and all the images of it under Δ-system types triples isomorphisms to those of M.
Note that no new ordinals were added in the process and only models that are images of A under Δ-system types isomorphisms for triples in X were added.

Suppose that A is not on the central line. In this case we are supposed to add to p the whole walk from $A^{0\kappa^+}$ to A. We can concentrate, using the induction, only on the case of a Δ-system triple. Namely given $F_0, F_1, F \in A^{1\kappa^{++}}$ of a Δ-system type with F_0 being the immediate predecessor of F in $C^{\kappa^+}(A^{0\kappa^+})$. We need to add F_1 (and probably also F_0, F if they are not inside) to p. F_0, F are on the central line, hence we may assume that they are in p. Let $\alpha_0, \alpha_1 \in F \cap A^{1\kappa^{++}}$ be so that $\alpha_0 \in F_0, \alpha_1 \in F_1$, $F_0 \cap F_1 = \alpha_0 \cap F_0 = \alpha_1 \cap F_1$ and either $\alpha_0 > \sup(F_1)$ or $\alpha_1 > \sup(F_0)$. By the argument above, we can assume that α_0 is already in p.
Note that $F_1 \notin p$ implies that $\alpha_1 \notin p$, since otherwise the walk to α_1 must be in p, by the definition of a suitable structure, but F_1 which is a part of this walk (actually the final model of it) is not in p. This provides a freedom to define the image of α_1 which will be crucial further in choosing the image of F_1.

Fix $n \geq l(p)$. We need to add F_1 to $\mathrm{dom}(a_n(p))$. Let $\mathrm{dom}(a_n(p)) = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\rangle$. We assume that $F_0, F \in C(\max(X))$ and $\alpha_0 \in Y$.
Note that $Y \cap [\alpha_1, \sup(F_1)] = \emptyset$, since if some $\xi \in Y \cap [\alpha_1, \sup(F_1)]$, then all models of the walk to ξ are in X, but F_1 is one of them.
Split into two cases.

Case 1. $\alpha_0 > \alpha_1$.

Then $\sup(F_1) < \alpha_0$. Consider the images F_0^*, F^* and M_0 of F_0, F and α_0 under a_n.

Let us deal first with a little bit simplified situation, but which still contains the main elements of the construction.

Subcase 1.A. No elements of $Y \cap (\sup(F_0 \cap \alpha_0), \alpha_0)$ are in $\mathrm{dom}(a_n) \cap F$.

By Definition 1.2, we have $\mathrm{cof}(\alpha_0) = \kappa^{++}$. Hence $\mathrm{cof}(M_0 \cap \kappa_n^{+n+3}) = \kappa_n^{+n+2}$. So, ${}^{\kappa_n^{+n+1}>}M_0 \subseteq M_0$. In particular, $M_0 \cap F_0^* \in M_0$. Clearly, $M_0 \cap F_0^* \in F^*$, as well. We assume that M_0 is k-good for k big enough. Hence there is a $k-1$-good $M_1 \in M_0$ realizing the same $k-1$ type over $M_0 \cap F_0^*$ as M_0 does. By elementarity, we can find such M_1 inside F^*. Finally, pick F_1^* to be an element of $F^* \cap M_0$ which realizes over $\langle M_0 \cap F_0^*, M_1\rangle$ the same $k-1$ type as F_0^* realizes over $\langle M_0 \cap F_0^*, M_0\rangle$.
Extend a_n by mapping F_1 to F_1^* and all the images of it under Δ-system types triples isomorphisms. In particular, M_1 is added as the image of M_0 under $\pi_{F_0^*,F_1^*}$.

Turn now to a general case.

Subcase 1.B. There are elements of $Y \cap (\sup(F_0 \cap \alpha_0), \alpha_0)$ in $\mathrm{dom}(a_n) \cap F$.

Let γ denotes the last such element below α_1 and β the first such element above α_1. If one of them does not exists, then the argument below applies with obvious simplifications. Note that, as was observed above, there is no elements of Y in the interval $[\alpha_1, \sup(F_1)]$.

Denote $a_n(\beta)$ by N and $a_n(\gamma)$ by γ^*. We assume that M_0 and N are k-good for k big enough. $\sup(F_0^* \cap M_0) \cap \kappa_n^{+n+3} < N \cap \kappa_n^{+n+3}$, hence $F_0^* \cap M_0 \cap \kappa_n^{+n+3} \in N$ (as a set of ordinals of small cardinality). There is a $k-1$-good $M_1 \in N$ realizing the same $k-1$ type over $F_0^* \cap M_0 \cap \kappa_n^{+n+3}$ as M_0 does and with $\gamma^* \in M_1$. By elementarity, we can find such M_1 inside F^*. Finally, pick F_1^* to be an element of $F^* \cap N$ which realizes over $\langle F_0^* \cap M_0 \cap \kappa_n^{+n+3}, M_1\rangle$ the same $k-1$ type as F_0^* realizes over $\langle M_0 \cap F_0^*, M_0\rangle$.
Extend a_n by mapping F_1 to F_1^* and all the images of it under Δ-system types triples isomorphisms. In particular, M_1 is added as the image of M_0

under $\pi_{F_0^*,F_1^*}$.

Case 1. $\alpha_0 < \alpha_1$.

The construction is similar. The only change is that we pick M_1 above M_0.

This completes the inductive construction, and hence the proof of the lemma.

$\square$

The ordering $\leq^*$ on $\mathcal{P}$ and $\leq_n$ on Q_{n0} seems to be not closed in the present situation. Thus it is possible to find an increasing sequence of $\aleph_0$ conditions $\langle\langle a_{ni}, A_{ni}, f_{ni}\rangle \mid i < \omega\rangle$ in Q_{n0} with no simple upper bound. The reason is that the union of maximal models of these conditions, i.e. $\bigcup_{i<\omega} \max(\operatorname{dom} a_{ni})$ need not be in $A^{1\kappa^+}$ for any $A^{1\kappa^+}$ in $G(\mathcal{P}')$. The next lemma shows that still $\leq_n$ and so also $\leq^*$ share a kind of strategic closure.

Lemma 2.17 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_n \rangle$ does not add new sequences of ordinals of the length $< \kappa_n$, i.e. it is (κ_n, ∞) – distributive.*

Proof. Let $\delta < \kappa_n$ and $\underset{\sim}{h}$ be a Q_{n0}-name of a function from δ to ordinals. Without loss of generality assume that δ is a regular cardinal.
Using genericity of $G(\mathcal{P}')$ (or stationarity of the set $\{A^{0\kappa^+} | A^{0\kappa^+}$ appears in an element of $G(\mathcal{P}')\}$) it is not hard to find elementary submodel M of some $H(\nu)$ for ν big enough so that

(a) Q_{n0}, $\underset{\sim}{h}$, $\mathcal{P}' \in M$,

(b) $|M| = \kappa^+$,

(c) there is $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ such that $A^{0\kappa^+} = M \cap H(\kappa^{+3})$ and $\max(A^{1\kappa^{++}} \cap \kappa^{+3}) = \sup(M \cap \kappa^{+3})$.

(d) $cf(M^* \cap \kappa^{++}) = \delta$,

(e) ${}^{\delta>}M \subseteq M$.

Note that for such M, $M^* = M \cap H(\kappa^{+3})$ must be a limit model, since by Definition 1.1(3) successor models are closed under κ sequences, but M^* is not by (d) above.

We have $C^{\kappa^+}(M^*) \setminus \{M^*\} \subseteq M^*$. Let $B \in C^{\kappa^+}(M^*) \setminus \{M^*\}$. We claim that then $C^{\kappa^+}(B) \in M$. Thus, by elementarity there are $B^{1\kappa^+}, D^{\kappa^+}, B^{1\kappa^{++}} \in M$ such that

$$\langle\langle B, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle \in G(\mathcal{P}') \cap M.$$

Note that $C^{\kappa^+} \upharpoonright B^{1\kappa^+}$ may be different from D^{κ^+}, but by the definition of order on $\mathcal{P}'$ (1.15) and since $B \in C^{\kappa^+}(M^*)$, there are $E_1, ..., E_n \in B^{1\kappa^+}$ such that the switch with $E_1, ..., E_n$ turns D^{κ^+} into $C^{\kappa^+} \upharpoonright B^{1\kappa^+}$. But $B^{1\kappa^+} \in M$ and $|B^{1\kappa^+}| \leq \kappa^+$. Hence $B^{1\kappa^+} \subseteq M$. So $E_1, ..., E_n \in M$, and then the corresponding switch is in M as well. This implies that its result $C^{\kappa^+} \upharpoonright B^{1\kappa^+}$ is in M.

The cofinality of $C^{\kappa^+}(M^*) \setminus \{M^*\}$ under the inclusion must be δ, since it is an $\in$-increasing continuous sequence of elements of M^* with limit M^* and by (d) above $cf(M^* \cap \kappa^{++}) = \delta$. Fix an increasing continuous sequence $\langle A_i \mid i < \delta\rangle$ of elements of $C^{\kappa^+}(M^*) \setminus \{M^*\}$ such that $\bigcup_{i<\delta} A_i = M^*$, A_0 is a successor model and for each limit model A_i in the sequence A_{i+1} is its immediate successor in $C^{\kappa^+}(M^*)$. By (e), each initial segment of it will be in M. Now we decide inside M one by one values of $\underset{\sim}{h}$ and put models from $\langle A_i \mid i < \delta\rangle$ to be maximal models of conditions used. This way we insure that unions of such conditions is a condition.

We define by induction an increasing sequence of conditions

$$\langle\langle a(i), A(i), f(i)\rangle | i \leq \delta\rangle.$$

and an increasing continuous subsequence

$$\langle A_{k_i} | i < \delta\rangle \text{ of } \langle A_i | i < \delta\rangle$$

such that for each $i < \delta$

(1) $\langle a(i), A(i), f(i)\rangle \in M$,

(2) $\langle a(i+1), A(i+1), f(i+1)\rangle$ decides $\underset{\sim}{h}(i)$,

(3) $A_{k_i}, A_{k_{i+1}} \in \operatorname{dom}(a(i))$, $A_{k_{i+1}}$ is the maximal model of $\operatorname{dom}(a(i))$ and $\langle\langle A_{k_{i+1}}, T, C^{\kappa^+} \restriction T\rangle, R\rangle \in G(\mathcal{P}') \cap M$ witnesses a generic suitability of $\operatorname{dom}(a(i))$, for some T, R, with $R \subseteq A_{k_{i+1}} \cup \sup(A_{k_{i+1}})$.

There is no problem with $A(i)$'s and $f(i)$'s in this construction. Thus we have enough completeness to take intersections of $A(i)$'s and unions of $f(i)$'s. The only problematic part is $a(i)$. So let us concentrate only on building of $a(i)$'s.

i=0

Then let us pick some $Z_0 \prec Z_1 \prec H(\chi^{+\omega}) \cap M$ of cardinality κ_n^{+n+1}, closed under κ_n^{+n} - sequences of its elements and $Z_0 \in Z_1$. Set $a(0) = \langle\langle A_0, Z_0\rangle, \langle A_1, Z_1\rangle\rangle$.

i+1

Then we first extend $\langle a(i), A(i), f(i)\rangle$ to a condition $\langle a(i)', A(i)', f(i)'\rangle \in M$ which decides $\underset{\sim}{h}(i)$. Then perform *swt* (see 1.13) to turn $\langle a(i)', A(i)', f(i)'\rangle$ into an equivalent condition $\langle a(i)'', A(i)', f(i)'\rangle$ with $A_{k_i} \in C^{\kappa^+}(\max(\operatorname{dom}(a(i)'')))$. Pick a successor model A_j (from the cofinal sequence $\langle A_i \mid i < \delta\rangle$) including $\max(\operatorname{dom}(a(i)''))$. Set $k_{i+1} = j$ and add it to $\operatorname{dom}(a(i)'')$, using *swt* inside A_j if necessary. Finally we add A_{j+1}.

i is a limit ordinal

Then we need to turn $a = \bigcup_{j<i} a(j)$ into condition. For this we will need to add to $\operatorname{dom}(a)$ models and ordinals which are limits of elements of $\operatorname{dom}(a)$. First we extend a by adding to it $\langle A_{k_i}, \bigcup_{j<i} a(A_{k_j})\rangle$, where $k_i = \cup_{j<i} k_j$. Then for each non decreasing sequence $\langle \alpha_j | j < i\rangle$ of ordinals in $\operatorname{dom}(a)$ we add the pair $\langle \cup_{j<i} \alpha_j, \cup_{j<i}(a(\alpha_j) \cap H(\chi^{+\ell}))\rangle$, if it is not already in the $\operatorname{dom}(a)$, where $\ell \leq \omega$ the maximal such that for unboundedly many j's in i $a(\alpha_j) \prec H(\chi^{+\ell})$, if the maximum exists or $\ell >> n$ otherwise. Finally, for each model $B \in \operatorname{dom}(a)$ if there is a nondecreasing sequence $\langle B_j | j < i\rangle$ of elements of $C^{\kappa^+}(B)$ in $\operatorname{dom}(a)$ and B is the least possible (under inclusion or with least sup) including the sequence, then we add the pair $\langle \cup_{j<i} B_j, \cup_{j<i}(a(B_j) \cap H(\chi^{+\ell}))\rangle$, if it is not already in the dom(a), where $\ell \leq \omega$ is the minimum between the least k such that $a(B) \subseteq H(\chi^{+k})$ and the maximal ℓ' such that for unboundedly many j's in i $a(B_j) \prec H(\chi^{+\ell'})$, if the maximum exists

or

it is k, if the maximum does not exist and $k < \omega$,

or

$\ell >> n$, if the maximum does not exist and $k = \omega$.

We will need to extend a bit more if the following hold:

1. $B \in \text{dom}(a)$,

2. $\langle B_j \mid j < i\rangle$ is a nondecreasing sequence of elements of $C^{\kappa^+}(B)$ in dom(a),

3. B is the least element of dom(a) such that $\bigcup_{j<i} B_j \in B$,

4. $\langle \alpha_j \mid j < i\rangle$ is a sequence of ordinals such that

   (a) $\alpha_j \in B_j$,

   (b) $\alpha_j \in \text{dom}(a)$,

   (c) $\bigcup_{j<i} \alpha_j \notin \text{dom}(a)$.

Set $\alpha = \bigcup_{j<i} \alpha_j$. Then $\alpha \in B$.

Let us consider two cases.

Case 1. $\alpha \notin \bigcup_{j<i} B_j$.

If B is the real immediate successor of $\bigcup_{j<i} B_j$, i.e. the one in $C^{\kappa^+}(A^{0\kappa^+})$ of $G(\mathcal{P}')$, then the extension made above suffices. Otherwise, we need to add the real successor of $\bigcup_{j<i} B_j$ in order to insure walks to ordinals closure. Denote such successor by E. We map it to a model E^* such that $\bigcup_{j<i}(a(B_j) \cap H(\chi^{+\ell})) \prec E^* \prec a(B) \cap H(\chi^{+\ell}))$ and E^* is good enough, where ℓ is as above. Note that each $\gamma \in B \cap \text{dom}(a)$ is already in B_j, for some $j < i$, by walks to ordinals closure of dom(a). Finally we map α to $\bigcup_{j<i}(a(B_j) \cap a(\alpha_j))$.

Case 2. $\alpha \in \bigcup_{j<i} B_j$.
Let E be the smallest model in $C'^{\kappa^+}(B)$ with $\alpha \in E$.
Subcase 2.1. E is the least (under the inclusion) element of $C'^{\kappa^+}(B)$.
If for some $j < i$ we have $\alpha_j \in E$, then by the walk closure of $\mathrm{dom}(a)$, the model E is in $\mathrm{dom}(a)$. It is easy now to extend a by adding only α which is mapped to an appropriate element of $a(E)$.
Suppose that for each $j < i$, $\alpha_j \notin E$. Consider α_0. Let D_0 be the largest model in $C'^{\kappa^+}(B)$ with $\alpha_0 \notin D_0$. By the walk closure of $\mathrm{dom}(a)$, we have $D_0 \in \mathrm{dom}(a)$. Assume that $D_0 \neq E$, otherwise proceed as above. Clearly $D_0 \supset E$, and hence $\alpha_0 < \alpha < \sup(D_0)$. Then $\alpha_{01} := \min(D_0 \cap \alpha_0) \in \mathrm{dom}(a)$. So, $\alpha_0 < \alpha_{01} < \alpha$. Let D_{01} be the largest model in $C'^{\kappa^+}(B)$ with $\alpha_{01} \notin D_0$. By the walk closure of $\mathrm{dom}(a)$, we have $D_{01} \in \mathrm{dom}(a)$. Again, we assume that $D_{01} \neq E$. Clearly $D_0 \supset D_{01} \supset E$, and hence $\alpha_{01} < \alpha < \sup(D_{01})$. Then $\alpha_{02} := \min(D_{01} \cap \alpha_{01}) \in \mathrm{dom}(a)$. So, $\alpha_0 < \alpha_{01} < \alpha_{02} < \alpha$. We continue and define D_{02} etc. The sequence of such D_{0k} will be $\in$-decreasing, and hence at certain stage $D_{0k} = E$.
Subcase 2.2. E is not the least (under the inclusion) element of $C'^{\kappa^+}(B)$.
Then E has the immediate predecessor E^- in $C'^{\kappa^+}(B)$. Suppose first that α is a limit point of E^-. Note that then necessarily E^- is a limit model, as successor ones are closed under $< \kappa^+$-sequences.
Claim 2.17.1 *There is an increasing sequence $\langle \alpha'_j \mid j < i \rangle$ in $E^- \cap \mathrm{dom}(a)$ with limit α.*
Proof. Let $j < i$. If $\alpha_j \in E^-$, then we take it. Suppose that $\alpha_j \notin E^-$. Pick D_j to be the largest model in $C'^{\kappa^+}(B)$ with $\alpha_j \notin D_j$. Then, $D_j \in \mathrm{dom}(a)$, and clearly, $D_j \supseteq E^-$. Also, $\alpha_j < \alpha$ and α is a limit point of E^-. Hence $\alpha_j < \sup(D_j)$. Then $\alpha_{j1} := \min(D_j \setminus \alpha_j) \in D_j \cap \mathrm{dom}(a)$. If $\alpha_{j1} \in E^-$, then we pick it. Otherwise, continue and consider D_{j1} the largest model in $C'^{\kappa^+}(B)$ with $\alpha_{j1} \notin D_{j1}$. Then, $D_{j1} \in \mathrm{dom}(a)$, and clearly, $D_{j1} \supseteq E^-$. Also, $\alpha_{j1} < \alpha$ and α is a limit point of E^-. Hence $\alpha_{j1} < \sup(D_j)$. Then $\alpha_{j2} := \min(D_{j1} \setminus \alpha_{j1}) \in D_{j1} \cap \mathrm{dom}(a)$. If $\alpha_{j2} \in E^-$, then we pick it. Otherwise,

continue. After finitely many steps we will reach some such $\alpha_{jk} \in E^-$.
$\square$ of the claim.
Let $\langle \alpha'_j \mid j < i \rangle$ be given by the claim. For each $j < i$ let K_j be the least model of $C^{\kappa^+}(B)$ with $\alpha'_j \in K_j$. Then $E^- = \bigcup_{j<i} K_j$, since, clearly $E^- \supseteq \bigcup_{j<i} K_j$ and if $E^- \not\supseteq \bigcup_{j<i} K_j$, then α will be in the immediate successor $K \in C^{\kappa^+}(B)$ of $\bigcup_{j<i} K_j$, but $K \subseteq E^-$ and $\alpha \notin E^-$. Now we are in situation of Case 1 with $\langle \alpha_j \mid i < j \rangle$ replaced by $\langle \alpha'_j \mid i < j \rangle$ and $\langle B_j \mid i < j \rangle$ by $\langle K_j \mid j < i \rangle$.

Suppose now that α is not a limit point of E^-. Pick $j^* < i$ such that for every $j, j^* \leq j < i$, $\sup(E^- \cap \alpha) < \alpha_j$. If for some $j, j^* \leq j < i$, $\alpha_j \in E$, then E will be the least model of $C^{\kappa^+}(B)$ with α_j inside, and hence $E, E^- \in \mathrm{dom}(a)$, due to the walk closure of $\mathrm{dom}(a)$. Suppose that for each $j, j^* \leq j < i$, $\alpha_j \notin E$. Fix such j. Pick D_j to be the largest model in $C^{\kappa^+}(B)$ with $\alpha_j \notin D_j$. Then, $D_j \in \mathrm{dom}(a)$, and clearly, $D_j \supseteq E$. If $D_j = E$, then $E \in \mathrm{dom}(a)$. Then, also $E^- \in \mathrm{dom}(a)$, since $\alpha_{j1} := \min(E \setminus \alpha_j) \in E \cap \mathrm{dom}(a)$, but E^- is the largest model in $C^{\kappa^+}(B)$ with α_{j1} not inside, and hence it must be in $\mathrm{dom}(a)$ by the walk closure.
Suppose that $D_j \neq E$. Consider $\alpha_{j1} := \min(D_j \setminus \alpha_j) \in D_j \cap \mathrm{dom}(a)$. Clearly, $\alpha_{j1} < \alpha$, since $E \subseteq D_j$ and $\alpha \in E$. If $\alpha_{j1} \in E$, then E will be the least model of $C^{\kappa^+}(B)$ with α_{j1} inside, since $\alpha_{j1} \notin E^-$. Then $E, E^- \in \mathrm{dom}(a)$.
If $\alpha_{j1} \notin E$, then we continue and pick D_{j1} to be the largest model in $C^{\kappa^+}(B)$ with $\alpha_{j1} \notin D_{j1}$. Then, $D_{j1} \in \mathrm{dom}(a)$, and clearly, $D_{j1} \supseteq E$. If $D_{j1} = E$, then $E \in \mathrm{dom}(a)$. Then, also $E^- \in \mathrm{dom}(a)$, since $\alpha_{j2} := \min(E \setminus \alpha_{j1}) \in E \cap \mathrm{dom}(a)$, but E^- is the largest model in $C^{\kappa^+}(B)$ with α_{j2} not inside, and hence it must be in $\mathrm{dom}(a)$ by the walk closure.
If $D_{j1} \neq E$, then we continue in the same fashion to define α_{j2}, D_{j2} etc. After finitely many steps we will have $E = D_{jk}$ or $\alpha_{jk} \in E$. Both imply $E, E^- \in \mathrm{dom}(a)$.

Finally denote the resulting extension of a by b.

Claim 2.17.2 $\mathrm{dom}(b)$ *is a suitable generic structure.*
Proof. Let as check the condition (6(6c)) of Definition 1.1. Thus let $A, \alpha \in$

dom(b), $A \in C(\max(\mathrm{dom}(b)))$ a non-limit model and $\sup(A) > \alpha$. We need to show that $\min(A \setminus \alpha) \in \mathrm{dom}(b)$.

Case 1. $A \in \mathrm{dom}(a(l))$ *for some* $l < i$.

If $\alpha \in \mathrm{dom}(a)$, then for some $j < i$ big enough we will have $A, \alpha \in \mathrm{dom}(a_j)$, and then $\min(A \setminus \alpha) \in \mathrm{dom}(a_j)$. Note that if α is a non-limit element of dom(b), then $\alpha \in \mathrm{dom}(a)$.

Suppose that α is a limit point of dom(b) and $\alpha \notin \mathrm{dom}(a)$. Let $\langle \alpha_j | j < i \rangle$ be a nondecreasing sequence from dom(a) converging to α. By (6(6c)) of Definition 1.1, $\gamma_j = \min(A \backslash \alpha_j) \in \mathrm{dom}(a)$. If $\langle \gamma_j | j < i \rangle$ is eventually constant, then the constant value will be as desired. Suppose otherwise. Then $\langle \gamma_j | j < i \rangle$ will be also a converging to α sequence. But remember that A is non-limit, hence ${}^{\kappa}A \subseteq A$, and so $\alpha \in A$. Then $\min(A \backslash \alpha) = \alpha \in \mathrm{dom}(b)$ and we are done.

Case 2. $A \notin \mathrm{dom}(a)$.

Assume that $\alpha \notin A$, just otherwise $\min(A \setminus \alpha) = \alpha$ and we are done. Denote $\min(A \setminus \alpha)$ by α^*

Subcase 2.1. $\alpha \in \mathrm{dom}(a)$.

Consider then the smallest model E_α in $C(\max(\mathrm{dom}(b)))$ with α inside. Let E_α^- be its immediate predecessor in $C(\max(\mathrm{dom}(b)))$. Then $A \subseteq E_\alpha^-$, since $\alpha \notin A$, and $A \neq E_\alpha^-$, since $E_\alpha^- \in \mathrm{dom}(a)$ and $A \notin \mathrm{dom}(a)$. Then $\sup(E_\alpha^-) > \alpha$, hence $\alpha_1 := \min(E_\alpha^- \setminus \alpha) > \alpha$ and $\alpha_1 \in \mathrm{dom}(a)$. $E_\alpha^- \supseteq A$ implies that $\alpha_1 \leq \alpha^*$. If $\alpha_1 = \alpha^*$, then $\alpha^* \in \mathrm{dom}(a)$ and we are done. Suppose otherwise. Then $\alpha_1 < \alpha^*$. Consider then the smallest model E_{α_1} in $C(\max(\mathrm{dom}(b)))$ with α_1 inside. Let $E_{\alpha_1}^-$ be its immediate predecessor in $C(\max(\mathrm{dom}(b)))$. Then $A \subseteq E_{\alpha_1}^-$, since $\alpha_1 \notin A$, and $A \neq E_{\alpha_1}^-$, since $E_{\alpha_1}^- \in \mathrm{dom}(a)$ and $A \notin \mathrm{dom}(a)$. Then $\sup(E_{\alpha_1}^-) > \alpha_1$, since $\alpha^* \in E_{\alpha_1}^-$ and $\alpha^* > \alpha_1$. Hence $\alpha_2 := \min(E_{\alpha_1}^- \setminus \alpha_1) > \alpha_1$ and $\alpha_2 \in \mathrm{dom}(a)$. If $\alpha_2 = \alpha^*$, then $\alpha^* \in \mathrm{dom}(a)$ and we are done. Otherwise, $\alpha_2 < \alpha^*$. We continue and consider $E_{\alpha_2}, E_{\alpha_2^-}$ etc. Note that the sequence of models E_{α_m} constructed this way is decreasing. So the process stops after finitely many steps. Which means that $\alpha^* \in \mathrm{dom}(a)$.

Subcase 2.2. $\alpha \notin \mathrm{dom}(a)$.

Then α is a limit of an increasing sequence $\langle \alpha_j \mid j < i \rangle$ of elements of $\mathrm{dom}(a)$. If an unbounded subsequence of the sequence $\langle \alpha_j \mid j < i \rangle$ is in A, then α will be in A as well, since A is a non-limit model and so is closed under δ sequences of its elements. Hence there is $j^* < i$ such that for every $j, j^* \leq j < i, \alpha_j \notin A$. Let $j^* \leq j < i$. We have $\sup(A) > \alpha > \alpha_j$. Set $\alpha_j^* = \min(A \backslash \alpha_j)$. By Subcase 2.1, $\alpha_j^* \in \mathrm{dom}(a)$. If $\alpha_j^* > \alpha$, then $\alpha_j^* = \alpha^*$ and we are done. Assume, hence that $\alpha_j^* < \alpha$, for every $j < i$. But the sequence $\langle \alpha_j^* \mid j < i \rangle$ is a sequence of elements of A which converges to α. So, $\alpha \in A$. Contradiction.

$\square$ of the claim.

The next claim is similar.

Claim 2.17.3 $\mathrm{rng}(b)$ *is a suitable structure over* κ_n.

We need to check that b is an isomorphism between the suitable structures $\mathrm{dom}(b)$ and $\mathrm{rng}(b)$. By Lemma 2.10, it is enough to show that the restriction of b is an isomorphism between between the corresponding weak suitable structures. But this is obvious, since no Δ-system type triples are added at limit stages.

$\square$

It is possible to work in V rather than in $V[G(\mathcal{P}')]$ or M. Combining arguments of 1.19 and the previous lemma it is not hard to show the following:

Lemma 2.18 $\mathcal{P}' * Q_{n0}$ *is* $< \kappa_n$*-strategically closed.*

Lemma 2.19 $\langle \mathcal{P}, \leq^* \rangle$ *does not add new sequences of ordinals of the length* $< \kappa_0$.

Proof. Repeat the argument of Lemma 2.17 with $\mathcal{P}$ replacing Q_{n0}.

$\square$

The argument of Lemma 2.17 can be used in a standard fashion to show the Prikry condition (i.e. the standard argument runs inside elementary submodel M with δ replaced by κ^+).

Lemma 2.20 $\langle \mathcal{P}, \leq^* \rangle$ *satisfies the Prikry condition.*

Finally we define $\rightarrow$ on $\mathcal{P}$ similar to those of [1] or [2].

Lemma 2.21 $\langle \mathcal{P}, \rightarrow \rangle$ *satisfies* κ^{++}*-c.c.*

Proof. Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++} \rangle$ be a name of an antichain of the length κ^{++}. Using 1.19 we find an increasing sequence $\langle \langle \langle A^{0\kappa^+}_\alpha, A^{1\kappa^+}_\alpha, C^{\kappa^+}_\alpha \rangle, A^{1\kappa^{++}}_\alpha \rangle \mid \alpha < \kappa^{++} \rangle$ of elements of $\mathcal{P}'$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ so that for every $\alpha < \kappa^{++}$ the following hold:

(a) $\langle \langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1} \rangle, A^{1\kappa^{++}}_{\alpha+1} \rangle \Vdash \underset{\sim}{p}_\alpha \leq \check{p}_\alpha$,

(b) $\bigcup_{\beta < \alpha} A^{0\kappa^+}_\beta = A^{0\kappa^+}_\alpha$, if α is a limit ordinal,

(c) ${}^\kappa A^{0\kappa^+}_{\alpha+1} \subseteq A^{0\kappa^+}_{\alpha+1}$,

(d) $A^{0\kappa^+}_{\alpha+1}$ is a successor model,

(e) $\langle A^{1\kappa^+}_\beta \mid \beta < \alpha \rangle \in A^{0\kappa^+}_{\alpha+1}$,

(f) for every $\alpha \leq \beta < \kappa^{++}$ we have

$$C^{\kappa^+}_\alpha(A^{0\kappa^+}_\alpha) \text{ is an initial segment of } C^{\kappa^+}_\beta(A^{0\kappa^+}_\beta),$$

(g) $p_\alpha = \langle p_{\alpha n} \mid n < \omega \rangle$,

(h) for every $n \geq l(p_\alpha)$, $A^{0\kappa^+}_{\alpha+1}$ is the maximal model of $\mathrm{dom}(a_{\alpha n})$ and $A^{0\kappa^+}_\alpha \in \mathrm{dom}(a_{\alpha n})$, where $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n} \rangle$.
Actually this condition is the reason for not requiring the equality in (a) above.

Let $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n} \rangle$ for every $\alpha < \kappa^{++}$ and $n \geq l(p_\alpha)$.
Let $\alpha < \kappa^{++}$. Fix some

$$\langle \langle B^{0\kappa^+}_{\alpha+1}, B^{1\kappa^+}_{\alpha+1}, D^{\kappa^+}_{\alpha+1} \rangle, B^{1\kappa^{++}}_{\alpha+1} \rangle \leq_{\mathcal{P}'} \langle \langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1} \rangle, A^{1\kappa^{++}}_{\alpha+1} \rangle$$

which witnesses a generic suitability of structure $\mathrm{dom}(a_{\alpha n})$ for each $n, l(p_\alpha) \leq n < \omega$, as in Definition 2.2. Note that $B^{0\kappa^+}_{\alpha+1}$ need not be in $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$ and even if it does, then $D^{\kappa^+}_{\alpha+1}(B^{0\kappa^+}_{\alpha+1})$ need not be an initial segment of $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$. By the definition of the order $\leq_{\mathcal{P}'}$ (Definition 1.15) there are $m < \omega$ and $E_1, ..., E_m \in A^{1\kappa^+}_{\alpha+1}$ such that

$$swt(\langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle, E_1, ..., E_m) \text{ and } \langle\langle B^{0\kappa^+}_{\alpha+1}, B^{1\kappa^+}_{\alpha+1}, D^{\kappa^+}_{\alpha+1}\rangle, B^{1\kappa^{++}}_{\alpha+1}\rangle$$

satisfy (1)-(3) of Definition 1.15.
By Lemma 2.16 it is possible to add all $E_i (i = 1, ..., m)$ to $\mathrm{dom}(a_{\alpha n})$, for a final segment of n's. By adding and taking non-direct extension if necessary, we can assume that E_i's are already in $\mathrm{dom}(a_{\alpha n})$, for every $n \geq l(p_\alpha)$.
Now we can apply the opposite switch (i.e. the one starting with E_m, then E_{m-1}, ...,and finally E_1) to $\mathrm{dom}(a_{\alpha n})$ (and the corresponding to it under $a_{\alpha n}$ to $\mathrm{rng}(a_{\alpha n})$). Denote the result still by $a_{\alpha n}$.
Finally, $\langle\langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1}\rangle, A^{1\kappa^{++}}_{\alpha+1}\rangle$ will witness a generic suitability of structure $\mathrm{dom}(a_{\alpha n})$ for each $n, l(p_\alpha) \leq n < \omega$.
In particular, we have now that the central line of $\mathrm{dom}(a_{\alpha n})$ is a part of $C^{\kappa^+}_{\alpha+1}(A^{0\kappa^+}_{\alpha+1})$ and $A^{0\kappa^+}_{\alpha}$ is on it, for every $n, l(p_\alpha) \leq n < \omega$.

Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^+$ the following holds:

(1) $\ell = \ell(p_\alpha) = \ell(p_\beta)$,

(2) for every $n < \ell \quad p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1} i.e. $p_{\alpha n} \cup p_{\beta n}$ is a function,

(3) for every n, $\ell \leq n < \omega$, $\quad \langle \mathrm{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$,

(4) for every n, $\omega > n \geq \ell$, $\quad \mathrm{rng}(a_{\alpha n}) = \mathrm{rng}(a_{\beta n})$.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let α be in S. For each $n, \ell \leq n < \omega$, there will be $\beta(\alpha, n) < \alpha$

such that

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} \subseteq A^{0\kappa^+}_{\beta(\alpha,n)}.$$

Just recall that $|a_{\alpha n}| < \kappa_n$. Shrink S to a stationary subset S^* so that for some $\alpha^* < \min S^*$ of cofinality κ^+ we will have $\beta(\alpha, n) < \alpha^*$, whenever $\alpha \in S^*, \ell \leq n < \omega$. Now, the cardinality of $A^{0\kappa^+}_{\alpha^*}$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\alpha, \beta \in S^*, \ell \leq n < \omega$

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} = \mathrm{dom}(a_{\beta n}) \cap A^{0\kappa^+}_{\beta}.$$

Let us add $A^{0\kappa^+}_{\alpha^*}$ to each p_α with $\alpha \in S^*$.

By 2.16(2), we can add it without adding ordinals and the only other models that probably were added are the images of $A^{0\kappa^+}_{\alpha^*}$ under Δ-system type isomorphisms. Denote the result for simplicity by p_α as well.

Let now $\beta < \alpha$ be ordinals in S^*. We claim that p_β and p_α are compatible in $\langle \mathcal{P}, \rightarrow \rangle$.

First extend p_α by adding $A^{0\kappa^+}_{\beta+2}$. This will not add other additional models or ordinals except the images of $A^{0\kappa^+}_{\beta+2}$ under isomorphisms to p_α, as was remarked above.

Let p be the resulting extension. Denote p_β by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$ and $p_n = \langle a_n, A_n, f_n \rangle$. Let $q_n = \langle b_n, B_n, g_n \rangle$. Without loss of generality we may assume that $a_n(A^{0\kappa^+}_{\beta+2})$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $\mathrm{rng}(b_n)$ inside $a_n(A^{0\kappa^+}_{\beta+2})$ over the common parts $\mathrm{dom}(b_n)$ and $\mathrm{dom}(a_n)$. This will produce $q'_n = \langle b'_n, B_n, g_n \rangle$ which is $k_n - 1$-equivalent to q_n and with $\mathrm{rng}(b'_n) \subseteq a_n(A^{0\kappa^+}_{\beta+2})$. Doing the above for all $n \geq \ell(p)$ we will obtain $q' = \langle q'_n \mid n < \omega \rangle$ equivalent to q (i.e. $q' \longleftrightarrow q$).

Extend q' to q'' by adding to it $\langle A^{0\kappa^+}_{\beta+2}, a_n(A^{0\kappa^+}_{\beta+2}) \rangle$ as the maximal set for every $n \geq \ell(p)$. Recall that $A^{0\kappa^+}_{\beta+1}$ was its maximal model. So we add a top model. Hence no additional models or ordinals are added at all. Let $q''_n = \langle b''_n, B_n, g_n \rangle$, for every $n \geq \ell(p)$.

Combine now p and q'' together. Thus for each $n \geq \ell(p)$ we add b''_n to a_n as well as all of its isomorphic images under Δ-system type isomorphisms of triples in a_n. The rest of the parts are combined in the obvious fashion (we put together the functions and intersect sets of measure one moving first to the same measure). Add if necessary $A^{0\kappa^+}_{\alpha+3}$ as a new top model in order to insure 2.11(2(2a)). Let $r = \langle r_n | n < \omega \rangle$ be the result, where $r_n = \langle c_n, C_n, h_n \rangle$, for $n \geq \ell(p)$.

Claim 2.21.1 For each $\gamma, \alpha + 3 < \gamma < \kappa^{++}$,

$$\langle \langle A^{0\kappa^+}_\gamma, A^{1\kappa^+}_\gamma, C^{\kappa^+}_\gamma \rangle, A^{1\kappa^{++}}_\gamma \rangle \Vdash r \in \underset{\sim}{\mathcal{P}}.$$

Proof. Let $\gamma \in (\alpha + 3, \kappa^{++})$ and $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$ with $\langle \langle A^{0\kappa^+}_\gamma, A^{1\kappa^+}_\gamma, C^{\kappa^+}_\gamma \rangle, A^{1\kappa^{++}}_\gamma \rangle \in G(\mathcal{P}')$.

Fix $n \geq \ell(p)$. The main points here are that b''_n and a_n agree on the common part and adding of b''_n to a_n does not required other additions of models or of ordinals except the images of b''_n under Δ -system type isomorphisms for triples in a_n.

We need to check that $\mathrm{dom}(c_n)$ is a suitable generic structure and $\mathrm{rng}(a_n)$ is a suitable structure. Let us deal with $\mathrm{dom}(c_n)$. The range is similar. By Lemma 2.10 it is enough to deal with a weak suitable structures. Let $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ be the corresponding redact of $\mathrm{dom}(c_n)$.

Clearly, $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ is a submodel of $\langle \langle A^{1\kappa^+}_\gamma, A^{1\kappa^{++}}_\gamma \rangle, C^{\kappa^+}_\gamma, \in, \subseteq \rangle$.

Let us check that the structures $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ and $\langle \langle A^{1\kappa^+}_\gamma, A^{1\kappa^{++}}_\gamma \rangle, C^{\kappa^+}_\gamma, \in, \subseteq \rangle$ agree about walks to members of X and to ordinals in Y. This will show, in particular that $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ is walks closed and, hence $\langle \langle \max(X), X, C \rangle, Y \rangle \in \mathcal{P}'$.

Fix $t \in X \cup Y$ (a model or an ordinal). Note that, by the choice of the top model $\max(X)$ of X we have $\max(X) \in C^{\kappa^+}_\gamma(A^{0\kappa^+}_\gamma)$. Hence, the walk from $A^{0\kappa^+}_\gamma$ to t will go via $\max(X)$. If t appears in $\mathrm{dom}(a_n)$, then the continuation of the walk will be inside $\mathrm{dom}(a_n)$, since $\max(a_n) = A^{0\kappa^+}_{\alpha+1} \in C(\max(X))$.

It will co-inside with the walk from $A^{0\kappa^+}_{\alpha+1}$ to t, since $\mathrm{dom}(a_n)$ is a suitable structure. Hence all the members of the walk are in $X \cup Y$.
Note that if t is in the common part, i.e. if t appears in both $\mathrm{dom}(a_n)$ and $\mathrm{dom}(b_n)$, then $t \in A^{0\kappa^+}_{\alpha^*}$. So the walk to t passes through $A^{0\kappa^+}_{\alpha^*}$, since $A^{0\kappa^+}_{\alpha^*} \in C^{\kappa^+}(A^{0\kappa^+}_{\gamma})$.
If t appears in $\mathrm{dom}(b''_n) = \mathrm{dom}(b_n) \cup \{A^{0\kappa^+}_{\beta+2}\}$, then the walk to t will proceed via $A^{0\kappa^+}_{\beta+2}$, since $t \in A^{0\kappa^+}_{\beta+2}$ and $A^{0\kappa^+}_{\beta+2} \in C(\max(X))$. Now, it will co-inside with the walk from $A^{0\kappa^+}_{\beta+1}$ to t, since $\mathrm{dom}(b_n)$ is a suitable structure and $A^{0\kappa^+}_{\beta+1} \in C(A^{0\kappa^+}_{\beta+2})$.

The agreement between the walks follows.

$\square$ of the claim.

Now we have $r \geq p, q''$. Hence, $p \to r$ and $q \to r$. Contradiction.

$\square$

References

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