More ultrafilters with Galvin property

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Abstract

We present a method of constructing models with $Q-$point ultrafilters which have a Galvin property but are not sums of P-points. This answers a question of Tom Benhamou [1].

1 Some general facts

Our basic setting will be the following:

Let W be a $\kappa-$complete ultrafilter over κ, $U = \{X \subseteq \kappa \mid \kappa \in j_W(X)\}$ and $k : M_U \to M_W$ the corresponding embedding. Let $\kappa_1 = j_U(\kappa)$. Suppose that $\kappa_1 = [id]_W$ and $\kappa_1 = \text{crit}(k)$.

The following lemma is well known:

Lemma 1.1 *$W \supseteq Cub_\kappa$.*

Lemma 1.2 *Suppose that $\{A_\alpha \mid \alpha < \kappa\} \subseteq W$ and $\bigcap_{\alpha<\kappa} A_\alpha \in W$. Then for every $B \in j_U(\{A_\alpha \mid \alpha < \kappa\}), \kappa_1 \in k(B)$.*

Proof. Follows from elementarity and since $j_W = k \circ j_U$.
□

Lemma 1.3 *For every $B \in j_U{}''W$, $\kappa_1 \in k(B)$.*

Proof. Let $B = j_U(A)$, for some $A \in W$. Then

$$\kappa_1 \in j_W(A) = k(j_U(A)) = k(B).$$

□

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Lemma 1.4 *There is $B \in j_U(W)$ such that $\kappa_1 \notin k(B)$.*

Proof. Let $f : \kappa \to \kappa$ be a function that represents κ in M_W, i.e. $j_W(f)(\kappa_1) = \kappa$. It is a regressive function which is not constant on a set in W. Then, for every $\eta < \kappa$,

$$A_\eta = \{\nu \in X \mid f(\nu) \neq \eta\} \in W.$$

Let

$$\langle A^1_\eta \mid \eta < \kappa_1 \rangle = k(\langle A_\eta \mid \eta < \kappa \rangle).$$

Then, $\kappa_1 \notin k(A^1_\kappa)$, since $j_W(f)(\kappa_1) = \kappa$.
□

The next lemma will be crucial for our further constructions.

Lemma 1.5 *Suppose that $\{A_\alpha \mid \alpha < \kappa^+\}, \{B_\alpha \mid \alpha < \kappa^+\} \subseteq W$ are such that*

1. *$\{A_\alpha \mid \alpha < \kappa^+\} \subseteq U$;*
2. *for every $A \in j_U(\{A_\alpha \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(A)$;*
3. *for every $B \in j_U(\{B_\alpha \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(B)$.*

Then there is $I \subseteq \kappa^+, |I| = \kappa$ such that

1. *$\bigcap_{\alpha \in I} A_\alpha \in U \cap W$,*
2. *$\bigcap_{\alpha \in I} B_\alpha \in W$.*

Proof. We repeat basically the Galvin proof simultaneously for $\{A_\alpha \mid \alpha < \kappa^+\}$ and $\{B_\alpha \mid \alpha < \kappa^+\}$.

Thus, define

$$H_{\alpha\xi} = \{\beta < \kappa^+ \mid A_\alpha \cap \xi = A_\beta \cap \xi \text{ and } B_\alpha \cap \xi = B_\beta \cap \xi\},$$

for every $\alpha < \kappa^+, \xi < \kappa$.
Then, as in the Galvin proof, there will be $\alpha^* < \kappa^+$ such that for every $\xi < \kappa$, $|H_{\alpha^*\xi}| = \kappa^+$.

Define by induction a sequence $\langle \eta_\xi \mid \xi < \kappa \rangle$ such that

$$\eta_\xi \in H_{\alpha^*\xi+1} \setminus \{\eta_{\xi'} \mid \xi' < \xi\}.$$

Set $I = \{\eta_\xi \mid \xi < \kappa\}$. Let us argue that such I is as desired.

Apply j_U and continue the inductive definition of the sequence $\langle \eta_\xi \mid \xi < \kappa \rangle$ in M_U. Let $\langle \eta^1_\xi \mid \xi < \kappa_1 \rangle$ be the resulting sequence. Denote $j_U(\{A_\alpha \mid \alpha < \kappa^+\})$ by $\{A^1_\alpha \mid \alpha < j_U(\kappa^+)\}$ and $j_U(\{B_\alpha \mid \alpha < \kappa^+\})$ by $\{B^1_\alpha \mid \alpha < j_U(\kappa^+)\}$.
Then, by the first assumptions of the lemma, $\kappa \in A^1_\alpha$, for every $\alpha \in j''_U \kappa^+$. In particular, $\kappa \in A^1_{\eta^1_\xi}$, for every $\xi < \kappa$.
For every $\xi, \kappa \leq \xi < \kappa_1$ we will have $A^1_{\eta^1_\xi} \cap \xi + 1 = A^1_{j_U(\alpha^*)} \cap \xi + 1$.
We have, $\kappa \in j_U(A_{\alpha^*})$, and so, $\kappa \in j_U(A_{\alpha^*}) \cap \xi + 1 = A^1_{\eta^1_\xi} \cap \xi + 1$.
So, $\kappa \in A^1_{\eta^1_\xi}$, for every $\xi < \kappa_1$, and then,

$$\kappa \in \bigcap_{\xi < \kappa_1} A^1_{\eta^1_\xi} = j_U(\bigcap_{\xi < \kappa} A_{\eta_\xi}).$$

Hence, $\bigcap_{\xi < \kappa} A_{\eta_\xi} \in U$.

By the second and the third assumptions of the lemma, $\kappa_1 \in k(A^1_{\eta^1_\xi})$ and $\kappa_1 \in k(B^1_{\eta^1_\xi})$, for every $\xi < \kappa_1$.
Apply k and continue the inductive definition of the sequence $\langle \eta^1_\xi \mid \xi < \kappa_1 \rangle$ in M_W. Let $\langle \eta^2_\xi \mid \xi < \kappa_2 \rangle$ be the resulting sequence. Then for every $\xi, \kappa_1 \leq \xi < \kappa_2$ we will have $A^2_{\eta^2_\xi} \cap \xi + 1 = A^2_{j_W(\alpha^*)} \cap \xi + 1$ and $B^2_{\eta^2_\xi} \cap \xi + 1 = B^2_{j_W(\alpha^*)} \cap \xi + 1$,
where $\langle A^2_{\eta^1_\xi} \mid \xi < \kappa_2 \rangle = j_W(\langle A_{\eta_\xi} \mid \xi < \kappa \rangle$ and $\langle B^2_{\eta^1_\xi} \mid \xi < \kappa_2 \rangle = j_W(\langle B_{\eta_\xi} \mid \xi < \kappa \rangle$.
We have $A^2_{j_W(\alpha^*)} = j_W(A_{\alpha^*})$ and $A_{\alpha^*} \in W$. The same holds with B_{α^*}. Hence, $\kappa_1 \in j_W(A_{\alpha^*})$, and so, $\kappa_1 \in j_W(A_{\alpha^*}) \cap \xi + 1 = A^2_{\eta^2_\xi} \cap \xi + 1$.
So, $\kappa_1 \in A^2_{\eta^2_\xi}$, for every $\xi < \kappa_2$, and then,

$$\kappa_1 \in \bigcap_{\xi < \kappa_2} A^2_{\eta^2_\xi} = j_W(\bigcap_{\xi < \kappa} A_{\eta_\xi}).$$

The same is true with B's instead of A's.
Hence, $\bigcap_{\xi < \kappa} A_{\eta_\xi} \in W$ and $\bigcap_{\xi < \kappa} B_{\eta_\xi} \in W$.
□

The next lemma is a slight generalization of 1.5.

Lemma 1.6 *Suppose that* $\{A_{\alpha n} \mid \alpha < \kappa^+, n < n^*\}, \{B_{\alpha,m} \mid \alpha < \kappa^+, m < m^*\} \subseteq W$*, for some* $n^*, m^* < \omega$*, are such that, for every* $n < n^*, m < m^*$*,*

1. $\{A_{\alpha n} \mid \alpha < \kappa^+\} \subseteq U$*;*

2. *for every* $A \in j_U(\{A_{\alpha n} \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(A)$;

3. *for every* $B \in j_U(\{B_{\alpha m} \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(B)$.

Then there is $I \subseteq \kappa^+, |I| = \kappa$ *such that, for every* $n < n^*, m < m^*$,

1. $\bigcap_{\alpha \in I} A_{\alpha n} \in U \cap W$,

2. $\bigcap_{\alpha \in I} B_{\alpha m} \in W$.

Proof. Similar to those of 1.5 only define $H_{\alpha\xi}$ as follows:

$$H_{\alpha\xi} = \{\beta < \kappa^+ \mid \forall n < n^*(A_{\alpha n} \cap \xi = A_{\beta n} \cap \xi) \text{ and } \forall m < m^*(B_{\alpha m} \cap \xi = B_{\beta m} \cap \xi)\},$$

for every $\alpha < \kappa^+, \xi < \kappa$.
□

The next lemma follows from Lemma 1.5.

Lemma 1.7 *Suppose that there are a family* $D \subseteq W$ *and a normal filter* $\mathcal{V} \subseteq W$ *such that*

1. *for every* $A \in W$ *there is* $B \in D$ *which is contained in* A *mod* $\mathcal{V}$,

2. *for every* $C \in j_U(D)$, $\kappa_1 \in k(C)$.

Then W *has the Galvin property.*

2 Construction

Assume GCH and let κ be a measurable cardinal. Let U be a normal ultrafilter over κ.

Define an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle.$$

Let $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal.
If $\beta < \kappa$ is an inaccessible cardinal then set $Q_\beta = Cohen(\beta)$.

Let G be generic subset of $P_{\kappa+1}$. The embedding $j_U : V \to M_U$ extends to $j^* : V[G] \to M_U[G^*]$ in a standard fashion.
Set

$$U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}.$$

Then

1. $U^* \supseteq U$,

2. $j_{U^*} = j^*$

3. $M_{U^*} = M_U[G^*]$.

We have $j_U(P) = P_{\kappa+1} * P_{(\kappa, j_U(\kappa)]}$.

Consider now $U \times U$. We have that Denote $j_U(\kappa)$ by κ_1 and $j_{U\times U}(\kappa) = j_{j_U(U)}(\kappa_1)$ by κ_2. Then $j_{j_U(U)} : M_U \to M_{U\times U}$ and κ_1 is its critical point.

Extend, in $V[G]$, $j_{U\times U}$ to $j^{**} : V[G] \to M_{U\times U}[G^{**}]$ as follows:

Set $G^{**} \cap P_{\kappa_1+1} = G^*$. Continue to define $G^{**} \cap P_{(\kappa_1, j_{U\times U}(\kappa))}$ in the standard fashion in order to insure that j^{**} is $j_{U^*\times U^*}$.
The main issue will be to define a Cohen function f_{κ_2}, where $\kappa_2 = j_{U\times U}(\kappa)$.
Set $f_{\kappa_2} \restriction \kappa_1 = f_{\kappa_1}$. Also, set $f_{\kappa_2}(\kappa_1) = \kappa$. This will insure U^* will be the normal ultrafilter Rudin-Keisler below the one which we will define.
Namely, define the continuation of f_{κ_2} arbitrary, but meeting the relevant dense sets.
Then, in $V[G]$, let

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j^{**}(X)\}.$$

Then W is a $\kappa-$complete ultrafilter over κ and $j_W = j^{**}$.
Also, $k = j_{j_U(U)} : M_U \to M_{U\times U}$ extends to $k^* : M_U[G^*] \to M_{U\times U}[G^{**}]$.

Lemma 2.1 $W \gneqq_{R-K} U^*$.

Proof. This follows since W includes Cub_κ and f_{κ_2} is a regressive function which is not constant mod W. Actually, it projects W to U^*.
□

The main issue thus will be to choose such W which is not a product, but still has a Galvin property $Gal(W, \kappa, \kappa^+)$.

Proceed as follows.
Fix in V an enumeration $\langle \underset{\sim}{D}_i \mid i < \kappa^+\rangle$ of all dense open subsets of $Cohen(\kappa_2)$ of $M_{U\times U}[G^{**} \cap j_{U\times U}(P_\kappa)]$.

We define a master condition sequence $\langle \underset{\sim}{s}_i \mid i < \kappa^+\rangle$ as follows:
if $i < \kappa^+$ and $\underset{\sim}{s}_i$ is defined, then let $\underset{\sim}{s}_{i+1}$ be an element of $\underset{\sim}{D}_{i+1}$ which is stronger than $\underset{\sim}{s}_i$ and $\mathrm{dom}(\underset{\sim}{s}_{i+1})$ is of the form $j_{U\times U}(h)(\kappa_1)$, for some $h : \kappa \to \kappa$, i.e. depends only on the second coordinate. Also require that it strictly includes those of $\underset{\sim}{s}_i$.

This is possible since for every $g : [\kappa]^2 \to \kappa$ there is $g' : \kappa \to \kappa$ such that for every $\alpha < \beta < \kappa, g(\alpha, \beta) < g'(\beta)$. Just define $g'(\beta) = \bigcup_{\alpha<\beta} g(\alpha, \beta) + 1$.

If i is a limit ordinal of cofinality $< \kappa$, then set $\underset{\sim}{s}'_i = \bigcup_{i'<i} \underset{\sim}{s}_{i'}$ and then let $\underset{\sim}{s}_i$ be an element of $\underset{\sim}{D}_i$ which is stronger than $\underset{\sim}{s}'_i$ and $\operatorname{dom}(\underset{\sim}{s}_i)$ is of the form $j_{U\times U}(h)(\kappa_1)$, for some $h : \kappa \to \kappa$, i.e. depends only on the second coordinate.

Finally, let us deal with the main case when i is a limit ordinal of cofinality κ.

We set first $\underset{\sim}{s}'_i = \bigcup_{i'<i} \underset{\sim}{s}_{i'}$.

For every $\alpha < i$, pick a function $t_\alpha : \kappa\times\kappa \to V_\kappa, t_\alpha(\mu, \nu) \in Cohen(\kappa)$, in V, which represents $\underset{\sim}{s}_\alpha$ in the ultrapower $M_{U\times U}$. Also, we can assume that $\operatorname{dom}(t_\alpha)(\mu, \nu)$ depends only on ν.

We have, for every $\alpha < \beta < i$,

$$\{(\mu, \nu) \in [\kappa]^2 \mid t_{\alpha+1}(\mu, \nu) \restriction \operatorname{dom}(t_\alpha(\mu, \nu)) = t_\alpha(\mu, \nu),$$

$$\operatorname{dom}(t_{\alpha+1}(\mu, \nu)) > \operatorname{dom}(t_\alpha(\mu, \nu))\} \in U \times U,$$

and so,

$$\{\mu < \kappa \mid \{\nu < \kappa \mid t_{\alpha+1}(\mu, \nu) \restriction \operatorname{dom}(t_\alpha(\mu, \nu)) = t_\alpha(\mu, \nu),$$

$$\operatorname{dom}(t_{\alpha+1}(\mu, \nu)) > \operatorname{dom}(t_\alpha(\mu, \nu))\} \in U\} \in U.$$

Using the dependence on the second coordinate only, we may assume that for every $\mu < \kappa$,

$$\{\nu < \kappa \mid t_{\alpha+1}(\mu, \nu) \restriction \operatorname{dom}(t_\alpha(\mu, \nu)) = t_\alpha(\mu, \nu), \operatorname{dom}(t_{\alpha+1}(\mu, \nu)) > \operatorname{dom}(t_\alpha(\mu, \nu))\} \in U.$$

Set $\langle t^1_\alpha \mid \alpha < \kappa_1\rangle = j_U(\langle t_\alpha \mid \alpha < \kappa\rangle)$. Then, in M_U, for every $\mu < \kappa_1$,

$$\{\nu < \kappa_1 \mid t^1_{\alpha+1} \restriction \operatorname{dom}(t^1_\alpha(\mu, \nu)) = t^1_\alpha(\mu, \nu), \operatorname{dom}(t^1_{\alpha+1}(\mu, \nu)) > \operatorname{dom}(t^1_\alpha(\mu, \nu))\} \in j_U(U).$$

In particular, for $\mu = \kappa$,

$$\{\nu < \kappa_1 \mid t^1_{\alpha+1} \restriction \operatorname{dom}(t^1_\alpha(\kappa, \nu)) = t^1_\alpha(\kappa, \nu), \operatorname{dom}(t^1_{\alpha+1}(\kappa, \nu)) > \operatorname{dom}(t^1_\alpha(\kappa, \nu))\} \in j_U(U).$$

Apply k and move to $M_{U\times U}$. Let $\langle t^2_\alpha \mid \alpha < \kappa_2\rangle = k(\langle t^1_\alpha \mid \alpha < \kappa_1\rangle)$. Then, in $M_{U\times U}$,

$$t^2_{\alpha+1}(\kappa, \kappa_1) \restriction \operatorname{dom}(t^2_\alpha(\kappa, \kappa_1)) = t^2_\alpha(\kappa, \kappa_1), \operatorname{dom}(t^2_{\alpha+1}(\kappa, \kappa_1)) > \operatorname{dom}(t^2_\alpha(\kappa, \kappa_1)).$$

Consider $\langle t^2_\alpha \mid \alpha < \kappa_1\rangle$. They are compatible. Take an upper bound for them in D_i and set it to be s_i.

The above construction allows to satisfy conditions of Lemma 1.5. Thus define

$$B_\alpha = \{(\mu, \nu) \in [\kappa]^2 \mid t_\alpha(\mu, \nu) \in G \cap Cohen(\kappa)\}.$$

Set $\langle B^1_\alpha \mid \alpha < \kappa_1\rangle = j^*(\langle B_\alpha \mid \alpha < \kappa\rangle)$.
Then, by elementarity of j^*,

$$B^1_\alpha = \{(\mu,\nu) \in [\kappa_1]^2 \mid t^1_\alpha(\mu,\nu) \in G^* \cap Cohen(\kappa_1)\}.$$

Then $(\kappa,\kappa_1) \in k(B^1_\alpha)$, for every $\alpha < \kappa_1$.

This completes the definition of the master condition sequence, and so, G^{**} and f_{κ_2}. Define W using it in the usual fashion.

Remember that W not supposed be a sum. However, we think that W as defined above is indeed a sum.
Let assume that f_{κ_2}, and so, G^{**} are in $M_U[G^*]$.
We redefine f_{κ_2} in order to prevent this, but still to keep the Galvin property.

Do the following:

at each limit stage i of cofinality κ such that i is of the form $\delta + \kappa$, for some $\delta \geq \kappa$, we replace the values on $\mathrm{dom}(s_i) \setminus \bigcup_{i'<i} \mathrm{dom}(s_{i'})$ from 1 to 0 and 0 to 1.

The rest is kept unchanged, only in the previous construction of s_i's we take care that such switches between 0's and 1's still keep conditions in the corresponding dense sets from the list.

Denote the resulting Cohen function by $\tilde{f}_{\kappa_2}$.

Lemma 2.2 $\tilde{f}_{\kappa_2}$ *cannot be in* $M_U[G^*]$.

Proof. Otherwise, compare $\tilde{f}_{\kappa_2}$ with f_{κ_2}. It will decode a cofinal in κ_2 sequence of order type κ^+, which is impossible since the cofinality of κ_2 in M_U is $\kappa_1^+ > \kappa^+$.
□

Let $G^{***} = (G^{**} \cap P_{\kappa_2}) * \tilde{f}_{\kappa_2}$. Define $\tilde{W}$ using G^{***}.

We would like now to argue that $\tilde{W}$ satisfies the conditions of Lemma 1.5, and so the Galvin property.

Let us specify relevant subsets of $\tilde{W}$.

First we deal with elements of P_{κ_2}.
For every $r \in P_{\kappa_2}$ pick a function $h_r : [\kappa]^2 \to P_\kappa$ (in V) which represents r mod $U \times U$, i.e., $r = (j_{U\times U}(h_r))(\kappa,\kappa_1)$. Set

$$C_r = \{(\mu,\nu) \in [\kappa]^2 \mid h_r(\mu,\nu) \in G \cap P_\kappa\}.$$

Lemma 2.3 $j_{\tilde{W}} \restriction V[G \cap P_\kappa] = j_{U^*\times U^*} \restriction V[G \cap P_\kappa]$ *and* $k^* \restriction M_U[G^* \cap P_{\kappa_1}] = k_{\tilde{W}} \restriction M_U[G^* \cap P_{\kappa_1}]$*, where* $k_{\tilde{W}} : M_{U^*} \to M_{\tilde{W}}$ *is the canonical embedding.*

Proof. This holds, since the ultrapowers $M_{U\times U}[G^{**}]$ by $U^*\times U^*$ and $M_{U\times U}[G^{**}\cap P_{\kappa_2},\tilde{f}_{\kappa_2}]$ by $\tilde{W}$ agree about generic set up to the final step, i.e. where the Cohen function is added to κ_2.
□

Lemma 2.4 $U^*\cap V[G\cap P_\kappa]=\tilde{W}\cap V[G\cap P_\kappa]$.

Proof. $A\in U^*\cap V[G\cap P_\kappa]$ iff $j_{U^*}(A)\in j_{U^*}\restriction M_U[G^*\cap P_{\kappa_1}]$ iff $\kappa_1\in k^*\restriction M_U[G^*\cap P_{\kappa_1}](j_{U^*}(A))$ iff $\kappa_1\in j_{U^*\times U^*}\restriction V[G\cap P_\kappa](A)$. By the previous lemma this is the same as $\kappa_1\in j_{\tilde{W}}\restriction V[G\cap P_\kappa](A)$. So we are done.
□

The next lemma follows from Lemma 2.4:

Lemma 2.5 *If $A\in\tilde{W}\cap V[G\cap P_\kappa]$, then both κ and κ_1 are in $j_{\tilde{W}}(A)$.*

Lemma 2.6 *If $X\in j_{U^*}(U^*\cap V[G\cap P_\kappa])$, then κ_1 is in $k_{\tilde{W}}(X)$.*

Proof. By elementarity, $X\in j_{U^*}(U^*)\cap M_U[G^*\cap P_{\kappa_1}]$. Then, $\kappa_1\in k^*(X)$, since $X\in j_{U^*}(U^*)$. By Lemma 2.3, $k^*(X)=k_{\tilde{W}}(X)$, since $X\in M_U[G^*\cap P_{\kappa_1}]$.
□

Lemma 2.7 *$\tilde{W}$ satisfies the Galvin property.*

Proof. Let $A\in\tilde{W}$ and $\underset{\sim}{A}$ be a name of it. Consider $x=||\kappa_1\in j_{U\times U}(\underset{\sim}{A}_\alpha)||$. By the definition of $\tilde{W}$, there are some $\langle r,s_\alpha\rangle\in(G^{**}\cap P_{\kappa_2})*Cohen(\kappa_2)$ (in $M_{U\times U}$) which are stronger than x (in the forcing sense, or alternatively, less than x in the corresponding Boolean algebra), where $\alpha<\kappa^+$ and s_α is from the master condition sequence.
Split r into $\langle r_1,r_2\rangle$, where $r_1\in P_{\kappa_1+1},r_2\in P_{\kappa_2}/P_{\kappa_1+1}$.
Pick in V functions h_{r_1} and h_{r_2} which represent r_1 and r_2 in the ultrapower.
We can assume that h_{r_1} is a function of the first coordinate only and, using $>\kappa_1$ completeness, h_{r_2} is a function of the second coordinate only.
Strengthening if necessary, pick some $i(\alpha)<\kappa^+$ such that

1. $i(\alpha)>\alpha$,

2. $\mathrm{cof}(i(\alpha))=\kappa$,

3. $i(\alpha)$ is not of the form $\delta+\kappa$, for some $\delta\geq\kappa$,

Then, in particular, $s_{i(\alpha)} \geq s_\alpha$.
Let $B_{i(\alpha)}$ denotes the set in $\tilde{W}$ defined by $s_{i(\alpha)}$, i.e.

$$B_{i(\alpha)} = \{(\mu, \nu) \in [\kappa]^2 \mid t_\alpha(\mu, \nu) \in G \cap Cohen(\kappa)\}.$$

Let

$$E_{r_1} = \{\mu < \kappa \mid h_{r_1}(\mu) \in G \cap P_\kappa\}$$

and

$$E_{r_2} = \{\nu < \kappa \mid h_{r_2}(\nu) \in G \cap P_{(\nu,\kappa)}\}.$$

Now, there is a set $C'_A \in U \times U$ such that
if $(\mu, \nu) \in C'_A \cap B_{i(\alpha)}, \mu \in E_{r_1}, \nu \in E_{r_2}$, then $\nu \in A$.
Note that U is a normal ultrafilter in V, so C'_A can be picked to be of the form $[C_A]^2$, for some $C_A \in U$.
Shrink $B_{i(\alpha)}$ to the following set in $\tilde{W}$:

$$B'_{i(\alpha)} = \{\nu < \kappa \mid (f_\kappa(\nu), \nu) \in B_{i(\alpha)}, \}.$$

Set

$$F_A = \{\nu < \kappa \mid f_\kappa(\nu) \in C_A \cap E_{r_1}\}.$$

Clearly, both $B'_{i(\alpha)}$ and F_A are in $\tilde{W}$.
Note that $U \subseteq \tilde{W}$ and $E_{r_2} \in \tilde{W}$. Hence, $C_A \cap B'_{i(\alpha)} \cap F_A \cap E_{r_2} \in \tilde{W}$. So,
if $\nu \in C_A \cap B'_{i(\alpha)} \cap F_A \cap E_{r_2}$, then $\nu \in A$.

We specified sets $C_A, E_{r_1}, E_{r_2}, B_{i(\alpha)}$ for every $A \in \tilde{W}$.
Note that $C_A, E_{r_1}, E_{r_2} \in U^* \cap \tilde{W}$, and so, by Lemma 2.4, $\kappa, \kappa_1 \in j_{\tilde{W}}(C_A)$ and $\kappa, \kappa_1 \in j_{\tilde{W}}(E_{r_1})$.
Denote E_{r_1} by E_{A1}, E_{r_2} by E_{A2} and $B_{i(\alpha)'}$ by B_A.

Now, we are ready to show the Galvin property of $\tilde{W}$.
Let $\{A_\gamma \mid \gamma < \kappa^+\} \subseteq \tilde{W}$. For every $\gamma < \kappa^+$, we pick $C_{A_\gamma}, E_{A_\gamma 1}, E_{A_\gamma 2}, B_{A_\gamma}$, as above. Apply Lemmas 1.5, 1.6 to the families $\{C_{A_\gamma} \mid \gamma < \kappa^+\}, \{E_{A_\gamma 1} \mid \gamma < \kappa^+\}, \{E_{A_\gamma 2} \mid \gamma < \kappa^+\}$ and $\{B_{A_\gamma} \mid \gamma < \kappa^+\}$.
Then there will be $I \subseteq \kappa^+, |I| = \kappa$ such that

1. $\bigcap_{\gamma \in I} C_{A_\gamma} \in U \cap W$,
2. $\bigcap_{\gamma \in I} E_{A_\gamma 1} \in U \cap W$,

3. $\bigcap_{\gamma\in I} E_{A_\gamma 2} \in U \cap W$,

4. $\bigcap_{\gamma\in I} B_{A_\gamma} \in W$.

Set

$$F = \{\nu < \kappa \mid f_\kappa(\nu) \in \bigcap_{\gamma\in I} C_{A_\gamma} \cap \bigcap_{\gamma\in I} E_{A_\gamma 1}\}.$$

Then for every $\alpha \in I$,
if $\nu \in \bigcap_{\gamma\in I} C_{A_\gamma} \cap \bigcap_{\gamma\in I} B_{A_\gamma} \cap F \cap \bigcap_{\gamma\in I} E_{A_\gamma 2}$, then $\nu \in A_\alpha$.
We have $\bigcap_{\gamma\in I} C_{A_\gamma} \cap \bigcap_{\gamma\in I} B_{A_\gamma} \cap F \cap \bigcap_{\gamma\in I} E_{A_\gamma 2} \in \tilde{W}$, so this completes the proof.
□

Remark 2.8 The idea used in the construction above works for variety of other forcing notions. The crucial point was a domination of functions $h(x, y)$ by functions $g(y)$ of the second variable.

3 Additional examples of non-Galvin ultrafilters

We show here that basic forcings over a measurable κ which preserve measurability, add non-Galvin ultrafilters extending Cub_κ.

Assume GCH and let κ be a measurable cardinal. Let U be a normal ultrafilter over κ. We will deal with $j_U : V \to M_U, j_{U\times U} : V \to M_{U\times U}, j_{j_U(U)} : M_U \to M_{j_U(U)} = M_{U\times U}$. Denote j_U by j_1, M_U by M_1, $j_U(\kappa)$ by κ_1, $j_{U\times U}$ by j_2, $M_{U\times U}$ by M_2, $j_{U\times U}(\kappa) = \kappa_2$ and $j_{j_U(U)}$ by k.

Let

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle$$

be an Easton support iteration of Cohen forcings $Cohen(\beta)$ which add a Cohen function $g_\beta : \beta \to 2$ to every regular $\beta \leq \kappa$. Let G be a generic subset of $P_{\kappa+1}$.

Then the embeddings j_1, j_2, k extend to $j_1^* : V[G] \to M_1[G_1], j_2^* : V[G] \to M_2[G_2]$,
$k^* : M_1[G_1] \to M_2[G_2]$.

Fix, in V, an increasing cofinal in κ_1 sequence $\langle \eta_\alpha \mid \alpha < \kappa^+\rangle$ and a sequence of functions $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ from κ to κ such that $[f_\alpha]_U = \eta_\alpha$, for every $\alpha < \kappa^+$.
Now, in $V[G]$, for every $\alpha < \kappa^+$, define

$$A_\alpha = \{\nu < \kappa \mid g_\kappa(f_\alpha(\nu)) = 1\}.$$

Now we would like to define a $\kappa-$complete ultrafilter W over κ which extends U, Cub_κ and such that the sets $\{A_\alpha \mid \alpha < \kappa^+\}$ witness that W is not Galvin.
The argument will be very similar to those of 2.6 of [2].

First we change g_{κ_1} by setting the values on each η_α to 1. Let g'_{κ_1} be the resulting function. Then the choice of η_α's insure that g'_{κ_1} is still generic over $M_1[G_1 \cap P_{\kappa_1}]$. Denote $G'_1 = G_1 \cap P_{\kappa_1} * g'_{\kappa_1}$ and let $j'_1 : V[G] \to M_1[G'_1]$ be the corresponding embedding.
We will have as a result that $\kappa \in j'_1(A_\alpha)$, for every $\alpha < \kappa^+$.

Apply now k and move to M_2. k extends naturally to $k' : M_1[G'_1] \to M_2[G'_2]$.

Let us change same values of g'_{κ_2}.
Let

$$\langle \eta^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle = j_1(\langle \eta_\gamma \mid \gamma < \kappa^+ \rangle).$$

Then by elementarity, $\langle \eta^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle$ will be a cofinal sequence in κ_2 in M_1.
Let

$$\langle f^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle = j_1(\langle f_\gamma \mid \gamma < \kappa^+ \rangle).$$

Then, f^1_γ will represent, mod $j_1(U)$, η^1_γ in M_1.
Set

$$\langle f^2_\gamma \mid \gamma < j_2(\kappa^+) \rangle = j_2(\langle f_\gamma \mid \gamma < \kappa^+ \rangle) = k(\langle f^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle).$$

Then, whenever $\gamma < \delta < j_1(\kappa^+)$,

$$f^2_{k(\gamma)}(\kappa_1) = k(f^1_\gamma)(\kappa_1) = \eta^1_\gamma < \eta^1_\delta = k(f^1_\delta)(\kappa_1) = f^2_{k(\delta)}(\kappa_1).$$

We change the value of $g'_{\kappa_2}(f^2_{j_2(\alpha)}(\kappa_1))$ to 1 , for every $\alpha < \kappa^+$. In addition, change $g'_{\kappa_2}(f^2_{k(\gamma)}(\kappa_1))$ to 0 , for every $\gamma \in j_1(\kappa^+) \setminus j''_1\kappa^+$.
Let $g^*_{\kappa_2}$ denotes the resulting function. As in 2.6 of [2], g'_{κ_1} is still generic over $M_2[G'_2 \cap P_{\kappa_2}]$. Denote $G^*_2 = G_2 \cap P_{\kappa_2} * g^*_{\kappa_2}$ and let $j^*_2 : V[G] \to M_2[G^*_2], k^* : M_1[G'_1] \to M_2[G^*_2]$ be the corresponding embeddings.
We will have as a result that $\kappa_1 \in j^*_2(A_\alpha)$, for every $\alpha < \kappa^+$ and $\kappa_1 \notin k^*(A^1_\gamma)$, for every $\gamma \in j_1(\kappa^+) \setminus j''_1\kappa^+$, where $\langle A^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle = j'_1(\langle A_\alpha \mid \alpha < \kappa^+ \rangle)$.
Thius hold, since by elementarity,

$$j^*_2(A_\alpha) = \{\nu < \kappa_2 \mid g^*_{\kappa_2}(f^2_{j_2(\alpha)}(\nu)) = 1\},$$

for every $\alpha < \kappa^+$ and

$$k^*(A^1_\gamma) = \{\nu < \kappa_2 \mid g^*_{\kappa_2}(f^2_{k(\gamma)}(\nu)) = 1\},$$

for every $\gamma \in j_1(\kappa^+)$.

Set

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j_2^*(X)\}$$

and

$$U^* = \{X \subseteq \kappa \mid \kappa \in j_2^*(X)\}.$$

Then $W >_{R-K} U^*$ both extend U, $W \supseteq Cub_\kappa$ and U^* is normal. Moreover, W is non-Galvin witnessed by $\{A_\alpha \mid \alpha < \kappa^+\} \subseteq U^*$.

Similar constructions can be used with iterations of other forcing notions. What is needed is possibilities to extend the elementary embeddings j_1, j_2, k and $\beta-$closure of iterants Q_β.

References

[1] T. Benhamou, Saturation properties of ultrafilters in canonical inner models.

[2] T. Benhamou and M. Gitik, On Cohen and Prikry forcing notions,