On countably closed mutually embeddable models

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Abstract

We show that a measurable cardinal is enough in order to construct two distinct countably closed mutually embeddable models. This answers a question from [1].

1 Introduction

Let M, N be two transitive models of ZFC. They called mutually embeddable iff there are elementary embeddings $j : M \to N$ and $i : N \to M$. Existence of this type of models was studied by Eskew, Friedman, Hayut and Schlutzenberg [1]. They asked the following question ([1], Question 2):

What is the consistency strength of the statement that there are two distinct countably closed mutually embeddable models?

It was shown in [1] that a $\mu-$measurable cardinal is enough for this.

The purpose of this note is to reduce the strength to a single measurable, and so, to obtain the equiconsistency.
Namely, we will show the following:

Theorem 1.1 *Assume GCH. Let κ be a measurable cardinal. Then in a generic cardinal preserving extension there are two distinct $\kappa-$closed mutually embeddable models.*

The ideas of the construction go back to [2].

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2 Construction

Assume GCH. Let κ be a measurable cardinal. Fix a normal ultrafilter U over κ.
Consider $j_U : V \to M_U \simeq \mathrm{Ult}(V,U), j_{U^2} : V \to M_{U^2} \simeq \mathrm{Ult}(V,U^2), j_{U^3} : V \to M_{U^3} \simeq \mathrm{Ult}(V,U^3)$.
Let $\kappa_0 = \kappa, \kappa_1 = j_U(\kappa), \kappa_2 = j_{U^2}(\kappa)$ and $\kappa_3 = j_{U^3}(\kappa)$. Also, consider $k : M_{U^2} \to M_{U^3}$ defined by setting $k(j_{U^2}(f)(\kappa,\kappa_1)) = j_{U^3}(f)(\kappa,\kappa_2)$. Then $\mathrm{crit}(k) = \kappa_1$ and $k(\kappa_1) = \kappa_2$.
We will further extend U^2, U^3, their embeddings and k in a special way.

Define an Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa\rangle$.
Let Q_β be trivial unless β is an inaccessible in V^{P_β} and if this is the case,
then let Q_β be the Cohen forcing adding a Cohen function from β to β, i.e.,

$$Q_\beta = \{h \mid h \text{ is a partial function from } \beta \text{ to } \beta, |h| < \beta\}.$$

Let $G \subseteq P_{\kappa+1}$ generic. Denote by f_β the Cohen function added by G at β, for every inaccessible $\beta \leq \kappa$.

We first extend U^2 and j_2 to $V[G]$.
Proceed as follows. Work in $V[G]$. Pick some $G_2 \subseteq j_2(P_{\kappa+1})$ such that

- G_2 is an $M_{U^2}-$generic subset of $j_2(P_{\kappa+1})$,
- $G_2 \cap P_{\kappa+1} = G$,
- $f_{\kappa_2} \restriction \kappa = f_\kappa$,
- $f_{\kappa_2}(\kappa) = \kappa_1$.

Extend, using such G_2, j_{U^2} to an elementary embedding $j_2^* : V[G] \to M_{U^2}[G_2]$.
Clearly,

$$U_2^* = \{X \subseteq [\kappa]^2 \mid (\kappa,\kappa_1) \in j_2^*(X)\} \supseteq U^2.$$

However, note that in contrast to U^2, U_2^* is isomorphic to a normal ultrafilter over κ which extends U. Namely, a typical element x of $M_{U^2}[G_2]$ is of the form $j_2^*(g)(\kappa,\kappa_1)$, but κ_1 itself can be represented by the function f_κ. Then $x = j_2^*(g)(\kappa, j_2^*(f_\kappa)(\kappa))$.

Now, turn to M_{U^3}. We will build an $M_{U^3}-$generic subset G_3 of $j_{U^3}(P_{\kappa+1})$ in $V[G]$ in a special way. First let $G_3 \cap P_{\kappa_2} = G_2 \cap P_{\kappa_2}$. Then let $f_{3,\kappa_2} : \kappa_2 \to \kappa_2$ be a Cohen generic over $M_{U^3}[G_3 \cap P_{\kappa_2}]$ such that

1. $f_{3,\kappa_2} \restriction \kappa_1 = f_{2,\kappa_1}$, where f_{2,κ_1} is the Cohen function from κ_1 to κ_1 of G_2,

2. for every $h : \kappa_2 \to \kappa_2$ which belongs to $M_{U^2}[G_2]$, $f_{3,\kappa_2} \neq h$.

Note that the total number of functions in $M_{U^2}[G_2]$ from κ_2 to κ_2 is κ^+. Hence it is easy to satisfy (2).
The embedding $k : M_{U^2} \to M_{U^3}$ extends to $k_* : M_{U^2}[G_2 \cap P_{\kappa_1+1}] \to M_{U^3}[(G_2 \cap P_{\kappa_2} * f_{3,\kappa_2}]$.
Let $R = P_{\kappa_3+1}/G_2 \cap P_{\kappa_2} * f_{3,\kappa_2}$ be the rest of the forcing over $M_{U^3}[(G_2 \cap P_{\kappa_2} * f_{3,\kappa_2}]$. Note that it is $\kappa_2^{++}-$closed forcing in $M_{U^3}[(G_2 \cap P_{\kappa_2} * f_{3,\kappa_2}]$.
Now, by standard means, k_*"$G_2 \cap P \restriction (\kappa_1, \kappa_2]$ generates $M_{U^3}[G_2 \cap P_{\kappa_2} * f_{3,\kappa_2}]-$generic subset H of R.
Let $G_3 = G_2 \cap P_{\kappa_2} * f_{3,\kappa_2} * H$.
Then k_* extends to $k^* : M_{U^2}[G_2] \to M_{U^3}[G_3]$ and j_{U^3} extends to $j_3^* : V[G] \to M_{U^3}[G_3]$, since $f_{3,\kappa_3} \restriction \kappa = f_{2,\kappa_2} \restriction \kappa = f_\kappa$.
Clealy,

$$U_3^* = \{X \subseteq [\kappa]^3 \mid (\kappa, \kappa_1, \kappa_2) \in j_3^*(X)\} \supseteq U^3.$$

Also its ultrapower will be $M_{U^3}[G_3]$.
The only generators of U_3^* are κ and κ_1, since in $M_{U^2}[G_2]$ we have $f_{2,\kappa_2}(\kappa) = \kappa_1$, k^* moves κ_1 to κ_2, f_{2,κ_2} to f_{3,κ_3} and κ does not move.

Note that $\mathcal{P}(\kappa_2)^{M_{U^2}[G_2]}$ differs from $\mathcal{P}(\kappa_2)^{M_{U^3}[G_3]}$, since, f_{3,κ_2} is not in $M_{U^2}[G_2]$.[1]

Now, we proceed as in Proposition 19 of [1]. Iterate $V[G]$ using U_3^* $\omega-$many times. The sequence $P = \langle (\kappa, \kappa_1), (\kappa_3, \kappa_4), (\kappa_6, \kappa_7), ... \rangle$ of the images of the generators of U_3^* will be a Prikry sequence over the final model M_ω of the iteration. The model $M_\omega[P]$ will be closed under $\kappa-$sequences. Also, $M_\omega[P] = M_\omega[j_3^*(P)]$, and hence, $j_3^* \restriction M_\omega[P] : M_\omega[P] \to M_\omega[P]$.

Now, we do the same, but over $M_{U^2}[G_2]$, i.e., we start with $M_{U^2}[G_2]$ (the ultrapower of $V[G]$ by U_2^*) and iterate the image $j_2^*(U_3^*)$ $\omega-$many times.
The sequence $P' = \langle (\kappa_2, \kappa_3), (\kappa_5, \kappa_6), (\kappa_7, \kappa_8), ... \rangle$ of the images of the generators of $j_2^*(U_3^*)$ will be a Prikry sequence over the final model N_ω of the iteration, by elementarity. Also, $N_\omega[P']$ will be closed under κ sequences in $M_{U^2}[G_2]$, and so in $V[G]$, since $M_{U^2}[G_2]$ is such.
Again, by elementarity,

$$j_2^* \restriction M_\omega[P] : M_\omega[P] \to N_\omega[P'].$$

Note that $M_\omega[P] \neq N_\omega[P']$, since, first - as it was observed above, $\mathcal{P}(\kappa_2)^{M_{U^2}[G_2]}$ differs from $\mathcal{P}(\kappa_2)^{M_{U^3}[G_3]}$,

[1]Using a further forcing, it is possible to get a difference already at the level of κ^+.

and second - $\mathcal{P}(\kappa_2)^{M_{U^2}[G_2]} = \mathcal{P}(\kappa_2)^{N_\omega[P']}, \mathcal{P}(\kappa_2)^{M_{U^3}[G_3]} = \mathcal{P}(\kappa_2)^{M_\omega[P]}$, as the critical points of the corresponding further iterations are above κ_2.

Finally, let us use the elementarity of $k^* : M_{U^2}[G_2] \to M_{U^3}[G_3]$ in order to conclude the argument. Thus,

$$k^* \restriction N_\omega[P'] : N_\omega[P'] \to M_\omega[P].$$

3 Strength

Suppose that M, N are two distinct countably closed mutually embeddible inner models. Then there is $j : M \to M$ which is not the identity. Let $\kappa = \text{crit}(j)$. Consider

$$U = \{X \in \mathcal{P}(\kappa)^M \mid \kappa \in j(X)\}.$$

Then U is an $M-$ultrafilter which is $\sigma-$complete (in V), and hence iterable.
Suppose that the core model K of M does not have a measurable cardinal.
Apply U to K $\omega-$many times. Then K will be moved to itself. Let κ_ω be the image of κ in such iteration. By elementarity, κ_ω will be a regular cardinal in K. However, the sequence $\langle \kappa_n \mid n < \omega \rangle$ will be in M, as M is countably closed.
Hence, κ_ω will be a regular cardinal of K which changed its cofinality in M. So, by the Dodd-Jensen Covering Lemma, there must be a measurable cardinal.

References

[1] M. Eskew, Sy-David Friedman, Y. Hayut, F. Schlutzenberg, Mutually embeddable models of ZFC,

[2] M. Gitik, On Non-Minimal p-Points Over a Measurable Cardinal. Annals of Math. Logic 20 (1981), 269-288.