# An application of the Silver theorem on decomposability

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Our aim is to prove the following:

**Theorem 0.1** *Suppose that $\aleph_\omega$ is a strong limit. Let $U$ be a uniform ultrafilter over a cardinal $\eta > \aleph_\omega$. Suppose that for some $n^* < \omega$, $U$ is $\aleph_n-$indecomposable, for all $\aleph_n \in [\aleph_{n^*}, 2^{\aleph_{n^*}}]$.*
*Let $K^U$ be a subset of $\eta$ which consists of regular cardinals $\rho$ such that*

1. $\sup(j_U{}''\rho)$ *exists.*
   *Note that $M_U$ is not well-founded, so it need not be the case always.*

2. $\sup(j_U{}''\rho) < j_U(\rho)$.
   *This means that $U$ is $\rho-$decompossible, i.e. $U_\rho = \{X \subseteq \rho \mid \sup(j_U{}''\rho) \in j_U(X)\}$ is a uniform ultrafilter over $\rho$ which is Rudin-Keisler below $U$.*

3. $M_U \models \mathrm{cof}(\sup(j_U{}''\rho)) < j_U(\aleph_\omega)$.
   *Equivaletly, $U_\rho$ concentrates on ordinals of cofinality less than $\aleph_\omega$.*

*Then $|K^U| < (2^{\omega_{n^*-1}})^+$. In particular, if $n^* = 1$, then $|K^U| < (2^\omega)^+$.*

**Remark 0.2** Note that by Kunen-Prikry theorem [3], $U$ is $\aleph_n-$indecompossible for every $n, n^* \leq n < \omega$.

*Proof.* Suppose otherwise. Fix $\langle \rho_i \mid i < (2^{\omega_{n^*-1}})^+\rangle$ an increasing sequence of consisting of elements of $K^U$.
Then by the theorem of Silver, see [2], there is an ultrafilter $D$ over some $\aleph_m, m < n^*$ such that $j_D(\omega) = j_U(\omega)$. Note that $j_D(\omega)$ is the first infinite cardinal in sense of $M_U$.
Denote it further by $\tilde{\omega}$. Its real cardinality (i.e. the cardinality of the set $\tilde{\omega}$ in $V$ is $\leq 2^{\aleph_m} < \aleph_\omega$. Denote it by $\delta$.

Consider $j_U(\aleph_\omega)$. By elementarity, $M_U \models j_U(\aleph_\omega) = \aleph_{\tilde{\omega}}$.
Then the number in $V$ of $M_U-$cardinals below $\aleph_{\tilde{\omega}}$ is $\delta$. We have

$$M_U \models \mathrm{cof}(\sup(j_U{}''\rho_i)) < \aleph_{\tilde{\omega}},$$

$i < (2^\omega)^+$. Hence, there will be $i < i' < (2^\omega)^+$, such that

$$M_U \models \mathrm{cof}(\sup(j_U{}''\rho_i)) = \mathrm{cof}(\sup(j_U{}''\rho_{i'})).$$

Pick then in $M_U$ a function $f$ such that

$$M_U \models f \text{ is an increasing function which maps a cofinal subset of } \sup(j_U{}''\rho_i)$$

$$\text{onto a cofinal subset of } \sup(j_U{}''\rho_{i'}).$$

Let us now define in $V$ an order preserving function $g$ from $\rho_{i'}$ to a subset of $\rho_i$. The existence of such function is clearly impossible and, so, will provide the desired contradiction.
Proceed by induction. Suppose that $\nu < \rho'$ and $g \restriction \nu$ is defined. By the inductive assumption, there is $\alpha_\nu < \rho$ such that $g''\nu \subseteq \alpha_\nu$.
There exists some $x_\nu$ such that

$$M_U \models j_U(\alpha_\nu) < x_\nu < \sup(j_U{}''\rho_i), x_\nu \in \mathrm{dom}(f), f(x_\nu) > j_U(\nu) \text{ and it is the least like this.}$$

Pick some $\beta_\nu, \alpha_\nu < \beta_\nu < \rho$ such that

$$M \models x_\nu < j_U(\beta_\nu).$$

Set $g(\nu) = \beta_\nu$.
This completes the construction of $g$, and so the proof of the theorem.
$\square$

**Theorem 0.3** *Indecomposable ultrafilters of Ben David -Magidor [1] satisfy the assumptions of 0.1.*

*Proof.* Let $U$ over $\mathcal{P}_\kappa(\lambda)$ be an indecomposable ultrafilter constructed as in Ben David - Magidor [1]. Note that the function $P \mapsto \sup(P)$ is one to one on a set in $U$, by Solovay, since $U$ extends a normal ultrafilter in the ground model.
Use the Prikry condition argument similar to [4] in order to show that for every function $f : \mathcal{P}_\kappa(\lambda) \to \aleph_{\omega+k}$ in $V[\langle \kappa_n \mid n < \omega\rangle, \langle F_n \mid n < \omega\rangle]$,
if $f(P) < \sup(P \cap \aleph_{\omega+k})$, then for some $\alpha < \aleph_{\omega+k}$ and $A \in U$, $f(P) < \alpha$, for all $\alpha \in A$.

# References


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