On $\kappa-$compact cardinals.

Moti Gitik*

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Abstract

We deal with some questions related to $\kappa-$compact cardinals.

1 Introduction

Definition 1.1 κ is κ-compact cardinal iff every κ complete filter over κ can be extended to a $\kappa-$complete ultrafilter over κ.

Clearly, if κ is 2^κ-supercompact or even 2^κ-strongly compact, then it is κ-compact.
In [7] W. Mitchell asked whether $o(\kappa) = \kappa^{++}$ is sufficient for model with a κ-compact cardinal. It was answered negatively in [1]. It was shown there that at least a strong cardinal is required. Here we will somewhat improve this result and also will address some related questions raised in [1].

2 An application to distributive forcing notion.

Let us argue first that κ-compact cardinal generates an extender suitable for Extender Based Prikry forcing.

Theorem 2.1 *Let κ be a κ-compact cardinal. Then there is an extender E over κ such that*

1. *${}^\kappa M_E \subseteq M$, where $i_E : V \to M_E \simeq Ult(V, E)$ is the corresponding elementary embedding,*
2. *every κ complete filter over κ can be extended to a $\kappa-$complete ultrafilter over κ of the form E_ξ, for some $\xi < i_E(\kappa)$, where $E_\xi = \{X \subseteq \kappa \mid \xi \in i_E(X)\}$.*

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Proof. Let $\lambda = (2^{2^\kappa})^+$. Let

$$\langle W_\alpha \mid \alpha < \lambda \rangle$$

be a list of all $\kappa-$complete non-principle ultrafilters over κ (with repetitions). Then for every $\kappa-$complete non-principle filter U over κ there is $\alpha < \lambda$ such that $U \subseteq W_\alpha$.
It is enough to construct an extender E with ultrapower closed under $\kappa-$sequences which has all W_α's among its measures.
The construction is similar to those from [1].
Denote by $\mathcal{P}_{\kappa^+}(\lambda)$ the set $\{a \subseteq \lambda \mid |a| \leq \kappa\}$. For every $\tau < \kappa^+$ let $f_\tau : \kappa \to \kappa$ be a canonical function representing τ. We shall define a $\kappa-$complete ultrafilter U_a concentrating over the set $X_a = \{\langle \alpha_\nu \mid \nu < f_{otp(a)}(\alpha_0)\rangle \mid \alpha_\nu < \kappa, \nu_1 < \nu_2 \to \alpha_{\nu_1} < \alpha_{\nu_2}\}$ for $a \in \mathcal{P}_{\kappa^+}(\lambda)$. Once $b \subseteq a$, U_b will be obtained from U_a, as follows. Let $\langle a_i \mid i < otp(a)\rangle$ be the increasing enumeration of a. Then for some increasing sequence $\langle i_j \mid j < otp(b)\rangle, b = \langle a_{i_j} \mid j < otp(b)\rangle$. Project U_a to the coordinates $\langle i_j \mid j < otp(b)\rangle$. Let π_{ab} be such a projection. Then U_b, will be the set of $\{\pi''_{ab}(X) \mid X \in U_a\}$. Let us turn to the definition of U_a's. Fix some enumeration $\langle a_\alpha \mid \alpha < \kappa^{++}\rangle$ of $\mathcal{P}_{\kappa^+}(\lambda)$. For every $\alpha < \kappa^{++}$, set $U_{\{\alpha\}} = W_\alpha$.
Suppose now that

1. for every $\beta < \alpha$, a $\kappa-$complete ultrafilter U_{a_β} is defined;

2. for every $b \in \mathcal{P}_{\kappa^+}(\lambda)$, if for some $\gamma < \alpha$, $b \subseteq a_\gamma$, then U_b is defined and it is the projection of U_{a_γ} by $\pi_{a_\gamma b}$.

Let us define U_{a_α}. The only nontrivial case is when there is no $\gamma < \alpha$ such that $a_\gamma \supseteq a_\alpha$. Define then first a $\kappa-$complete filter U concentrating over $X_{opt(a_\alpha)}$. Set $X \in U$ iff for some $\gamma < \alpha$, some $b \subseteq a_\alpha \cap a_\gamma$ there exists $X_b \in E_b$, such that $X = \pi^{-1}_{a_\alpha b}{}''X_b$. Using the inductive assumptions (1), (2) and the commutativity of the projection function π_{cd}, it is not hard to see that a so-defined U is a $\kappa-$complete filter. Let U_{a_α} be a $\kappa-$complete ultrafilter extending U. For every $b \subseteq a_\alpha$, if U_b is still not defined, define it to be the projection of U_{a_α} by $\pi_{a_\alpha b}$. This completes the construction of $\langle U_{a_\alpha} \mid \alpha < \lambda\rangle$.

Let N_a be the ultrapower of V by U_a and $i_a : V \to N_a$, the canonical embedding. The projection π_{ab} induces the elementary embedding $i_{ba} : N_b \to N_a$. $\langle N_a, i_{ab} \mid a \subseteq b, a, b \in \mathcal{P}_{\kappa^+}(\lambda)\rangle$ forms a directed system, where $N_\emptyset = V$ and $i_{\emptyset a} = i_a$. The direct limit of this system is well-founded and closed under $\kappa-$sequences. Let E be the derived extender. Then it is as desired.
□

Let us now use such an extender E to define a variation Extender Based Prikry forcing.

Theorem 2.2 *Assume GCH. Let κ be a $\kappa-$compact cardinal. Then there is a cardinal preserving extension in which for every $\kappa-$distributive forcing notion $Q \in V$ of cardinality κ there is a $V-$generic subset.*

Remark 2.3 Note that it is easy to obtain such generics once κ is a κ^+-strongly compact cardinal, but κ^+ is collapsed in the extension.

Proof. Fix an extender E given by 2.1. We assumed GCH, so E can be picked to be an extender over κ of the length κ^{++}.
Let Q be $\kappa-$distributive forcing notion of cardinality κ. Replace by an isomorphic one over κ.
Consider the filter F_Q of its dense open subsets. Then F_Q is a $\kappa-$complete filter over κ. Hence, for some $\eta < \kappa^{++}$, $F_Q \subseteq E_\eta$. Denote the least such η by η_Q.
It is possible to force now with Extender E based Prikry forcing in the Merimovich style [6] or, after an additional forcing turning E into a P-point, with the original extender based Prikry forcing, as in [3]. This will produce Prikry sequences for each F_Q as above, i.e. sequences $\langle q_n \mid n < \omega\rangle$ such that for every dense open $D \subseteq Q$, $q_n \in D$, for all but finitely many n's.
However, it is not enough to produce a generic subset of Q, since such q_n's need not be compatible.
Let us modify slightly the extender based forcing used, in order to overcome this difficulty.

Denote by Q_q, for every $q \in Q$, the set

$$\{q' \in Q \mid q' \geq q\}.$$

Then Q_q is a $\kappa-$distributive forcing notion of cardinality κ as well. So, F_{Q_q} is defined. In addition, for every $D \in F_Q$, the set

$$\{q' \in Q \mid q' \geq q \text{ and } q' \in D\}$$

is in F_{Q_q}.
Without loss of generality we can assume that each Q under the consideration is nowhere atomic forcing notion. Then, for every $A \subseteq Q, |A| < \kappa$, there is a dense open $D \subseteq Q$ with $A \cap D = \emptyset$. Just for each $q \in A$ consider

$$D_q = \{q' \in Q \mid q' > q \text{ or } q' \not\parallel q\}.$$

Then every D_q is a dense open and $\bigcap_{q \in A} D_q$ is a dense open disjoint from A.

A typical condition p in the extender based Prikry forcing with E is of the form

$$\langle\langle p^\gamma \mid \gamma \in supp(p)\rangle, \langle p^{mc}, T^p\rangle\rangle.$$

The support of p, $supp(p)$ is a subset of κ^{++} of cardinality $\leq \kappa$, $\kappa \in supp(p)$. The maximal coordinate $mc = mc(p)$ is an ordinal $\alpha < \kappa^{++}$ which is above (in the order $\leq_E$ of the extender) every $\beta \in supp(p)$. Each $p^\gamma, \gamma \in supp(p)$ and p^{mc} is a finite increasing sequence of ordinals. They are initial segments of the Prikry sequences for γ's and mc respectively. The set T^p is responsible for potential extensions.
Let us make the following changes:

1. if for some Q, $\eta_Q \in supp(p) \cup \{mc(p)\}$, then for every $q \in Q$, $\eta_{Q_q} \in supp(p) \cup \{mc(p)\}$, as well;
2. p^{η_Q} is increasing also in the order of Q;
3. once extending a condition p, p^{η_Q} extends by a member of a set of measure one for $E_{\eta_{Q_{\max(p^{\eta_Q})}}}$ instead of a member of a set of measure one for E_{η_Q}.

The basic properties of the forcing remain valid after this changes.
The last condition insures that the generic $\omega-$sequences growing over a coordinate η_Q will be increasing in the order of Q, and so will generate a $V-$generic subset of Q.
$\square$

3 Strength of $\kappa-$compact cardinals.

It was shown in [1] that an inner model with a strong cardinal is a lower bounds on a strength of $\kappa-$compact cardinals. Here we would like to improve this lower bound.

Theorem 3.1 *Suppose κ is $\kappa-$compact then there is a inner model with a Woodin cardinal*

Proof. Suppose otherwise. Then by Jensen-Steel [5], the core model K exists.
Define

$$E := (E^\kappa_{\aleph_0})^K = \{\nu < \kappa \mid (cf(\nu))^K = \omega\}$$

and let

$$F_0 = (Cub \restriction E)^K.$$

Then F_0 is a normal filter on κ in K.

Let, in V,

$$\mathcal{F} = \{X \subseteq \kappa \mid \exists A \in F_0 (A \subseteq X)\}.$$

Lemma 3.2 *$\mathcal{F}$ is a κ-complete filter in V.*

Proof. By [5], K satisfies GCH, in particular, $2^\kappa = \kappa^+$. So, in K, there is a sequence $\langle A_\alpha \mid \alpha < \kappa^+ \rangle$ such that

1. $A_\alpha \in Cub_\kappa \restriction E$, for every $\beta < \alpha < \kappa^+$,
2. $A_\alpha \subseteq^* A_\beta$ (i.e. $|A_\alpha \backslash A_\beta| < \kappa$) for every $\beta < \alpha < \kappa^+$,
3. $\forall A \in Cub_\kappa \restriction E$ there is $\alpha < \kappa^+$ such that $A_\alpha \subseteq A$.

Now,work in V. We have, by [5], $(\kappa^+)^K = \kappa^+$.
Let

$$\langle X_\tau \mid \tau < \delta \rangle$$

be the sequence of members of $\mathcal{F}$ for some $\delta < \kappa$. Let us show that

$$\bigcap_{\tau<\delta} X_\delta \in \mathcal{F}.$$

For each $\tau < \delta$ there is $A \in (Cub_\kappa \restriction E)^K$ such that $X_\tau \supseteq A$. Then there is $\alpha_\tau < \kappa^+$ such that $A_{\alpha_\tau} \subseteq^* A$. Pick $\rho_\tau < \kappa$ with $A \supseteq A_{\alpha_\tau} \backslash \rho_\tau$. Then,

$$X_\tau \supseteq A_{\alpha_\tau} \backslash \rho_\tau.$$

Let

$$\rho^* = \sup(\{\rho_\tau \mid \tau < \delta\}) < \kappa^+$$

and

$$\alpha^* = \sup(\{\alpha_\tau \mid \tau < \delta\}) < \kappa^+.$$

Then for all $\tau < \delta$ we have

$$A_{\alpha^*} \subseteq^* A_{\alpha_\tau}.$$

Pick $\xi_\tau < \kappa$ such that $A_{\alpha^*} \backslash \xi_\tau \subseteq A_{\alpha_\tau}$. Set

$$\xi^* = sup_{\tau<\delta} \xi_\tau < \kappa.$$

Then

$$A_{\alpha^*} \setminus \max(\rho^*, \xi^*) \subseteq X_\tau,$$

for each $\tau < \delta$. Clearly,

$$A_{\alpha^*} \setminus \max(\rho^*, \xi^*) \in \mathcal{F}.$$

Hence,

$$\bigcap_{\tau<\delta} X_\tau \in \mathcal{F}.$$

$\square$ of the lemma.

There is a $\mathcal{F}^* \supseteq \mathcal{F}$ that is a κ-complete ultrafilter, since κ is a $\kappa-$compact cardinal. Consider

$$i^* : V \longrightarrow M \simeq V^\kappa / F^*.$$

Let $\tilde{i} = i \restriction K$. By R. Schindler [8], $\tilde{i}$ is an iterated ultrapower along the cofinal branch of an iteration tree.
Let $\delta = [id]_{\mathcal{F}^*}$

Claim 1 *δ can not be of the form κ_α, where κ_α is one of the images of κ along the iteration $\tilde{i}$.*

Proof. Just otherwise, δ will be regular in $(K)^M$, but $\delta \in \tilde{i}(E)$.
$\square$ of the claim.

So, δ is not one of κ_α's. Then, unless there is an extender involved of a super-strong type, there will be $n < \omega, f : [\kappa]^n \to \kappa$, generators $\mu_1 < ... < \mu_n < \delta$ such that

$$\tilde{i}(f)(\mu_1, ..., \mu_n) \geq \delta.$$

Consider in K the following set

$$C = \{\nu < \kappa \mid \forall a_1, ..., a_n \in [\nu]^{<\omega} f(a_1, ..., a_n) < \nu\}.$$

Then, C is a club. Hence, $C \in F_0 \subseteq \mathcal{F}$. So, $\delta \in i(C) = \tilde{i}(C)$, which is impossible. Contradiction.
$\square$

4 Some weakening.

Let us consider the following natural weakening of $\kappa-$compactness:

Definition 4.1 κ is weakly $\kappa-$compact iff for every stationary $S \subseteq \kappa$, the filter $Cub \restriction S$ can be extended to a κ complete ultrafilter over κ.

Let us recall the following notion:

Definition 4.2 (Mitchell) Let $\langle U(\kappa, \beta) \mid \beta < \rho \rangle$ be a sequence of measures over κ. We say $\beta^* < \rho$ is a weak repeat point for the sequence iff for every $A \in U(\kappa, \beta^*)$ there is some $\gamma < \beta^*$ such that $A \in U(\kappa, \gamma)$.

Note that under GCH, the first weak repeat point is an ordinal of cofinality κ^+, above κ^+ and below κ^{++}.

The next lemma is well known and likely is due to W. Mitchell.

Lemma 4.3 *Let* $\vec{U}= \langle U(\alpha, \beta) \mid \alpha \leq \kappa, \alpha \in dom(\vec{U}) \& \beta < o^{\vec{U}}(\alpha) \rangle$ *be a coherent sequence* $\kappa \in dom(U)$, $\kappa = maxdom(U)$. *Suppose that* $\beta^* < o^{\vec{U}}(\kappa)$ *is the first weak repeat point for* $\langle U(\kappa, \beta) \mid \beta < o^{\vec{U}}(\kappa) \rangle$. *Then there is a sequence* $\langle A_\beta \mid \beta < \beta^* \rangle$ *such that for every* $\beta' \neq \beta < \beta^*$, $A_\beta \in U(\kappa, \beta) \setminus U(\kappa, \beta')$.

Proof. Let $\beta < \beta^*$. Then there is $B_\beta \in U(\kappa, \beta)$ such that for every $\gamma < \beta$,

$$B_\beta \notin U(\kappa, \gamma).$$

Consider

$$X_\beta = \{\nu < \kappa \mid \forall \xi < o^{\vec{U}}(\nu)(B_\beta \cap \nu \notin U(\nu, \xi))\}.$$

Then $X_\beta \in U(\kappa, \beta)$, since by coherence

$$M_{\kappa,\beta} \models \forall \xi < o^{i_\beta(\vec{U})}(\kappa) = \beta(i_{\kappa,\beta}(B_\beta) \cap \kappa) = B_\beta \notin U(\kappa, \xi)).$$

It follows that $\kappa \in i_{\kappa,\beta}(X_\beta)$ and then $X_\beta \in U(\kappa, \beta)$, where

$$i_{\kappa,\beta} := i_{U(\kappa,\beta)} : V \longrightarrow M_{\kappa,\beta} \simeq V^\kappa / U(\kappa, \beta).$$

Take $A_\beta = B_\beta \cap X_\beta$. Then $A_\beta \in U(\kappa, \beta)$, $A_\beta \notin U(\kappa, \gamma)$ for every $\gamma < \beta$, but, also, we can check that $A_\beta \in U(\kappa, \gamma)$ for $\beta < \gamma < o^{\vec{U}}(\kappa)$.

Thus, if $A_\beta \in U(\kappa,\gamma), \beta < \gamma < o^{\vec{U}}(\kappa)$, then $\kappa \in i_{\kappa,\gamma}(X_\beta)$ and, consequently,

$$M_{\kappa,\gamma} \models \forall \xi < \gamma \ (i_{\kappa,\gamma}(B_\beta \cap \kappa) = B_\beta \cap \kappa \notin U(\kappa,\xi)\ ),$$

but $\beta < \gamma$ and $B_\beta \in U(\kappa,\beta)$

□

Theorem 4.4 *Suppose that κ is a weakly $\kappa-$compact, then there is a weak repeat point for the coherent sequence of measures over κ in the core model K.*

Proof. Pick be a normal measure W over κ.
Let

$$i : V \longleftrightarrow M \simeq V^\kappa/W.$$

Let $\tilde{i} = i \restriction K$. Then

$$\tilde{i} : K \longrightarrow (K)^M$$

is an iterate of K.
Let $U(\kappa,\eta)$ be the first measure used in $\tilde{i}$.
Assume that there is no repeat point over κ in K. Then there will be a set $A_\eta \in K$, $A_\eta \in U(\kappa,\eta)$ such that

$$\forall \xi \neq \eta \ (A_\eta \notin U(\kappa,\xi)).$$

Lemma 4.5 *Suppose that $B \subseteq A_\eta$, $B \in K$ and $B \notin U(\kappa,\eta)$. Then, in V, B is non-stationary.*

Proof. Suppose otherwise. Then there is $B \subseteq A_\eta$,$B \in K$,$B \notin U(\kappa,\eta)$ stationary in V.
Work in V. Let $F := Cub_\kappa \restriction B$.
By the assumption, there is a κ-complete ultrafilter F^* over κ such that $F^* \supseteq F$.
Let

$$i^* : V \longrightarrow M \simeq V/F^*$$

and

$$i^* \restriction K = \tilde{i}^* : K \longrightarrow (K)^{M^*}.$$

Then $\tilde{i}^*$ it is an iterate ultrapower of K. Let

$$\delta = [id]_{F^*}.$$

Then $\delta \in i^*(B) = \tilde{i}^*(B)$, also, for every $C \subseteq \kappa$ club in K, $\delta \in i^*(C)$

Claim 2 δ can not be any of the images κ_α of κ obtained during the iteration $\tilde{i}^*$.

Proof. Suppose otherwise. Then there is α such that $\delta = \kappa_\alpha$. Write $i_{\geq\delta} \circ i_{<\delta} = \tilde{i}^*$ where $\delta = \kappa_\alpha = critc(i_{\geq\delta})$. So there is $\gamma < o^{(K)^{M^*}}(\kappa_\alpha)$ such that $U(\kappa_\alpha, \gamma)$ is used in the iteration. Then $\delta \in \tilde{i}^*(B)$ implies $i_{<\delta}(B) \in U(\kappa_\alpha, \gamma)$. But

$$K \models \forall \xi < o(\kappa) \ \Big(B \notin U(\kappa, \xi)\Big).$$

Then by elementarity of $i_{<\delta}$, we have that

$$i_{<\delta}(B) \notin U(\kappa_\alpha, \gamma).$$

Contradiction.

□ of the claim.

So, δ is not an image of κ during the iteration. Then there are $f : [\kappa]^n \longrightarrow \kappa$ and $\kappa_{\alpha_1}, ..., \kappa_{\alpha_n} < \delta$ such that

$$\tilde{i}^*(f)(\kappa_{\alpha_1}, ..., \kappa_{\alpha_n}) \geq \delta.$$

Consider

$$C := \{\nu \in \kappa \mid \forall \rho_1, .., \rho_n < \nu \ f(\rho_1, ..., \rho_n) < \nu\}.$$

Then C is a club in κ, $C \in K$, but $\delta \in \tilde{i}^*(C)$. Contradiction.

□ of the lemma.

Let us conclude now the proof of the theorem.

Notice that $Cub \restriction A_\eta \in M$, since $(\mathcal{P}(\kappa))^V = (\mathcal{P}(\kappa))^M$. By the lemma, it follows that

$$(\mathcal{P}(\kappa))^K \cap Cub \restriction A_\eta = U(\kappa, \eta) \in M,$$

this is a contradiction since $U(\kappa, \eta)$ coheres with K^M and is in M (it implies that it is in K^M by its maximality) but is not in K^M.

□

Remark 4.6 1. The above proof actually shows that non of $U(\kappa, \eta)$'s with η below a weak repeat point can be extended to a normal $\kappa-$complete ultrafilter and every stationary $X \subseteq \kappa, X \in K$ must have measure one in one of the measures over κ in K.

2. If we assume that only the following:

for every stationary $A \subseteq \kappa, A \in K$ the filter $Cub \restriction A$ extends to a $\kappa-$complete ultrafilter, then the argument goes through and the conclusion will be the same.

The next result is strengthening a bit the previous one.

Theorem 4.7 *Suppose that for every $A \subseteq \kappa, A \in K$ such that $A \cap Regular$ is stationary, the filter generated by $(Cub_\kappa \restriction A \cap Regular)^K$ extends to a $\kappa-$complete ultrafilter. Then there is a repeat point for the coherent sequence of measures over κ in the core model K.*

Proof. Proceed as in 4.4. Let W be a normal measure over κ.
Let

$$i : V \longleftrightarrow M \simeq V^\kappa/W.$$

Consider $\tilde{i} = i \restriction K$. Then

$$\tilde{i} : K \longrightarrow (K)^M$$

is an iterate of K.
Let $U(\kappa, \eta)$ be the first measure used in $\tilde{i}$.
Assume that there is no repeat point over κ in K. Then there will be a set $A_\eta \in K$, $A_\eta \in U(\kappa, \eta)$ such that

$$\forall \xi \neq \eta \ (A_\eta \notin U(\kappa, \xi)).$$

Some of the elements of A_η may be singular in V, still $A \cap Regular$ is stationary since $A \in W$ and W is a normal measure.

The following analog of Lemma 4.5:

Lemma 4.8 *Suppose that $B \subseteq A_\eta$, $B \in K$ and $B \notin U(\kappa, \eta)$. Then, in V, $B \cap Regular$ is non-stationary.*

Proof. Suppose otherwise. Then there is $B \subseteq A_\eta$,$B \in K$,$B \notin U(\kappa, \eta)$ stationary in V.
Work in V.
Let F be the filter generated by $(Cub_\kappa \restriction B)^K$, i.e.

$$F = \{X \subseteq \kappa \mid (\exists C \in K \text{ a club })(X \supseteq B \cap C)\}.$$

By the assumption, there is a κ-complete ultrafilter F^* over κ such that $F^* \supseteq F$.

Continue now exactly as in Lemma 4.5. Note that the club C defined there at the final stage is in K, hence $\delta \in \tilde{i}^*(C)$. Contradiction.
□ of the lemma.

Let us conclude now the proof of the theorem.
Notice that $Cub_\kappa \restriction A_\eta \cap Regular \in M$, since V and M agree about regularity of cardinals below κ (just κ is the critical point) and $(\mathcal{P}(\kappa))^V = (\mathcal{P}(\kappa))^M$. By the lemma, it follows that

$$(\mathcal{P}(\kappa))^K \cap (Cub \restriction A_\eta \cap Regular) = U(\kappa, \eta) \in M,$$

this is a contradiction since $U(\kappa, \eta)$ coheres with K^M and is in M (it implies that it is in K^M by its maximality) but is not in K^M.
□

5 Forcing constructions-regular cardinals.

In this section we would like to provide an upper bound on consistency strength of a weakly $\kappa-$compact cardinal κ and weaker properties considered in the previous section.
Let us start with the following observation.

Theorem 5.1 *Suppose that there is a weak repeat point over κ in the core model. Then there is a cofinality preserving extension in which for every $X \subseteq \kappa, X \in K$ stationary and consisting of regular cardinals, the filter generated $(Cub_\kappa \restriction X)^K$ extends to a $\kappa-$complete ultrafilter. However there is $X \subseteq \kappa, X \in K$ stationary and consisting of regular cardinals, such that the filter $Cub_\kappa \restriction X$ does not extend to a $\kappa-$complete ultrafilter.*

Proof. Let
$$\vec{U} = \langle U(\kappa, \alpha) \mid \alpha \leq \eta \rangle$$
be a coherent sequence of measures over κ in K, $o(\kappa) = \eta + 1$ and η is the least weak repeat point for $\vec{U}$. It is well known (see for example [2]) that then $\mathrm{cof}(\eta) = \kappa^+$ and for every $X \in U(\kappa, \eta)$ the set
$$\{\xi < \eta \mid X \in U(\kappa, \xi)\}$$
is unbounded in η. Denote by $\mathcal{F}_\eta$ the following set:

$$\{X \subseteq \kappa \mid \exists \gamma < \eta \forall \beta (\gamma \leq \beta < \eta \longrightarrow X \in U(\kappa, \beta))\}.$$

Then it is a $\kappa-$copmplete filter over κ and $U(\kappa, \eta) \supseteq \mathcal{F}_\eta$, since otherwise there will be a set $Y \in \mathcal{F}_\eta \setminus U(\kappa, \eta)$. But, then $\kappa \setminus Y \in U(\kappa, \eta)$, which is impossible, since the set $\{\xi < \eta \mid \kappa \setminus Y \in U(\kappa, \xi)\}$ is unbounded in η.

Define now a Backward Easton iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \beta \leq \kappa, \alpha \leq \kappa + 1 \rangle.$$

Suppose that $\alpha < \kappa + 1$ and P_α is defined. Define $\underset{\sim}{Q}_\alpha$. Set $\underset{\sim}{Q}_\alpha$ to be a trivial forcing unless $o(\alpha) > 0$ is a limit ordinal.
Once $o(\alpha) > 0$ and it is a limit ordinal, then let $\underset{\sim}{Q}_\alpha$ be the less than $\alpha-$support iteration of the standard forcing notion for adding a club into $X \cup Singular$, for every $X \subseteq \alpha$ such that for some $\gamma < \alpha$,

$$X \in \bigcap_{\gamma \leq \beta < o(\alpha)} U(\alpha, \beta).$$

Now, the elementary embedding $i_\eta : K \to K_\eta \simeq K^\kappa / U(\kappa, \eta)$ extends, but non of i_ξ for $\xi < \eta$. However, we will extend the embeddings by $U(\kappa, \eta) \times U(\kappa, \xi)$, for $\xi < \eta$. This way it will be insured that for each $X \subseteq \kappa \cap Regular, X \in K$ which is stationary in the extension there will be a $\kappa-$complete ultrafilter including $(Cub_\kappa \restriction X)^K$. Such ultrafilter will be an extension of $U(\kappa, \eta) \times U(\kappa, \xi)$ with $X \in U(\kappa, \xi)$.

Let $G(P_\kappa) * G(Q_\kappa)$ be a generic subset of $P_\kappa * \underset{\sim}{Q}_\kappa$.

Lemma 5.2 *The elementary embedding*

$$i_\eta : K \to K_\eta \simeq K^\kappa / U(\kappa, \eta)$$

extends to an elementary embedding

$$i^*_\eta : K[G(P_\kappa) * G(Q_\kappa)] \to K_\eta[G(P_{i_\eta(\kappa)} * G(Q_{i_\eta(\kappa)})],$$

for some $K_\eta-$generic subsets $G(P_{i_\eta(\kappa)} * G(Q_{i_\eta(\kappa)})$ of $i_\eta(P_\kappa * \underset{\sim}{Q}_\kappa)$.

Proof. Note that $\mathcal{F}_\eta \in M_\eta$ by the coherency of the sequence $\vec{U}$. The club subsets are added over κ to $X \cup Singular$, for every $X \in \mathcal{F}_\eta$. But each X like this is necessary in $U(\kappa, \eta)$. So i_η extends in a standard way. Note that the adding of singulars provides enough closure in order to construct a master condition sequence.
$\square$ of the lemma.

Let

$$i_{\eta\xi} : K \to K_{\eta,\xi} \simeq K^{\kappa^2} / U(\kappa, \eta) \times U(\kappa, \xi)$$

be the elementary embedding corresponding to $U(\kappa, \eta) \times U(\kappa, \xi)$, where $\xi < \eta$.
Then, similar to 5.2, we obtain the following:

Lemma 5.3 *Let $\xi < \eta$. Then the elementary embedding*

$$i_{\eta\xi} : K \to K_{\eta\xi}$$

extends to an elementary embedding

$$i^*_{\eta\xi} : K[G(P_\kappa) * G(Q_\kappa)] \to K_{\eta\xi}[G(P_{i_\eta(\kappa)} * G(Q_{i_\eta(\kappa)})],$$

for some $K_{\eta\xi}$–generic subsets $G(P_{i_{\eta\xi}(\kappa)} * G(Q_{i_{\eta\xi}(\kappa)})$ of $i_{\eta\xi}(P_\kappa * \underset{\sim}{Q}_\kappa)$.

Suppose now that $X \subseteq \kappa \cap Regular, X \in K$ is stationary in the extension. Then there is $\xi < \eta$ such that $X \in U(\kappa, \xi)$. Hence,

$$\kappa \in i_\xi(X) \text{ and } i_\eta(\kappa) \in i_{\eta\xi}(X).$$

Also, if $C \subseteq \kappa, C \in K$ is a club, then

$$\kappa \in i_\xi(C) \text{ and } i_\eta(\kappa) \in i_{\eta\xi}(C).$$

Consider now in $K[G(P_\kappa) * G(Q_\kappa)]$ the following κ–complete ultrafilter:

$$U_\xi := \{Y \subseteq \kappa \mid i_\eta(\kappa) \in i^*_{\eta\xi}(Y)\}.$$

Then

$$(Cub \restriction X)^K \subseteq U_\xi.$$

It remains to give an example of a stationary (in the extension) set $X \subseteq \kappa \cap Regular, X \in K$ such that the filter $Cub_\kappa \restriction X$ does not extend to a κ–complete ultrafilter.
Let X be any $\mathcal{F}_\eta$–positive set in K which does not belong to $U(\kappa, \eta)$.
Suppose that the filter $Cub_\kappa \restriction X$ extends to a κ–complete ultrafilter W. Consider

$$i_W : K[G(P_\kappa) * G(Q_\kappa)] \to M_W \simeq K[G(P_\kappa) * G(Q_\kappa)]^\kappa / W.$$

Let

$$\delta = [id]_W \text{ and } \tilde{i} = i_W \restriction K.$$

Then

$$\tilde{i} : K \to K^{M_W}.$$

The forcing used was cofinality preserving forcing. Then, also, K^{M_W} and M_W agree on cofinality of ordinals. In addition, M_W is closed under κ–sequences of its elements, as an ultrapower by a κ–complete ultrafilter. Hence, $\tilde{i}$ is finite iterated ultrapower of K.
It follows, as in 4.5, that δ is κ or one its images in this iteration.

Claim 3 $\delta \neq \kappa$.

Proof. Suppose otherwise. Then W is normal. The iteration $\tilde{i}$ starts with a normal measure $U(\kappa, \xi)$, for some $\xi \leq \eta = o(\kappa) - 1$, and $W \supseteq U(\kappa, \xi)$. But $X \notin U(\kappa, \eta)$, hence $\xi < \eta$. Recall that η is the first weak repeat point. So, there is $A_\xi \in U(\kappa, \xi)$ which does not belong to any other $U(\kappa, \xi')$ with $\xi' \neq \xi$. Then the forcing Q_κ adds a club C disjoint with A_ξ. Hence, $C \in W$, but also $X \cap A_\xi \in U(\kappa, \xi) \subseteq W$. Contradiction.
$\square$ of the claim.

So, $\delta \neq \kappa$. In addition, $W \not\supseteq U(\kappa, \eta)$. Hence there is $\xi < \eta$ such that δ is the critical point of the iteration at a step where an image of $U(\kappa, \xi)$ was applied. Then $W \supseteq U(\kappa, \xi)$, but such possibility was already ruled out in the claim above. Hence we obtain a contradiction.

In order to finish the proof, we need to show that the set X as above remains stationary. Suppose otherwise. Then the forcing Q_κ over $K[G(P_\kappa)]$ adds a club C disjoint to X. Recall that Q_κ is a $< \kappa-$support iteration of forcings of cardinality κ of the length κ^+. So, there $\beta < \kappa^+$ such that already $Q_\kappa \restriction \beta$ adds C.
Pick now $\rho < \eta$ such that

1. $X \in U(\kappa, \rho)$,

2. for every $\rho', \rho \leq \rho' < \eta$, for every $Y \in U(\kappa, \rho')$ there is no forcing shooting a club through Y in the iteration $Q_\kappa \restriction \beta$.

This is possible since $\mathrm{cof}(\eta) = \kappa^+$ and $X \in U(\kappa, \zeta)$ for unboundedly many $\zeta < \eta$. But, the elementary embedding

$$i_\rho : K \to K_\rho \simeq K^\kappa / U(\kappa, \rho)$$

extends to an elementary embedding

$$i^*_\rho : K[G(P_\kappa) * (G(Q_\kappa \restriction \rho)] \to K_\eta[G(P_{i_\rho(\kappa)} * G(Q_{i_\rho(\kappa)}))],$$

as in Lemma 5.2. This is clearly impossible, since we will have that both

$$\kappa \in i_\rho(X) = i^*_\rho(X) \text{ and } \kappa \in i^*_\rho(C).$$

Contradiction. So, we are done.
$\square$

Let us deal now with an other filter and extend the previous result to filters of the form $Cub_\kappa \restriction X$ where X is as in 5.1.

Theorem 5.4 *Suppose that there is a weak repeat point over κ in the core model. Then there is a cofinality preserving extension in which for every $X \subseteq \kappa, X \in K$ stationary and consisting of regular cardinals, the filter $Cub_\kappa \upharpoonright X$ extends to a $\kappa-$complete ultrafilter.*

Proof. Let

$$\vec{U} = \langle U(\kappa,\alpha) \mid \alpha \leq \eta \rangle$$

be a coherent sequence of measures over κ in K, $o(\kappa) = \eta + 1$ and η is the least weak repeat point for $\vec{U}$. It is well known (see for example [2]) that then $\operatorname{cof}(\eta) = \kappa^+$ and for every $X \in U(\kappa,\eta)$ the set

$$\{\xi < \eta \mid X \in U(\kappa,\xi)\}$$

is unbounded in η. Denote by $\mathcal{G}_\eta$ the filter

$$\bigcap_{\alpha<\eta} U(\kappa,\alpha).$$

We have

$$\bigcap_{\alpha<\eta} U(\kappa,\alpha) = \bigcap_{\alpha\leq\eta} U(\kappa,\alpha) \text{ and } U(\kappa,\eta) \supseteq \bigcap_{\alpha\leq\eta} U(\kappa,\alpha),$$

since η is a weak repeat point.

Define a Backward Easton iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \beta \leq \kappa, \alpha \leq \kappa+1 \rangle.$$

Suppose that $\alpha < \kappa + 1$ and P_α is defined. Define $\underset{\sim}{Q}_\alpha$. Set $\underset{\sim}{Q}_\alpha$ to be a trivial forcing unless $o(\alpha) > 0$ is a limit ordinal.
Once $o(\alpha) > 0$ and it is a limit ordinal, then let $\underset{\sim}{Q}_\alpha$ be the less than $\alpha-$support iteration of the standard forcing notion for adding a club into $X \cup Singular$, for every $X \subseteq \alpha$ such that

$$X \in \bigcap_{\beta<o(\alpha)} U(\alpha,\beta).$$

Let $G(P_\kappa) * G(Q_\kappa)$ be a generic subset of $P_\kappa * \underset{\sim}{Q}_\kappa$.

The proof of the next lemma is the same as those of 5.2.

Lemma 5.5 *The elementary embedding*

$$i_\eta : K \to K_\eta \simeq K^\kappa/U(\kappa,\eta)$$

extends to an elementary embedding

$$i^*_\eta : K[G(P_\kappa) * G(Q_\kappa)] \to K_\eta[G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})],$$

for some $K_\eta-$generic subsets $G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})$ of $i_\eta(P_\kappa * \underset{\sim}{Q}_\kappa)$.

Let $\xi < \eta$. Consider

$$i_{\eta\xi} : K \to K_{\eta,\xi} \simeq K^{\kappa^2}/U(\kappa,\eta) \times U(\kappa,\xi)$$

the elementary embedding corresponding to $U(\kappa,\eta) \times U(\kappa,\xi)$. It can be written also as

$$K \longrightarrow^{i_\eta} K_\eta \longrightarrow^{k_{\eta\xi}} K_{\eta,\xi},$$

where $k_{\eta\xi}$ is the canonical embedding of K_η into its ultrapower by $i_\eta(U(\kappa,\xi))$.

Similar to 5.2, we have the following:

Lemma 5.6 *Let $\xi < \eta$. Then the elementary embedding*

$$i_{\eta\xi} : K \to K_{\eta\xi}$$

extends to an elementary embedding

$$i^*_{\eta\xi} : K[G(P_\kappa) * G(Q_\kappa)] \to K_{\eta\xi}[G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})],$$

for some $K_{\eta\xi}-$generic subsets $G(P_{i_{\eta\xi}(\kappa)}) * G(Q_{i_{\eta\xi}(\kappa)})$ of $i_{\eta\xi}(P_\kappa * \underset{\sim}{Q}_\kappa)$.

Let us argue that in the present situation also $k_{\eta\xi}$, and so, the all diagram extends.

Lemma 5.7 *Let $\xi < \eta$. Then the diagram*

$$K \longrightarrow^{i_\eta} K_\eta \longrightarrow^{k_{\eta\xi}} K_{\eta,\xi}$$

extends to

$$K[G(P_\kappa) * G(Q_\kappa)] \longrightarrow^{i^*_\eta} K_\eta[G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})] \longrightarrow^{k^*_{\eta\xi}} K_{\eta\xi}[G(P_{i_{\eta\xi}(\kappa)}) * G(Q_{i_{\eta\xi}(\kappa)})],$$

for some $K_{\eta\xi}-$generic subsets $G(P_{i_{\eta\xi}(\kappa)}) * G(Q_{i_{\eta\xi}(\kappa)})$ of $i_{\eta\xi}(P_\kappa * \underset{\sim}{Q}_\kappa)$.

Proof. The new point here is that the forcing $Q_{i_\eta(\kappa)}$ used at $i_\eta(\kappa)$ over $K_\eta[G(P_{i_\eta(\kappa)})]$ shoots clubs only to sets which belong to

$$i_\eta(\mathcal{G}_\eta) = \bigcap_{\alpha < i_\eta(\eta)} U(i_\eta(\kappa),\alpha).$$

In particular, every subset of $i_\eta(\kappa)$ into which $Q_{i_\eta(\kappa)}$ shoots a club belongs to $U(i_\eta(\kappa), i_\eta(\xi))$. Also,

$$i_\eta(\kappa) \in k_{\eta\xi}(X) \text{ iff } X \in U(i_\eta(\kappa), i_\eta(\xi)).$$

Hence, we can add $i_\eta(\kappa)$ to $k_{\eta\xi}{}''C = C$ and keep it a condition in $Q_{i_{\eta\xi}(\kappa)}$, for every generic (i.e. in $G(Q_{i_\eta(\kappa)})$) club $C \subseteq i_\eta(\kappa)$. So, $k_{\eta\xi}$ extends as well as the diagram.
$\square$ of the lemma.

Lemma 5.8 *Let $\xi < \eta$ and*

$$i^*_{\eta\xi} : K[G(P_\kappa) * G(Q_\kappa)] \to K_{\eta\xi}[G(P_{i_\eta(\kappa)} * G(Q_{i_\eta(\kappa)})],$$

be as in the previous lemma (5.7). Then for every club $C \subseteq \kappa$ in $K[G(P_\kappa) * G(Q_\kappa)]$, we have

$$i_\eta(\kappa) \in i^*_{\eta\xi}(C).$$

Proof. By Lemma 5.7,

$$i^*_{\eta\xi} = k^*_{\eta\xi} \circ i^*_\eta.$$

The critical point of $k^*_{\eta\xi}$ is $i_\eta(\kappa)$ and $i^*_\eta(C)$ is unbounded in $i_\eta(\kappa)$. Hence,

$$i_\eta(\kappa) \in k^*_{\eta\xi}(i^*_\eta(C)) = i^*_{\eta\xi}(C).$$

□ of the lemma.

Suppose now that $X \subseteq \kappa \cap Regular, X \in K$ is stationary in the extension. Then there is $\xi < \eta$ such that $X \in U(\kappa, \xi)$. Hence,

$$\kappa \in i_\xi(X) \text{ and } i_\eta(\kappa) \in i_{\eta\xi}(X).$$

Also, if $C \subseteq \kappa$ is a club, then, by Lemma 5.8,

$$i_\eta(\kappa) \in i^*_{\eta\xi}(C).$$

Consider now in $K[G(P_\kappa) * G(Q_\kappa)]$ the following $\kappa-$complete ultrafilter:

$$U_\xi := \{Y \subseteq \kappa \mid i_\eta(\kappa) \in i^*_{\eta\xi}(Y)\}.$$

Then

$$Cub \restriction X \subseteq U_\xi.$$

□

Remark 5.9 It is possible to show that in $K[G(P_\kappa) * G(Q_\kappa)]$,

$$U^*(\kappa, \eta) := \{Y \subseteq \kappa \mid \kappa \in i^*_\eta(Y)\}$$

is the only normal measure and each U_ξ, with $\xi < \eta$, is a non-normal $Q-$point measure.

Let us now remove the restriction $X \in K$ from the previous theorem.

Theorem 5.10 *Suppose that there is a weak repeat point over κ in the core model. Then there is a cofinality preserving extension in which for every stationary $X \subseteq \kappa$ consisting of regular cardinals, the filter $Cub_\kappa \restriction X$ extends to a $\kappa-$complete ultrafilter.*

Proof. We proceed as in 5.4.
Let
$$\vec{U} = \langle U(\kappa, \alpha) \mid \alpha \leq \eta \rangle$$
be a coherent sequence of measures over κ in K, $o(\kappa) = \eta + 1$ and η is the least weak repeat point for $\vec{U}$. It is well known (see for example [2]) that then $\mathrm{cof}(\eta) = \kappa^+$ and for every $X \in U(\kappa, \eta)$ the set
$$\{\xi < \eta \mid X \in U(\kappa, \xi)\}$$
is unbounded in η. Denote by $\mathcal{G}_\eta$ the filter
$$\bigcap_{\alpha<\eta} U(\kappa, \alpha).$$
We have
$$\bigcap_{\alpha<\eta} U(\kappa, \alpha) = \bigcap_{\alpha\leq\eta} U(\kappa, \alpha) \text{ and } U(\kappa, \eta) \supseteq \bigcap_{\alpha\leq\eta} U(\kappa, \alpha),$$
since η is a weak repeat point.
Let us first continue further as in 5.4. So, we define a Backward Easton iteration
$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \beta \leq \kappa, \alpha \leq \kappa + 1 \rangle.$$

Let $G(P_\kappa) * G(Q_\kappa)$ be a generic subset of $P_\kappa * \underset{\sim}{Q}_\kappa$.

Suppose now that $X \subseteq \kappa$ is stationary in $V[G(P_\kappa) * G(Q_\kappa)]$ which consists of regular cardinals.

Consider first extensions of
$$i_\eta : V \to M_\eta \simeq V^\kappa / U(\kappa, \eta).$$
If there are condition $p \in G(P_\kappa) * G(\underset{\sim}{Q}_\kappa)$ and $\underset{\sim}{q} \in i_\eta(P_\kappa * \underset{\sim}{Q}_\kappa)/P_\kappa * \underset{\sim}{Q}_\kappa$ such that
$$(p, \underset{\sim}{q}) \Vdash \kappa \in i_\eta(\underset{\sim}{X}),$$

then X will belong to a normal ultrafilter which extends $U(\kappa, \eta)$.
Suppose that this is not the case.
Then, there is $p \in G(P_\kappa) * G(\underset{\sim}{Q}_\kappa)$ such that

$$(p, \underset{\sim}{0}) \Vdash \eta \notin i_\eta(\underset{\sim}{X}).$$

We can alter the name $\underset{\sim}{X}$ of X such that for every $\nu < \kappa$, if a condition $(s, \underset{\sim}{t}) \in P_{\kappa+1}$ is incompatible with $f(\nu)$, then

$$(\check{\nu}, (s, \underset{\sim}{t})) \notin \underset{\sim}{X},$$

where f is a function which represents $(p, \underset{\sim}{0})$ in M_η. So, using such name, we will have

$$0_{P_{i_\eta(\kappa)+1}} \Vdash \eta \notin i_\eta(\underset{\sim}{X}).$$

Set

$$Y_\eta = \{\nu < \kappa \mid 0_{P_{\kappa+1}} \Vdash \nu \notin \underset{\sim}{X}\}.$$

Then $Y_\eta \in U(\kappa, \eta)$ and, in $V[G(P_\kappa) * G(\underset{\sim}{Q}_\kappa)]$,

$$Y_\eta \cap X = \emptyset.$$

Now, let us do a similar thing for every $\xi < \eta$.
Consider

$$i_{\eta\xi} : V \to M_{\eta,\xi} \simeq V^{\kappa^2}/U(\kappa, \eta) \times U(\kappa, \xi)$$

the elementary embedding corresponding to $U(\kappa, \eta) \times U(\kappa, \xi)$. It can be written as

$$V \longrightarrow^{i_\eta} M_\eta \longrightarrow^{k_{\eta\xi}} M_{\eta,\xi},$$

where $k_{\eta\xi}$ is the canonical embedding of M_η into its ultrapower by $i_\eta(U(\kappa, \xi))$.
By Lemma 5.6, the elementary embedding

$$i_{\eta\xi} : V \to M_{\eta\xi}$$

extends to an elementary embedding

$$i^*_{\eta\xi} : V[G(P_\kappa) * G(Q_\kappa)] \to M_{\eta\xi}[G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})],$$

for some $M_{\eta\xi}-$generic subsets $G(P_{i_{\eta\xi}(\kappa)}) * G(Q_{i_{\eta\xi}(\kappa)})$ of $i_{\eta\xi}(P_\kappa * \underset{\sim}{Q}_\kappa)$. Also, $k_{\eta\xi}$ extends to

$$k^*_{\eta\xi} : M_\eta[G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})] \to M_{\eta\xi}[G(P_{i_{\eta\xi}(\kappa)}) * G(Q_{i_{\eta\xi}(\kappa)})],$$

for some $M_{\eta\xi}-$generic subsets $G(P_{i_{\eta\xi}(\kappa)}) * G(Q_{i_{\eta\xi}(\kappa)})$ of $i_{\eta\xi}(P_\kappa * \underset{\sim}{Q}_\kappa)$. If there are condition $p = (r, \underset{\sim}{s}) \in G(P_\kappa) * G(\underset{\sim}{Q}_\kappa)$ and $\underset{\sim}{q} \in i_{\eta\xi}(P_\kappa * \underset{\sim}{Q}_\kappa)/P_\kappa * \underset{\sim}{Q}_\kappa$ which extends $i_\eta{}'' \underset{\sim}{s}$ and such that

$$(p, \underset{\sim}{q}) \Vdash i_\eta(\kappa) \in i_{\eta\xi}(\underset{\sim}{X}),$$

then X will belong to a normal ultrafilter which extends $U(\kappa, \xi)$.
Suppose that it is not the case. Then there is a condition $p = (r, \underset{\sim}{s}) \in G(P_\kappa) * G(\underset{\sim}{Q}_\kappa)$ such that for every $\underset{\sim}{q} \in i_{\eta\xi}(P_\kappa * \underset{\sim}{Q}_\kappa)/P_\kappa * \underset{\sim}{Q}_\kappa$ which extends $i_\eta{}'' \underset{\sim}{s}$, we have

$$(p, \underset{\sim}{q}) \Vdash i_\eta(\kappa) \notin i_{\eta\xi}(\underset{\sim}{X}).$$

Consider

$$i_\xi : V \to M_\xi \simeq V^\kappa / U(\kappa, \xi).$$

Claim *In* M_ξ,

$$(p, \underset{\sim}{0}) \Vdash \kappa \notin i_\xi(\underset{\sim}{X}).$$

Proof. Suppose otherwise. Then there is some $\underset{\sim}{t} \in i_\xi(P_\kappa * \underset{\sim}{Q}_\kappa)/P_\kappa * \underset{\sim}{Q}_\kappa$, $\underset{\sim}{t} \geq i_\xi{}'' \underset{\sim}{s}$ such that

$$(p, \underset{\sim}{t}) \Vdash \kappa \in i_\xi(\underset{\sim}{X}).$$

We would like to use now the elementary embedding

$$\sigma_{\xi\eta} : M_\xi \to M_{\eta\xi}$$

which is defined as follows:

$$\sigma_{\xi\eta}(i_\xi(g)(\kappa)) = (i_{\eta\xi}(g))(i_\eta(\kappa)).$$

Apply $\sigma_{\xi\eta}$ to $(p, \underset{\sim}{t})$. Then, by elementarity, in $M_{\eta\xi}$,

$$(p, \sigma_{\xi\eta}(\underset{\sim}{t})) \Vdash \kappa \in \sigma_{\xi\eta}(i_\xi(\underset{\sim}{X})) = i_{\eta\xi}(\underset{\sim}{X}).$$

The condition $\underset{\sim}{t} \geq i_\xi{}'' \underset{\sim}{s}$ translates into $\sigma_{\xi\eta}(\underset{\sim}{t}) \geq i_\eta{}'' \underset{\sim}{s}$. But this is impossible.
Contradiction.
$\square$ of the claim.

Now, as above with η, we can alter the name $\underset{\sim}{X}$ and find $Y_\xi \in U(\kappa, \xi)$ such that in $V[G(P_\kappa) * G(\underset{\sim}{Q}_\kappa)]$,

$$Y_\xi \cap X = \emptyset.$$

Set $Y = \bigcup_{\xi \leq \eta} Y_\xi$. Then $Y \cap X = \emptyset$ and for every $\xi \leq \eta$, $Y \supseteq Y_\xi \in U(\kappa, \xi)$.
Hence, if $Y \in V$ then a club was added to $Y \cup Singular$.
We have $2^\kappa = \kappa^+$ and the forcing $P_{\kappa+1}$ satisfies κ^+-c.c., hence, there is a sequence

$$\langle Z_\xi \mid \xi \leq \eta \rangle \in V$$

such that $Z_\xi \in U(\kappa, \xi)$ and $|Z_\xi \cap X| < \kappa$.
However this does not guarantee that there will be a set in $\mathcal{F}_\eta = \bigcap_{\xi < \eta} U(\kappa, \xi)$ disjoint with X.
In order to deal with this problem, let us modify the forcing a bit: if at some stage of the iteration a set X as above appears, then let us force a club disjoint to it.
Such modified version shares the properties of the original forcing, but in the final extension there will be no stationary sets X as above and so for every stationary set S consisting of regular cardinals the filter $Cub_\kappa \restriction S$ extends to a $\kappa-$complete ultrafilter.
$\square$

6 Forcing constructions-singular cardinals.

Let us extend now the previous results in order to include stationary sets consisting of singular ordinals as well.

Theorem 6.1 *Suppose that there is a weak repeat point over κ in the core model. Then there is cardinal preserving extension in which for every $X \subseteq \kappa, X \in K$ stationary, the filter $Cub_\kappa \restriction X$ extends to a $\kappa-$complete ultrafilter.*

Proof. Let

$$\vec{U} = \langle U(\nu, \alpha) \mid \nu \leq \kappa, o(\nu) > 0, \alpha < o(\nu) \rangle$$

be a coherent sequence of measures in K. Assume that $o(\kappa) = \eta + 1$ and η is the least weak repeat point for

$$\langle U(\kappa, \alpha) \mid \alpha \leq \eta \rangle$$

.

Force with Easton iteration of Prikry-Magidor forcings and change cofinality of each $\nu < \kappa$ such that $o(\nu) > 0$ and $\mathrm{cof}(o(\nu)) < \nu^+$. This way ν's below κ with $\mathrm{cof}(o(\nu)) = \nu^+$ remain measurable.

Let $V = K$ and denote the generic extension above $V_1 = V[G]$.
Fix an extension $U_1(\kappa, \eta)$ of $U(\kappa, \eta)$ in V_1. Let

$$i^1_\eta : V_1 = V[G] \to M^1_\eta = \tilde{M}_\eta[\tilde{G}]$$

be the corresponding embedding. Note that $\tilde{M}_\eta$ is not M_η, but rather its iterated ultrapower. Consider the set $\mathcal{R}_\xi$ of all possible extensions of $U(\kappa, \xi)$ in M^1_η or equivalently in $\tilde{M}_\eta[G]$, for every $\xi < \eta$. Set

$$\mathcal{R}(\eta) = \bigcup_{\xi < \eta} \mathcal{R}_\xi.$$

Lemma 6.2 *Let $X \in U_1(\kappa, \eta)$. Then $X \in W$, for some normal measure $W \in \mathcal{R}(\eta)$.*

Proof. It is enough to proof the statement for sets of the form

$$X_p := \{\nu < \kappa \mid p \restriction \nu^\frown f_p(\nu) \in G\},$$

where $p \in \tilde{G}$ and f_p represents (mod $U_1(\kappa, \eta)$) the part of p above κ.
Clearly there are many $W \in \mathcal{R}$ with $X_p \in W$.
$\square$ of the lemma.

Define now over V_1 a Backward Easton iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \beta \leq \kappa, \alpha \leq \kappa + 1 \rangle.$$

Suppose that $\alpha < \kappa + 1$ and P_α is defined. Define $\underset{\sim}{Q}_\alpha$. Suppose first that $\alpha < \kappa$. Set $\underset{\sim}{Q}_\alpha$ to be a trivial forcing unless in K, $\mathrm{cof}(o(\alpha)) = \kappa^+$.
Once it is, then let $\underset{\sim}{Q}_\alpha$ be the less than $\alpha-$support iteration of the standard forcing notion for adding a club into X, for every $X \subseteq \alpha$ such that

$$X \in \bigcap \mathcal{R}(\alpha),$$

where $\mathcal{R}(\alpha)$ is is the intersection of all $\alpha-$complete ultrafilters over α in V_1, i.e. of all extensions of $U(\alpha, \beta), \beta < o(\alpha)$.
Note that such Q_α preserves cardinals (and cofinality), since we have here closed chunks of Magidor sequences of arbitrary length below α.
If $\alpha = \kappa$, then let $\underset{\sim}{Q}_\alpha$ be the less than $\alpha-$support iteration of the standard forcing notion for adding a club into X, for every $X \subseteq \alpha$ such that

$$X \in \bigcap \mathcal{R}(\eta).$$

Again, such Q_κ preserves cardinals (and cofinality), since we have here closed chunks of Magidor sequences of arbitrary length below κ.

Let $G(P_\kappa) * G(Q_\kappa)$ be a generic subset of $P_\kappa * \underset{\sim}{Q}_\kappa$.

It is natural now to try to extend the elementary embedding

$$i^1_\eta : V_1 = V[G] \to M^1_\eta = \tilde{M}_\eta[\tilde{G}].$$

However, the forcing Q_κ seems to have not enough closure for this. So, instead of dealing directly with i^1_η, let us choose an other embedding.
Consider in V the sequence

$$\langle U(\kappa, \beta) \mid \beta < \kappa^+ \rangle.$$

The first forcing turns it into a Rudin-Keisler increasing. More precisely, there is a sequence

$$\langle U_1(\kappa, \beta) \mid \beta < \kappa^+ \rangle$$

in V_1 (i.e. before forcing clubs) of extensions which is a Rudin-Keisler increasing. Also, there is such a sequence consisting of elements of $\mathcal{R}(\eta)$. Let

$$\langle U_1(\kappa, \beta) \mid \beta < \kappa^+ \rangle$$

be such a sequence.

Consider now the following sequence

$$\langle U_1(\kappa, \eta) \times U_1(\kappa, \beta) \mid \beta < \kappa^+ \rangle.$$

It is still a Rudin-Keisler increasing. Let

$$i^*_\eta : V_1 \to M^*_\eta$$

be the corresponding embedding into its direct limit. Then M^*_η is closed under $\kappa-$sequences of its elements and its core model, which we denote by K^*_η, is a further iteration of $\tilde{M}_\eta$ which uses measures from

$$i^1_\eta(\langle U_1(\kappa, \beta) \mid \beta < \kappa^+ \rangle).$$

We claim that the embedding i^*_η extends.

Lemma 6.3 *The elementary embedding*

$$i^*_\eta : V_1 \to M^*_\eta$$

extends to an elementary embedding

$$i^{**}_\eta : V_1[G(P_\kappa) * G(Q_\kappa)] \to M^*_\eta[G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})],$$

for some $M^*_\eta-$*generic subsets* $G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})$ *of* $i^*_\eta(P_\kappa * \underset{\sim}{Q}_\kappa)$.

Proof. The proof is rather standard and similar to those of Lemma 5.5. The new point here is to use the critical points measures

$$i^1_\eta(\langle U_1(\kappa,\beta) \mid \beta < \kappa^+\rangle)$$

in order to proceed κ^+-many steps in the process of constructing of a master condition sequence.

□ of the lemma.

Let $\xi < \eta$. Consider

$$i_{\eta\xi} : K \to K_{\eta,\xi} \simeq K^{\kappa^2}/U(\kappa,\eta) \times U(\kappa,\xi)$$

the elementary embedding corresponding to $U(\kappa,\eta) \times U(\kappa,\xi)$. It can be written also as

$$K \longrightarrow^{i_\eta} K_\eta \longrightarrow^{k_{\eta\xi}} K_{\eta,\xi},$$

where $k_{\eta\xi}$ is the canonical embedding of K_η into its ultrapower by $i_\eta(U(\kappa,\xi))$.

Now, instead of extending this diagram directly, as in 5.4, let us add a Rudin -Keisler increasing sequences of the length κ^+ to both η and ξ.

Proceed as follows. Let $U_1(\kappa,\xi)$ be an extension in V_1 of $U(\kappa,\xi)$ which belongs to $\mathcal{R}(\eta)$. Let

$$i^1_\xi : V_1 = V[G] \to M^1_\xi = \tilde{M}_\xi[\tilde{G}_\xi]$$

be the corresponding elementary embedding.

Let

$$\langle U_1(\kappa,\beta) \mid \beta < \kappa^+\rangle$$

be as above. We will use

$$\langle U_1(\kappa,\eta) \times U_1(\kappa,\beta) \mid \beta < \kappa^+\rangle,$$

its elementary embedding

$$i^*_\eta : V_1 \to M^*_\eta$$

and an extension

$$i^{**}_\eta : V_1[G(P_\kappa) * G(Q_\kappa)] \to M^*_\eta[G(P_{i_\eta(\kappa)}) * G(Q_{i_\eta(\kappa)})],$$

given by Lemma 6.3.
Add $U_1(\kappa,\xi)$ in the following fashion. Consider

$$\langle U_1(\kappa,\eta)\times U_1(\kappa,\beta)\times U_1(\kappa,\xi)\times U_1(\kappa,\beta)\mid \beta<\kappa^+\rangle.$$

It is still Rudin-Keisler increasing. Let

$$i^*_{\eta\xi}:V_1\to M^*_{\eta\xi}$$

be its elementary embedding into the direct limit.
It can be written also as

$$V_1 \longrightarrow^{i^*_\eta} M^*_\eta \longrightarrow^{k^*_{\eta\xi}} M^*_{\eta\xi},$$

where $k^*_{\eta\xi}$ is the canonical embedding of M^*_η into its ultrapower by the system

$$i^*_\eta(\langle U_1(\kappa,\xi)\times U_1(\kappa,\beta)\mid \beta<\kappa^+\rangle).$$

Then the following analog of Lemma 5.7 holds:

Lemma 6.4 *Let $\xi<\eta$. Then the diagram*

$$V_1 \longrightarrow^{i^*_\eta} M^*_\eta \longrightarrow^{k^*_{\eta\xi}} M^*_{\eta,\xi}$$

extends to

$$V_1[G(P_\kappa)*G(Q_\kappa)] \longrightarrow^{i^{**}_\eta} M^*_\eta[G(P_{i^*_\eta(\kappa)})*G(Q_{i^*_\eta(\kappa)})] \longrightarrow^{k^{**}_{\eta\xi}} M^*_{\eta\xi}[G(P_{i^*_{\eta\xi}(\kappa)})*G(Q_{i^*_{\eta\xi}(\kappa)})],$$

for some $M^*_{\eta\xi}-$generic subsets $G(P_{i^*_{\eta\xi}(\kappa)})*G(Q_{i^*_{\eta\xi}(\kappa)})$ of $i^*_{\eta\xi}(P_\kappa*\underset{\sim}{Q}_\kappa)$.

Proof. The proof just combines the arguments of 5.7 and 6.3.
□ of the lemma.

Lemma 6.5 *Let $\xi<\eta$ and*

$$i^{**}_{\eta\xi}:V_1[G(P_\kappa)*G(Q_\kappa)]\to M^*_{\eta\xi}[G(P_{i_\eta(\kappa)}*G(Q_{i_\eta(\kappa)})],$$

be as in the previous lemma (6.4). Then for every club $C\subseteq\kappa$ in $V_1[G(P_\kappa)*G(Q_\kappa)]$, we have

$$i^*_\eta(\kappa)\in i^{**}_{\eta\xi}(C).$$

Proof. By Lemma 6.4,

$$i^{**}_{\eta\xi} = k^{**}_{\eta\xi} \circ i^{**}_{\eta}.$$

The critical point of $k^{**}_{\eta\xi}$ is $i^*_\eta(\kappa)$ and $i^{**}_\eta(C)$ is unbounded in $i^*_\eta(\kappa)$. Hence,

$$i^*_\eta(\kappa) \in k^{**}_{\eta\xi}(i^{**}_\eta(C)) = i^{**}_{\eta\xi}(C).$$

□ of the lemma.

Suppose now that $X \subseteq \kappa, X \in K$ is stationary in the final extension $V_1[G(P_\kappa) * G(Q_\kappa)]$. Then there is $\xi < \eta$ such that $X \in U(\kappa, \xi)$. Hence,

$$\kappa \in i_\xi(X) \text{ and } i^*_\eta(\kappa) \in i^{**}_{\eta\xi}(X).$$

Also, if $C \subseteq \kappa$ is a club, then, by Lemma 6.5,

$$i^*_\eta(\kappa) \in i^{**}_{\eta\xi}(C).$$

Consider now in $V_1[G(P_\kappa) * G(Q_\kappa)]$ the following $\kappa-$complete ultrafilter:

$$U_\xi := \{Y \subseteq \kappa \mid i^*_\eta(\kappa) \in i^{**}_{\eta\xi}(Y)\}.$$

Then

$$Cub \restriction X \subseteq U_\xi.$$

□

In order to deal with arbitrary stationary sets which may be not in K, combine the previous construction (6.1) with one of 5.10. We obtain the following:

Theorem 6.6 *Suppose that there is a weak repeat point over κ in the core model. Then there is cardinal preserving extension in which for every $X \subseteq \kappa$ stationary, the filter $Cub_\kappa \restriction X$ extends to a $\kappa-$complete ultrafilter.*

7 Open problems.

Let us conclude with the following questions.

Question 1. *What is the exact consistency strength of a $\kappa-$compact cardinal κ?*

We think that it should be somewhere beyond a superstrong.

Question 2. *What is the exact consistency strength of the following statement: every normal $\kappa-$complete filter over a cardinal κ extends to a $\kappa-$complete ultrafilter?*

By previous results at least a weak repeat is needed. But may be the upper bound is below $o(\kappa) = \kappa^{++}$?

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