Collapsing generators.

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February 26, 2020

Abstract

We show that it is possible to combine the long extenders forcing of Section 2 of [1] with collapses in order to move the structure to $\aleph_\omega$. A new element here is that many generators of extenders involved are collapsed.

1 Basic setting.

Let E be an extender (or a measure). We denote by $j_E : V \to M_E \simeq \mathrm{Ult}(V, E)$ the corresponding elementary embedding. Also, we assume that $j_E(\mathrm{crit}(E)) > length(E)$.

Let us recall the basic setting for the forcing with long extenders from Section 2 of [1].

Assume GCH. Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of cardinals, $\kappa_\omega = \bigcup_{n<\omega} \kappa_n$. Assume that for every $n < \omega$, there is a $(\kappa_n, \kappa_\omega^{++})-$extender $E(n)$ over κ_n such that ${}^{\kappa_n} M_{E(n)} \subseteq M_{E(n)}$.

For every $\alpha < \kappa_\omega^{++}$, denote by $E_\alpha(n)$ the $\alpha-$th measure of $E(n)$, i.e.,

$$E_\alpha(n) = \{X \subseteq \kappa_n \mid \alpha \in j_{E(n)}(X)\}.$$

Note that $E_{\kappa_n}(n)$ is a normal measure. Given $\nu < \kappa_n$, denote by ν^0 the projection of ν to the normal measure $E_{\kappa_n}(n)$.

Let $a_n \in [\kappa_\omega^{++}]^{<\omega}$ and $s_n : [\kappa_n]^{|a_n|} \mapsto \kappa_n$ be such that $j_{E(n)}(s)(a_n) = \kappa_\omega^{++}$. Suppose for simplicity that $a_n = \{\kappa_n\}$, i.e., $j_{E(n)}(s_n)(\kappa_n) = \kappa_\omega^{++}$.

The forcing of Section 2 of [1], so called - a long extenders forcing, blows up the power of κ_ω to κ_ω^{++} by adding many $\omega-$sequences (actually Prikry sequences for measures of extenders $E(n)$'s). It preserves all cofinalities and GCH below κ_ω.

*We are grateful to Assaf Rinot for asking us about the possibility to collapse generators in the present context and to James Cummings and Sittinon Jirattikansakul for their comments and corrections. The work was partially supported by Israel Science Foundation Grant No. 1216/18.

Note that for every $n < \omega$, generators of $E(n)$ are unbounded in every cardinal in the interval $[\kappa_n^{++}, \kappa_\omega^{++}]$. So, if one likes to move down, say to $\aleph_\omega$, then generators must be collapsed.
By S. Shelah [2], there are bounds on $2^{\aleph_\omega}$, and this implies that 2^{κ_ω} should be relatively small, if we do not want to collapse cardinals above κ_ω.

We deal here with the situation where $2^{\kappa_\omega} = \kappa_\omega^{++}$. The same method will work with $2^{\kappa_\omega} = \kappa_\omega^{+k}$, for any $k, 1 \leq k < \omega$.

2 The main forcing.

Our purpose will be to define a forcing that simultaneously blows up the power of κ_ω to λ and to turn κ_ω into $\aleph_\omega$.

Let start with pure conditions.

Definition 2.1 The set $\mathcal{P}^*$ consists of elements $p = \langle p_n \mid n < \omega \rangle$ such that

1. $p_0 = \langle \langle a_0, A_0, f_0 \rangle, F_0^0, F_0^1, F_0^2 \rangle$, where
   (a) $\langle a_0, A_0, f_0 \rangle$ is a condition in Q_0 defined as in [1],
   (b) $\kappa_0, \kappa_\omega^+ \in a_0$,
   (c) $a_0 \cap \kappa_\omega^+$ has a maximal element,
   (d) for every $i < 3$, $\mathrm{dom}(F_0^i) = A_0 \restriction \kappa_\omega^+$, i.e., the projection of A_0 from $\max(a_0)$ to $\max(a_0 \cap \kappa_\omega^+)$.
   For every $\nu \in A_0 \restriction \kappa_\omega^+$,
      i. $F_0^0(\nu) \in Col(\omega_1, < \nu^0)$,
      ii. $F_0^1(\nu) \in Col(\nu^0, s_0(\nu^0)^+)$, i.e., the cardinal that corresponds to κ_ω^+ is collapsed to the one that corresponds to κ_0,
      iii. $F_0^2 \in Col(s_0(\nu^0)^{+3}, < \kappa_1)$.
      The meaning of this is that the further collapse starts already above the things that correspond to the generators of $E(0)$. The generators between κ_ω^+ and κ_ω^{++} are left untouched.

2. Let $n > 0$. Then $p_n = \langle \langle a_n, A_n, f_n \rangle, F_n^0, F_n^1, F_n^2 \rangle$, where
   (a) $\langle a_n, A_n, f_n \rangle$ is a condition in Q_n defined as in [1],

(b) $\kappa_n, \kappa_\omega^+ \in a_n$,

(c) $a_n \cap \kappa_\omega^+$ has a maximal element,

(d) for every $i < 3$, $\mathrm{dom}(F_n^i) = A_n \restriction \kappa_\omega^+$, i.e., the projection of A_n from $\max(a_n)$ to $\max(a_n \cap \kappa_\omega^+)$.

For every $\nu \in A_n \restriction \kappa_\omega^+$,

i. $F_n^0(\nu) \in Col(\kappa_{n-1}^+, < \nu^0)$,

ii. $F_n^1(\nu) \in Col(\nu^0, s_n(\nu^0)^+)$, i.e., the cardinal that corresponds to κ_ω^+ is collapsed to the one that corresponds to κ_n,

iii. $F_n^2(\nu) \in Col(s_n(\nu^0)^{+3}, < \kappa_{n+1})$.
The meaning of this is that the further collapse starts already above the things that correspond to the generators of $E(n)$. The generators between κ_ω^+ and κ_ω^{++} are left untouched.

Define the direct extension order on $\mathcal{P}^*$.

Definition 2.2 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle$ be two elements of $\mathcal{P}^*$. Set $q \leq^* p$ iff for every $n < \omega$, $q_n = \langle \langle b_n, B_n, h_n \rangle, H_n^0, H_n^1, H_n^2 \rangle \leq_n p_n = \langle \langle a_n, A_n, f_n \rangle, F_n^0, F_n^1, F_n^2 \rangle$, where $\leq_n$ means the following

1. $\langle a_n, A_n, f_n \rangle$ extends $\langle b_n, B_n, h_n \rangle$, as in [1],

2. for every $i < 3$, for every $\nu \in A_n \restriction \kappa_\omega^+$, $F_n^i(\nu) \supseteq H_n^i(\pi_{\max(A_n \restriction \kappa_\omega^+), \max(B_n \restriction \kappa_\omega^+)}(\nu))$.

Define now a non-direct extension order.

Definition 2.3 Let $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}^*$ and $p_0 = \langle \langle a_0, A_0, f_0 \rangle, F_0^0, F_0^1, F_0^2 \rangle$. Let $\rho \in A_0$. Define $p^\frown\rho$ to be $\langle p_0{}^\frown\rho, p_1, p_2, ..., p_n, ... \rangle$, where $p_0{}^\frown\rho = \langle \langle a_0, A_0, f_0 \rangle^\frown\rho, h_0^0, h_0^1, h_0^2 \rangle$ is such that

1. $\langle a_0, A_0, f_0 \rangle^\frown\rho$ is defined as in [1],

2. $h_0^i = F_0^i(\nu)$, for every $i < 3$, where $\nu = \pi_{\max(a_0), \max(a_0 \cap \kappa_\omega^+)}(\rho)$.

Given $n < \omega$ and a sequence $\langle \rho_k \mid k \leq n \rangle \in \prod_{k \leq n} A_k$, define $p^\frown\langle \rho_k \mid k \leq n \rangle$ similar.
Set $\mathcal{P}$ to be the set of all such $p^\frown\langle \rho_k \mid k \leq n \rangle$.

Definition 2.4 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle$ be two elements of $\mathcal{P}$. We define $p \geq q$ iff there are $n < \omega$ and a sequence $\langle \rho_k \mid k \leq n \rangle \in \prod_{k \leq n} A_k^q$ such that $p \geq^* q^\frown\langle \rho_k \mid k \leq n \rangle$.

Lemma 2.5 *The forcing notion $\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. Let $p \in \mathcal{P}$ and σ be a statement in the forcing language. We need to show that there is $p^* \geq^* p$ such that $p^* \parallel \sigma$.

Suppose for simplicity that $p \in \mathcal{P}^*$ and assume that no direct extension of p decides σ. Let $p = \langle p_n \mid n < \omega \rangle$ and for every $n < \omega$, $p_n = \langle \langle a_n, A_n, f_n \rangle, F_n^0, F_n^1, F_n^2 \rangle$.
As usual we go by induction through all $\rho \in A_0 \subseteq \kappa_0$ and ask wether a condition $p(\rho)$ from a stage ρ extended non-directly by ρ has a direct extension which decides σ.
So suppose that $\rho \in A_0$ and an $\leq^*$ $-$increasing sequence $\langle p(\rho') \mid \rho' \in A_0 \cap \rho \rangle$ of direct extensions of p is already defined. We assume that $p(\rho')_0 = p_0$, for every $\rho' \in A_0 \cap \rho$.
Let $p(\rho)'$ to be the upper bound of $\langle p(\rho') \mid \rho' \in A_0 \cap \rho \rangle$.
It exists, since for every $n, 1 \leq n < \omega$, all the collapses in $p(\rho')_n$'s are κ_n-closed, as well as all the measures $E_n(\max(a_n))$ are κ_n-complete.
Next, we ask whether $p(\rho)'^\frown \rho$ has a direct extension which decides σ. If there is no such extension, then set $p(\rho) = p(\rho)'$. Otherwise, we pick a direct extension $q(\rho)$ of $p(\rho)'^\frown \rho$ which decides σ.
Set $p(\rho)_0 = \langle \langle a_0, A_0, q(\rho) \restriction \mathrm{dom}(q(\rho)) \setminus a_0 \rangle, F_0^0, F_0^1, F_0^2 \rangle$
and for every $n, 1 \leq n < \omega$, set $p(\rho)_n = q(\rho)_n$.

This completes the inductive construction[1] .
Suppose for simplicity that there is $A \subseteq A_0, A \in E_0(\max(a_0))$ such that for every $\rho \in A$, $q(\rho)$ decides σ. Without loss of generality, say $q(\rho) \Vdash \sigma$. Abusing notation a bit, let us identify between A and A_0.
Define a direct extension p^* of p as follows.
First, for every $n, 1 \leq n < \omega$, we take p_n^* to be the upper bound of $p_n(\rho)$'s, $\rho \in A_0$, it is possible due to κ_0^+-completeness of all components involved.
Finally, let us define p_0^*. Let $a_0(p^*) = a_0)$, $A_0(p^*) = A_0$ and $f_0(p^*)$ be the union of the relevant Cohen parts.
Turn to collapses. This is the crucial point here.
We have, for every $\rho \in A_0(p^*)$, collapsing functions
$h_0^0(q(\rho)) \in Col(\omega_1, < \rho^0), h_0^1(q(\rho)) \in Col(\rho^0, s_0(\rho^0)^+), h_0^2(q(\rho)) \in Col(\rho^0)^{+3}, < \kappa_1)$. It is tempting to make them into a condition by using the map

$$\rho \mapsto \langle h_0^0(q(\rho)), h_0^1(q(\rho)), h_0^2(q(\rho)) \rangle.$$

[1]The general case requires a repeating of the above construction $\omega-$many times, however the main argument is the same.

However, the requirement is that this should depend only on $A_0(p^*) \restriction \kappa_\omega^+$, and not on $A_0(p^*)$.
Note that actually h_0^0 and h_0^1 are problematic, since $h_0^2(q(\rho)) \in Col(s_0(\rho^0)^{+3}, < \kappa_1)$ and $s_0(\rho^0)^{+3}$ corresponds to κ^{+3} which is above all the generators of E_0. The treatment here is a standard one, see [1].
Let us deal so with h_0^0 and h_0^1 only.
Denote the function $\rho \mapsto \langle h_0^0(q(\rho)), h_0^1(q(\rho))\rangle$ by t.
Then, in M_{E_0}, $j_{E_0}(t)(\max(a_0(p^*)) \in V_{\kappa_\omega+1}^{M_{E_0}}$.
The next claim solves this matter.

Claim 1 Suppose that $\alpha < \kappa_\omega^{++}, r : \kappa_0 \to V_{\kappa_0}$ such that $j_{E_0}(r)(\alpha) \in V_{\kappa_\omega+1}^{M_{E_0}}$.
Then there are $\alpha' < \kappa_\omega^+$ and $r' : \kappa_0 \to V_{\kappa_0}$ such that $j_{E_0}(r)(\alpha) = j_{E_0}(r')(\alpha')$.

Proof. Consider $E' = E_0 \restriction \kappa_\omega^+$. It is a $(\kappa_0, \kappa_\omega^+)-$extender and there is a natural embedding $k : M_{E'} \to M_E$ defined by setting $k(j_{E'}(f)(a) = j_E(f)(a)$.
The critical point of k is $((\kappa_\omega^+)^+)^{M_{E'}} > \kappa_\omega^+$.
In particular, $j_{E_0}(r)(\alpha) \in M_{E'}$. Hence, $\alpha' < \kappa_\omega^+$ and $r' : \kappa_0 \to V_{\kappa_0}$ such that

$$j_{E_0}(r)(\alpha) = j_{E_0'}(r')(\alpha') = k(j_{E_0'}(r')(\alpha')) = j_{E_0}(r')(\alpha').$$

□ of the claim.

Applying the claim, we can replace now the function $\rho \mapsto \langle h_0^0(q(\rho)), h_0^1(q(\rho))\rangle$ defined on $A_0(p^*)$ by an equivalent function defined on $A_0(p^*) \restriction \kappa_\omega^+$.

This completes the argument.

□

It follows now by standard arguments that in a generic extension $V^{\langle \mathcal{P}, \leq \rangle}$, κ_ω is turned into $\aleph_\omega$ and below it only ω, ω_1 and $\eta_n, s_n(\eta_n)^{++}, s_n(\eta_n)^{+3}, \kappa_n, \kappa_n^+$, for $n < \omega$, are preserved, where $\langle \eta_n \mid n < \omega\rangle$ denotes the Prikry sequence for the normal measures $\langle E_n(\kappa_n) \mid n < \omega\rangle$.

The cardinal structure above κ_ω does not change. First κ_ω^+ is preserved, since otherwise it changes its cofinality to a cardinal $< \kappa_\omega$ and this rules out by a standard arguments, see [1].
A preservation of κ_ω^{++} and bigger cardinals follows from the following lemma.

Lemma 2.6 *The forcing $\langle \mathcal{P}, \leq \rangle$ satisfies $\kappa_\omega^{++}-c.c.$*

Proof. Again the proof is rather standard. Just note that the collapse components depend only on $E_n \restriction \kappa_\omega^+$'s. This implies that the size in V, of the collapses forcing in the iterated ultrapowers by E_n's is only κ_ω^+. But then it is impossible to have κ_ω^{++} incompatible conditions. Moreover, given κ_ω^{++} many conditions it is possible to find a subfamily of size κ_ω^{++} which consists of pairwise compatible elements.
□

References

[1] M. Gitik, Prikry type forcings, in Handbook of Set Theory, Foreman, Kanamori, eds. v.2, pages 1351-1448, Springer, 2010.

[2] S. Shelah, Cardinal Arithmetic.