Adding Cohen functions to an ultrapower

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April 30, 2021

Abstract

Starting with two measurable cardinals, we construct a model with a $\sigma-$complete uniform ultrafilter over a cardinal which is not a strong limit.

1 Introduction

By a classical result of A. Tarski and S. Ulam a measurable cardinal must be a strong limit. In other words, if a cardinal λ carries a uniform $\lambda-$complete ultrafilter, then for every $\mu < \lambda$, $2^\mu < \lambda$. If we replace $\lambda-$completeness by just $\sigma-$completeness, then this need not be true anymore. For example if there is a supercompact cardinal κ which is indestructible under $\kappa-$directed closed forcing, then for every given regular $\lambda > \kappa$ there is a forcing extension in which λ is not strong limit but still carries a uniform $\kappa-$complete ultrafilter.

The purpose of this paper is to show that actually two measurable cardinals are enough in order to construct a model with a uniform $\sigma-$complete ultrafilter over a cardinal which is not a strong limit. Note that if there is a uniform $\sigma-$complete ultrafilter over λ with the critical point $< \lambda$, then there is an inner model with at least two measurable cardinals.

Let us explain the basic idea behind the construction.
Suppose that λ is a measurable cardinal and U is a normal ultrafilter over λ. Let $\mu < \lambda$ be a regular cardinal and $\delta \geq \lambda$. Force δ Cohen functions $\langle r_i \mid i < \delta\rangle, r_i : \mu \to 2$. Then, in order to extend the elementary embedding $j_U : V \to M_U \simeq {}^\lambda V/U$ to one from $V[\langle r_i \mid i < \delta\rangle]$ one will need more Cohen functions. Namely, the sequence $\langle r_i \mid i < \delta\rangle$ must be stretched to a sequence $\langle r'_i \mid i < j_U(\delta)\rangle$ such that for every $\alpha < \delta$, $r_\alpha = r'_{j_U(\alpha)}$. This requires more Cohen functions over $V[\langle r_i \mid i < \delta\rangle]$. For example r'_λ should be Cohen generic over $V[\langle r_i \mid i < \delta\rangle]$. Clearly, we do not have such r'_λ in $V[\langle r_i \mid i < \delta\rangle]$.

*The work was partially supported by Israel Science Foundation Grant No. 1216/18.

Let us assume now that instead of a single measurable λ, we have two measurable cardinals $\kappa < \lambda$. Fix normal ultrafilters U_κ over κ and U_λ over λ. Let M be the ultrapower by U_κ and j be the corresponding elementary embedding. Then, in M, consider $j(U_\lambda)$. It is a normal ultrafilter over λ generated by U_λ. Take the ultrapower of M by $j(U_\lambda)$. Let $\pi : M \to N$ be the corresponding elementary embedding.
Suppose now that there are (in an extension V^* of V) δ Cohen over M functions $\langle r_i \mid i < \delta \rangle$, for some $\delta \geq \lambda$. As above we will try to extend π to an embedding $\pi' : M[\langle r_i \mid i < \delta \rangle] \to N[\langle r'_i \mid i < \pi(\delta) \rangle]$. Again, more Cohen functions are needed in order to preform the tusk. However, now they need be generic only over the model $M[\langle r_i \mid i < \delta \rangle]$ and not over V^*. So, there is a chance to find "the missing" Cohen's inside V^*.
We will show that indeed this can be done.

The paper is organized as follows. In Section 2 we describe the main idea of the paper by dealing with a measurable κ and building a single Cohen function from κ^+ to 2 over the ultrapower. In Section 3, this is generalized to many Cohen functions. The main result is shown there. We use the result of P. Lücke and S. Müller [1] on fresh subsets in the last section in order to replace κ^+ by arbitrary regular cardinal.

2 A basic construction

Assume GCH. Let κ be a measurable cardinal. Force κ^+-many Cohen functions $\langle r_i \mid i < \kappa^+ \rangle$ to κ^+, i.e. we force with

$$Cohen(\kappa^+, \kappa^+) = \{p \mid p : \kappa^+ \times \kappa^+ \to 2 \text{ is a partial function of cardinality at most } \kappa\}.$$

Let G be a generic subset. Then $r_i(\alpha) = p(i, \alpha)$, where $p \in G$ and $(i, \alpha) \in \text{dom}(p)$.

Let U be a normal ultrafilter over κ in V or the same in $V[\langle r_i \mid i < \kappa^+ \rangle]$, since no new subsets are added to κ be this forcing.
Consider $j_U : V \to M_U \simeq {}^\kappa V/U$. Denote j_U by j and M_U by M.
Then j extends canonically to

$$j' : V[\langle r_i \mid i < \kappa^+ \rangle] \to M[\langle r'_i \mid i < j(\kappa^+) \rangle] \simeq {}^\kappa V[\langle r_i \mid i < \kappa^+ \rangle]/U.$$

Note that $j''G$ generates the corresponding generic over M and so $\langle r'_i \mid i < j(\kappa^+) \rangle$.

Our aim here will be to show that there is (in $V[G]$) $r : j(\kappa^+) \to 2$'s which are Cohen generic over $M[\langle r'_i \mid i < j(\kappa^+) \rangle]$.

Let us use $j''\kappa^+$ in order to define such r.

Define $r(\beta)$ as follows:

if $\beta = j(\alpha)$, then let $r(\beta) = r'_\beta(\beta)$;

if $\beta \notin j''\kappa^+$, then let γ be the least such that $\beta < j(\gamma)$. Set $r(\beta) = r'_{j(\gamma)}(\beta)$.

Note that each initial segment $r \restriction \beta, \beta < j(\kappa^+)$ of r is in M, since its definition requires only κ members from the set $\{r'_{j(\gamma)} \mid \gamma < \kappa^+\}$ and $M[\langle r'_i \mid i < j(\kappa^+)\rangle]$ is closed under $\kappa-$sequences of its elements.

Let us argue that such r is a Cohen generic over $M[\langle r'_i \mid i < j(\kappa^+)\rangle]$.

Suppose otherwise. Then there is a dense open $D \subseteq Cohen(\kappa^+)$ in $V[G]$ such that $r \restriction \beta \notin j'(D)$, for every $\beta < j(\kappa^+)$, where $Cohen(\kappa^+)$ is the Cohen forcing for adding a single function, i.e.,

$$Cohen(\kappa^+) = \{p \mid \mathrm{dom}(p) \in \kappa^+ \mid p : \mathrm{dom}(p) \to 2\}.$$

Note that due to $< \kappa^+-$closure of the forcing $Cohen(\kappa^+)$, every dense open set in the ultrapower contains an image of one from $V[G]$.

Let $\underset{\sim}{D}$ and $\underset{\sim}{r}$ be $Cohen(\kappa^+, \kappa^+)-$names of D and r respectively. We assume for simplicity that the empty condition forces this.

For every $\alpha < \kappa^+$, pick some $d_\alpha \in j'(D)$ such that $d_\alpha \geq r \restriction j(\alpha) + 1$.

Then, clearly, $d_\alpha \in M$, since G' does not add new bounded subsets of $j(\kappa^+)$. Fix a function $f_\alpha : \kappa \to D$ which represents d_α.

We would like to change one value of some of f_α's.

Thus, suppose that $\mathrm{cof}(\alpha) = \tau, \kappa > \tau > \aleph_0$ and assume that there is a club $C_\alpha \subseteq \alpha$ such that for every $\tilde{\nu}, \tilde{\rho} \in C_\alpha$, if $\tilde{\nu} \leq \tilde{\rho}$, then

$$f_{\tilde{\nu}}(\tau) \restriction \tilde{\nu} + 1 = f_{\tilde{\rho}}(\tau) \restriction \tilde{\nu} + 1.$$

In this case we set $f_\alpha(\tau)$ to be an element of D stronger than $\bigcup_{\tilde{\nu} \in C_\alpha}(f_{\tilde{\nu}}(\tau) \restriction \tilde{\nu} + 1)$.

For every $\alpha < \kappa^+$, pick $p_\alpha \in G$ which decides $f_\alpha(\tau)$ and forces that it is in D, for every $\tau < \kappa$.

Define

$$h(\alpha) = \max(\sup(\bigcup_{\nu < \kappa} \mathrm{dom}(f_\alpha(\nu)), \mathrm{dom}(p_\alpha) + 1)).$$

So, $h : \kappa^+ \to \kappa^+$.

Then there is a club $C \subseteq \kappa^+$ such that for every $\alpha \in C$ and every $\beta < \alpha$,

$$\alpha > \max(h(\beta), \sup(\{\max(\tau, \rho) \mid (\tau, \rho) \in \mathrm{dom}(p_\beta)\})).$$

Let $\langle \alpha_\xi \mid \xi \leq \kappa \rangle$ be the first $\kappa+1-$th elements of C.
Note that this sequence is constructed using only $\langle r_\nu \mid \nu < \alpha_\kappa \rangle$, due to its continuity. This leaves r_{α_κ} free.

Define a function $f : \kappa \to D$ by setting $f(\nu) = f_{\alpha_\nu}(\nu)$.
Consider $f' = j(f)(\kappa)$. Clearly, by elementarity, it is in $j'(D)$.

Claim. *For every $\nu < \kappa$, $f' \restriction j(\alpha_\nu)+1 = d_{\alpha_\nu} \restriction j(\alpha_\nu)+1$.*

Proof. For every $\nu < \rho < \kappa$, $d_{\alpha_\rho} \restriction j(\alpha_\rho)+1 = r \restriction j(\alpha_\rho)+1$ and $d_{\alpha_\nu} \restriction j(\alpha_\nu)+1 = r \restriction j(\alpha_\nu)+1$, and hence by elementarity, the set

$$B_{\nu\rho} = \{\tau < \kappa \mid f_{\alpha_\nu}(\tau) \restriction \alpha_\nu + 1 = f_{\alpha_\rho}(\tau) \restriction \alpha_\nu + 1\} \in U.$$

Set $E = \Delta_{\nu<\rho<\kappa} B_{\nu\rho}$. Then, by normality, E is in U as well.
Let $\tau \in E$ be say an inaccessible. Consider α_τ. Then $Z = \{\alpha_\nu \mid \nu < \tau\}$ is a club in α_τ.
Let $\nu < \rho < \tau$. Then $\tau \in B_{\nu\rho}$, and so, $f_{\alpha_\nu}(\tau) \restriction \alpha_\nu + 1 = f_{\alpha_\rho}(\tau) \restriction \alpha_\nu + 1$.
Due to the change made in $f_{\alpha_\tau}(\tau)$, then

$$f_{\alpha_\tau}(\tau) \restriction \alpha_\tau = \bigcup_{\nu<\tau} f_{\alpha_\nu}(\tau) \restriction \alpha_\nu + 1.$$

By the definition of f, then for every $\tau \in E$,

$$f(\tau) \restriction \alpha_\tau = f_{\alpha_\tau}(\tau) \restriction \alpha_\tau = \bigcup_{\nu<\tau} f_{\alpha_\nu}(\tau) \restriction \alpha_\nu + 1.$$

Now, in the ultrapower,

$$f' \restriction j(\langle \alpha_\tau \mid \tau < \kappa \rangle)(\kappa) = j(f)(\kappa) \restriction j(\langle \alpha_\tau \mid \tau < \kappa \rangle)(\kappa) = \bigcup_{\nu<\kappa} d_{\alpha_\nu} \restriction j(\alpha_\nu) + 1.$$

$\square$ of the claim.

Note that $\bigcup_{\tau<\kappa} j(\alpha_\tau) = j(\langle \alpha_\tau \mid \tau < \kappa \rangle)(\kappa)$, due to the continuity of the sequence. Denote $\bigcup_{\tau<\kappa} j(\alpha_\tau)$ by α^*.
Then, $j(f)(\kappa) \restriction \alpha^* = r \restriction \alpha^*$. Also, $j(f)(\kappa) \in j'(D)$. Clearly, $\alpha^* < j(\alpha_\kappa)$.
Recall that by the definition of h, we have that $j'(h)(j(\alpha_\nu)) < j(\alpha_{\nu+1}) < \alpha^*$ and $j'(h)(\alpha^*) < j(\alpha_\kappa)$.

We have $f(\tau) = f_{\alpha_\tau}(\tau)$ and $h(\alpha_\tau) > \sup(\mathrm{dom}(f_{\alpha_\tau}(\tau)))$, for every $\tau < \kappa$.
Hence, $j'(h)(\alpha^*) > \sup(\mathrm{dom}(j(f)(\kappa))$.

Let us now explore the fact that we are free to choose r_{α_κ}.
Construct it inductively such that for every $\tau < \kappa$, $r_{\alpha_\kappa} \restriction [\alpha_\tau, \alpha_{\tau+1}) \geq f(\tau) \setminus \alpha_\tau$.

Then, in M', $r'_{j(\alpha_\kappa)} \restriction [\alpha^*, j(\alpha_\kappa)) \geq j(f)(\kappa)\setminus\alpha^*$. By the definition of r, then $r \restriction j(\alpha_\kappa) \in j'(D)$, as forced by $r_{\alpha_\kappa} \restriction \alpha_\kappa$ over $V[\langle r_\beta \mid \beta < \alpha_\kappa\rangle]$.
Contradiction.

3 Adding many Cohen functions

Suppose that for some regular $\lambda > \kappa^+$, instead of κ^+-many Cohen functions $\langle r_i \mid i < \kappa^+\rangle$ we add $\lambda-$many, i.e. we force with $Cohen(\kappa^+, \lambda)$ instead of $Cohen(\kappa^+, \kappa^+)$.

Lemma 3.1 *For every $a \subseteq \lambda$ and $\beta < j(\kappa^+)$ such that $a \in M$ and $M \models |a| \leq j(\kappa)$, $\langle r(\alpha) \restriction \beta \mid \alpha \in a\rangle \in M$.*

Proof. Note first that $j''\beta \in M$, since M is closed under $\kappa-$sequences of ordinals.

Define (in V) the set

$$a^* = \{\tau \in j''\lambda \mid \exists \alpha \in a(\tau = \min(j''\lambda \setminus \alpha)\}.$$

We claim that $|a^*| \leq \kappa$, and hence, a^* is in M. The rest follows by the definition of $r(\alpha)$'s.
Let us proceed by induction and argue that for every $\eta \leq \sup(a^*)$, $|a^* \cap \eta| \leq \kappa$.
Suppose otherwise. Then pick the least $\eta \leq \sup(a^*)$, such that $|a^* \cap \eta| > \kappa$. Clearly, such η should be a limit ordinal of cofinality κ^+.
Now, by the definition of a^*, $a \cap \eta$ should be unbounded in η. $M \models |a| \leq j(\kappa)$, hence $M \models \mathrm{cof}(\eta) \leq j(\kappa)$.
Let b be the pre-image of $a^* \cap \eta$, i.e.,

$$b = \{\nu < \lambda \mid j(\nu) \in a^* \cap \eta\}.$$

Then $\sup(j''b) = \sup(j(b))$, since we can write b as an increasing sequence $\langle \mu_i \mid i < \kappa^+\rangle$ of ordinals less than λ, each element ζ of $a^* \cap \eta$ is then of the form $j(\mu_i)$, for some $i < \kappa^+$. So,

$$j(b) = j(\{\mu_i \mid i < \kappa^+\} = \{\mu'_\zeta \mid \zeta < j(\kappa^+)\}.$$

We have $\sup(j''\kappa^+) = j(\kappa^+)$, hence $\sup(j''b) = \sup(a^* \cap \eta) = \eta$.
However $M \models \mathrm{cof}(\sup(j(b)) = j(\mathrm{cof}(\sup(b))) = j(\kappa^+)$. Contradiction.
□

Lemma 3.2 *Let $\eta \leq \lambda$ be a cardinal of cofinality $> \kappa$. Suppose that $a \subseteq j(\eta)$ is such that $a \in M$ and $M \models |a| \leq j(\kappa)$. Set $a^* = \{\tau \in j''\eta \mid \exists \alpha \in a(\tau = \min(j''\eta \setminus \alpha)\}$. Then $|a^*| \leq \kappa$. In particular, $a^* \in M$.*

Proof. Let us proceed by induction and argue that for every $\mu \le \sup(a^*)$, $|a^* \cap \mu| \le \kappa$. Suppose otherwise. Then pick the least $\mu \le \sup(a^*)$, such that $|a^* \cap \mu| > \kappa$. Clearly, such μ should be a limit ordinal of cofinality κ^+.
Clearly, $a \cap \mu$ should be unbounded in μ.
Now, $M \models |a| \le j(\kappa)$, hence $M \models \mathrm{cof}(\mu) \le j(\kappa)$.
Let b be the pre-image of $a^* \cap \eta$, i.e.,

$$b = \{\nu < \eta \mid j(\nu) \in a^* \cap \mu\}.$$

Then $\sup(j''b) = \sup(j(b))$, since we can write b as an increasing sequence $\langle \xi_i \mid i < \kappa^+ \rangle$ of ordinals less than η, each element ζ of $a^* \cap \eta$ is then of the form $j(\xi_i)$, for some $i < \kappa^+$. So,

$$j(b) = j(\{\xi_i \mid i < \kappa^+\} = \{\xi'_i \mid i < j(\kappa^+)\}.$$

We have $\sup(j''\kappa^+) = j(\kappa^+)$, hence $\sup(j''b) = \sup(a^* \cap \mu) = \mu$.
However $M \models \mathrm{cof}(\sup(j(b)) = j(\mathrm{cof}(\sup(b))) = j(\kappa^+)$. Contradiction.
□

Let us show a similar statement for cardinals of cofinality κ.

Lemma 3.3 *Let $\eta \le \lambda$ be a cardinal $\ge \kappa$. Suppose that $a \subseteq (j(\eta)^{+\kappa+1})^M$ is such that $a \in M$ and $M \models |a| \le j(\kappa)$. Let $\langle f_i \mid i < \eta^{+\kappa+1} \rangle$ be a scale in $\prod_{\nu<\kappa} \eta^{+\nu+1}$.*
Set $a^* = \{\tau \in (j(\eta)^{+\kappa+1})^M \mid \exists \alpha \in a \exists i < \eta^{+\kappa+1}(\tau = j(f_i)(\kappa) > \alpha \wedge \tau$ is the least like this $\}$.
Then $|a^*| \le \kappa$. In particular, $a^* \in M$.

Remark 3.4 The first η like this is $\kappa^{+\kappa} = \bigcup_{\nu<\kappa} \kappa^{+\nu}$.

Proof. Note that every $\alpha < (j(\eta)^{+\kappa+1})^M$ is of the form $j(f)(\kappa)$, for some $f \in \prod_{\nu<\kappa} \eta^{+\nu+1}$. Since, if $f : \kappa \to On$ is a function which represents α, then, by elementarity,

$$\{\nu < \kappa \mid f(\nu) \in \eta^{+\nu+1}\} \in U.$$

Let i_α be the least i such that $f_i \ge f$ and $\alpha^* = j(f_{i_\alpha})(\kappa)$. Then α^* in a^*, as witness by α and i_α.
Let $\vec{f} = \langle f_i \mid i < \eta^{+\kappa+1} \rangle$. Consider $\vec{f'} = \langle f'_i \mid i < j(\eta^{+\kappa+1}) \rangle$. Then

$$a^* \subseteq \{f'_{j(i)}(\kappa) \mid i < \eta^{+\kappa+1}\}.$$

Suppose otherwise. Then pick the least $\mu \le \sup(a^*)$, such that $|a^* \cap \mu| > \kappa$. Clearly, such μ should be a limit ordinal of cofinality κ^+.

Clearly, $a \cap \mu$ should be unbounded in μ.
Now, $M \models |a| \leq j(\kappa)$, hence $M \models \mathrm{cof}(\mu) \leq j(\kappa)$.
Let

$$b = \{i_\alpha \mid \alpha \in a^* \cap \mu\}.$$

Then $\sup(j''b) = \sup(j(b))$.We have $\sup(j''\kappa^+) = j(\kappa^+)$, hence $\sup(\{f'_{j(i)}(\kappa) \mid i \in b) = \sup(a^* \cap \mu) = \mu$.
However $M \models \mathrm{cof}(\sup(j(b)) = j(\mathrm{cof}(\sup(b))) = j(\kappa^+)$. Contradiction.
□

Lemma 3.5 *Let $\eta \leq \lambda$ be a cardinal of cofinality κ. Let $\langle \eta_\nu \mid \nu < \kappa\rangle$ be a club in η such that $\mathrm{cof}(\eta_\nu) = \nu$, for every $\nu < \kappa$. Let $\langle f_i \mid i < \eta^+\rangle$ be a scale in $\prod_{\nu<\kappa} \eta_\nu^+$. Let $\langle \eta'_\nu \mid \nu < j(\kappa)\rangle = j(\langle \eta_\nu \mid \nu < \kappa\rangle)$.*
Suppose that $a \subseteq ({\eta'_\kappa}^+)^M$ is such that $a \in M$ and $M \models |a| \leq j(\kappa)$.
Set $a^* = \{\tau \in ({\eta'_\kappa}^+)^M \mid \exists \alpha \in a \exists i < \eta^+(\tau = j(f_i)(\kappa) > \alpha \wedge \tau$ is the least like this $\}$. Then $|a^*| \leq \kappa$. In particular, $a^* \in M$.

Proof. Note that every $\alpha < ({\eta'_\kappa}^+)^M$ is of the form $j(f)(\kappa)$, for some $f \in \prod_{\nu<\kappa} \eta_\nu^+$. Since, if $f : \kappa \to On$ is a function which represents α, then, by elementarity,

$$\{\nu < \kappa \mid f(\nu) \in \eta_\nu^+\} \in U.$$

Let i_α be the least i such that $f_i \geq f$ and $\alpha^* = j(f_{i_\alpha})(\kappa)$. Then α^* in a^*, as witness by α and i_α.
Let $\vec{f} = \langle f_i \mid i < \eta^+\rangle$. Consider $\vec{f'} = \langle f'_i \mid i < j(\eta^+)\rangle$. Then

$$a^* \subseteq \{f'_{j(i)}(\kappa) \mid i < \eta^+\}.$$

Suppose otherwise. Then pick the least $\mu \leq \sup(a^*)$, such that $|a^* \cap \mu| > \kappa$. Clearly, such μ should be a limit ordinal of cofinality κ^+.
Clearly, $a \cap \mu$ should be unbounded in μ.
Now, $M \models |a| \leq j(\kappa)$, hence $M \models \mathrm{cof}(\mu) \leq j(\kappa)$.
Let

$$b = \{i_\alpha \mid \alpha \in a^* \cap \mu\}.$$

Then $\sup(j''b) = \sup(j(b))$.We have $\sup(j''\kappa^+) = j(\kappa^+)$, hence $\sup(\{f'_{j(i)}(\kappa) \mid i \in b) = \sup(a^* \cap \mu) = \mu$.
However $M \models \mathrm{cof}(\sup(j(b)) = j(\mathrm{cof}(\sup(b))) = j(\kappa^+)$. Contradiction.
□

Lemma 3.6 *Let $\eta \leq \lambda$ be a cardinal of cofinality κ. Let $\eta_\kappa = \sup(j''\eta)$. Let $\rho, \eta_\kappa \leq \rho < j(\eta)$ be a regular cardinal in M. Let h_ρ be a one-to-one function which represents ρ, such that for every $\nu < \kappa$, $h(\nu)$ is a regular cardinal.*
Let $\langle f_i \mid i < \eta^+ \rangle$ be a scale in $\prod_{\nu<\kappa} h(\nu)$.
Suppose that $a \subseteq \rho$ is such that $a \in M$ and $M \models |a| \leq j(\kappa)$.
Set $a^* = \{\tau \in \rho \mid \exists \alpha \in a \exists i < \eta^+ (\tau = j(f_i)(\kappa) \geq \alpha \wedge \tau$ is the least like this $\}$. Then $|a^*| \leq \kappa$. In particular, $a^* \in M$.

Proof. Note that every $\alpha < \rho = j(h)(\kappa)$ is of the form $j(f)(\kappa)$, for some $f \in \prod_{\nu<\kappa} h(\nu)$. Since, if $f : \kappa \to On$ is a function which represents α, then, by elementarity,

$$\{\nu < \kappa \mid f(\nu) \in h(\nu)\} \in U.$$

Let i_α be the least i such that $f_i \geq f$ and $\alpha^* = j(f_{i_\alpha})(\kappa)$. Then α^* in a^*, as witness by α and i_α.
Let $\vec{f} = \langle f_i \mid i < \eta^+ \rangle$. Consider $\vec{f'} = \langle f'_i \mid i < j(\eta^+) \rangle$. Then

$$a^* \subseteq \{f'_{j(i)}(\kappa) \mid i < \eta^+\}.$$

Suppose otherwise. Then pick the least $\mu \leq \sup(a^*)$, such that $|a^* \cap \mu| > \kappa$. Clearly, such μ should be a limit ordinal of cofinality κ^+.
Clearly, $a \cap \mu$ should be unbounded in μ.
Now, $M \models |a| \leq j(\kappa)$, hence $M \models \mathrm{cof}(\mu) \leq j(\kappa)$.
Let

$$b = \{i_\alpha \mid \alpha \in a^* \cap \mu\}.$$

Then $\sup(j''b) = \sup(j(b))$.We have $\sup(j''\kappa^+) = j(\kappa^+)$, hence $\sup(\{f'_{j(i)}(\kappa) \mid i \in b) = \sup(a^* \cap \mu) = \mu$.
However $M \models \mathrm{cof}(\sup(j(b)) = j(\mathrm{cof}(\sup(b))) = j(\kappa^+)$. Contradiction.
□

Lemma 3.7 *Let $\zeta \leq \lambda$ be a cardinal in M. Then either*

1. *for some cardinal $\xi, j(\xi) = \zeta$,*
   or

2. *there is a cardinal ξ such that*

   (a) $\mathrm{cof}(\xi) = \kappa$,

(b) $\sup(j''\xi) \leq \zeta < j(\xi)$.

Proof. Suppose that ζ is not in the image of j. Pick then ξ to be the least ordinal such that $\sup(j''\xi) \leq \zeta < j(\xi)$. Then $\mathrm{cof}(\xi) = \kappa$, since U is a normal ultrafilter over κ. If ξ is not a cardinal, then $|\xi| < \xi$. By elementarity, in M, $j(|\xi|) = |j(\xi)|$. But

$$j(|\xi|) < \sup(j''\xi) \leq \zeta < j(\xi)$$

and ζ is a cardinal in M. Contradiction.
□

Let us build in $V[\langle r_i \mid i < \lambda\rangle]$ $\lambda-$many Cohen functions mutually generic over $M[\langle r'_i \mid i < \lambda\rangle]$.
Proceed as follows. Split λ into disjoint intervals of length κ^+. Let $I_0 = [0, \kappa^+)$ and by induction define $I_\alpha = [\bigcup_{\beta<\alpha} \sup(I_\beta), \bigcup_{\beta<\alpha} \sup(I_\beta) + \kappa^+)$, for every $\alpha < \lambda$. Denote $\bigcup_{\beta<\alpha} \sup(I_\beta)$ by η_α.

In M, let $\langle I'_\alpha \mid \alpha < j(\lambda) = \lambda\rangle = j(\langle I_\alpha \mid \alpha < \lambda\rangle)$.So, each I'_α is of the form $[\eta'_\alpha, \eta'_\alpha + j(\kappa^+))$. Define $r(0)$ using I'_0 as in the previous section. Continue similar - use I'_1 to define $r(1)$, I'_2 to define $r(2)$, and so on up to $j(\kappa)$. The argument of the previous section provides mutual genericity, i.e. a genericity for $Cohen(j(\kappa^+), j(\kappa))$. Now, let $\alpha \in [j(\kappa), j(\kappa^+))$. Pick the least $\beta < \kappa^+$ such that $\sup(j''\beta) \leq \alpha < j(\beta)$. The cardinality, in M, of the interval $[\sup(j''\beta), j(\beta))$ is $\leq j(\kappa)$. Let us use $I'_{j(\beta)}, ..., I'_{j(\beta)+j(\kappa)}$ in order to build every $r(\gamma), \sup(j''\beta) \leq \gamma < j(\beta)$. This way, using $j(\kappa^+)-$many blocks we obtain generic functions for $Cohen(j(\kappa^+), j(\kappa^+))$. The process continues similar up to $j(\kappa)^{+\kappa}$. In order to define $\langle r(\gamma) \mid j(\kappa)^{+\kappa} \leq \gamma < j(\kappa^{+\kappa})\rangle$ we use Lemma 3.6.
Continue all the way to λ using Lemmas 3.6 and 3.7.

The next lemma shows that restrictions of $\vec{r} = \langle r(\alpha) \mid \alpha < \lambda\rangle$ are in M.

Lemma 3.8 *For every $a \subseteq \lambda$ and $\beta < j(\kappa^+)$ such that $a \in M$ and $M \models |a| \leq j(\kappa)$, $\langle r(\alpha) \restriction \beta \mid \alpha \in a\rangle \in M$.*

Proof. The proof follows from the previous lemmas.
□

We need to show that $\vec{r} = \langle r(\alpha) \mid \alpha < \lambda\rangle$ are generic for the forcing $Cohen(j(\kappa^+), \lambda)$ over $M[\langle r'_i \mid i < \lambda\rangle]$.
Proceed as in the previous section. Suppose otherwise. Then there is a dense open $D \subseteq Cohen(\kappa^+, \lambda)$ in $V[G]$ such that $\vec{r} \restriction (a, \beta) = \langle r(\alpha) \restriction \beta \mid \alpha \in a\rangle \notin j'(D)$, for every $a \subseteq \lambda$ and

$\beta < j(\kappa^+)$ such that $a \in M$ and $M \models |a| \leq j(\kappa)$. Just due to $< \kappa^+-$closure of the forcing $Cohen(\kappa^+, \lambda)$, every dense open set in the ultrapower contains an image of one from $V[G]$. Now, by $\kappa^{++}-$c.c. of the forcing $Cohen(\kappa^+, \lambda)$, there is $J \subseteq \lambda, |J| \leq \kappa^+$ such that D depends only on Cohen's with indexes in J. Enlarging J if necessary, we can assume that $J = j'(J^*)$, for some $J^* \subseteq \lambda$ of cardinality κ^+. Then for every $\alpha \in J$ there will be $\alpha^* \in J^*$ such that $j(\alpha^*) \geq \alpha$.
The rest basically repeats the argument of the previous section.

Now, using the above construction, we can deduce the following:

Theorem 3.9 *Suppose that $\kappa < \mu$ are measurable cardinals. Assume GCH. Then after forcing with $Cohen(\kappa^+, \mu^+)$ there will be a $\kappa-$complete uniform ultrafilter over λ.*

Remark 3.10 Note if κ was a supercompact, then it is standard to construct a generic extension in which λ is not a strong limit and still there is a uniform $\kappa-$complete ultrafilter over λ. However, GCH will break down below κ at many places.

Proof. We preserve the notation used above with $\lambda = \mu^+$. Let U_μ be a normal ultrafilter over μ. Clearly, it generates a normal ultrafilter over μ in M and we will denote the generate ultrafilter by U_μ, as well.
So, we have

$$V \to {}^{j} M \to {}^{j_{U_\mu}} \tilde{M} = M^M_{U_\mu}.$$

Now, force over V with $Cohen(\kappa^+, \lambda)$. Let $\langle r_i \mid i < \lambda \rangle$ be the corresponding Cohen functions. Then j extends to

$$j' : V[\langle r_i \mid i < \lambda \rangle] \to M[\langle r'_i \mid i < \lambda \rangle].$$

We have $\langle r(\alpha) \mid \alpha < \lambda \rangle \in V[\langle r_i \mid i < \lambda \rangle]$ which are $Cohen(\kappa^+, \lambda)-$generic over $M[\langle r'_i \mid i < \lambda \rangle]$.

Extend j_{U_μ} to an embedding

$$j^* : M[\langle r'_i \mid i < \lambda \rangle] \to \tilde{M}[\langle r''_i \mid i < j_{U_\mu}(\lambda) \rangle].$$

Set $r''_{j_{U_\mu}(i)} = r'(i)$, for every $i < \lambda$. At places i which are not images under j_{U_μ} use $\langle r(\alpha) \mid \alpha < \lambda \rangle$.

Let us argue that such defined sequence $\langle r''_i \mid i < j_{U_\mu}(\lambda) \rangle$ is a $Cohen(j(\kappa^+), j_{U_\mu}(\lambda))-$generic over $\tilde{M}$.
By the chain condition of the forcing, it is enough to deal with $j(\kappa^+)-$many coordinates in

$\tilde{M}$ only. Recall that $\tilde{M}$ is an internal ultrapower of M. So this set is in M. Finally, over M, all Cohen's involved are mutually generic. So, we are done.

Define an extension U^* of U in $V[\langle r_i \mid i < \lambda \rangle]$ using $j^* \circ j'$:

$$X \in U^* \text{ iff } \lambda \in j^*(j'(X)).$$

Clearly, it is as desired.
□

Corollary 3.11 *The existence of a $\sigma-$complete uniform ultrafilter over a cardinal which is not strong limit is equiconsistent with existence of two measurable cardinals.*

Corollary 3.12 *Assume GCH. Let $\kappa < \mu$ be measurable cardinals. Then in a generic extension the following hold:*

1. *there is an uniform $\kappa-$complete ultrafilter over μ,*
2. $2^\eta = \eta^+$*, for every* $\eta \leq \kappa$*,*
3. $2^{\kappa^+} > \mu$*.*

The above situation seems to be new, at least we do not know how to do this by standard methods form a supercompact cardinal.

4 Beyond the successor of a measurable

Our aim here will be to replace adding Cohen functions to the successor of a measurable by adding them to arbitrary regular cardinal.

Let κ be a measurable cardinal and $\lambda > \kappa^+$ be a regular cardinal. Assume GCH.

We start by adding a single Cohen function to the ultrapower.
Force with $Cohen(\lambda, \lambda)$. Let $\langle r_i \mid i < \lambda \rangle$ be the corresponding generic functions. As in the previous section, consider

$$j' : V[\langle r_i \mid i < \lambda \rangle] \to M[\langle r'_i \mid i < \lambda \rangle].$$

Note that $j(\lambda) = \lambda$.
We would like to construct $r \in V[\langle r_i \mid i < \lambda \rangle]$ which is $Cohen(\lambda)-$generic over $M[\langle r'_i \mid i < \lambda \rangle]$. The problem with the previous approach is that the initial segments of $j''\lambda$ are not in M, for example $j''\kappa^+ \notin M$.

We will need a fresh subset of λ over M. By P. Lücke and S. Müller [1], there such subsets provided that $\square(\lambda)$ holds. They actually used the principal $\square^{\text{ind}}(\lambda, \kappa)$ and we will relay on a generic version of it.

Let us state the definition.

Definition 4.1 *(Lambie-Hanson) A $\square^{ind}(\lambda, \kappa)-$sequence is a matrix*

$$\langle C_{\gamma\xi} \mid \gamma < \lambda, i(\gamma) \leq \xi < \kappa \rangle$$

satisfying the following statements:

1. *If $\gamma \in Lim \cap \lambda$, then $i(\gamma) < \kappa$.*
2. *If $\gamma \in Lim \cap \lambda$ and $i(\gamma) \leq \xi < \kappa$, then $C_{\gamma\xi}$ is a closed unbounded subset of γ.*
3. *If $\gamma \in Lim \cap \lambda$ and $i(\gamma) \leq \xi' < \xi < \kappa$, then $C_{\gamma\xi'} \subseteq C_{\gamma\xi}$.*
4. *If $\beta, \gamma \in Lim \cap \lambda$ and $i(\gamma) \leq \xi < \kappa$ with $\beta \in Lim(C_{\gamma\xi})$, then $\xi \geq i(\beta)$ and $C_{\beta\xi} = C_{\gamma\xi} \cap \beta$.*
5. *If $\beta, \gamma \in Lim \cap \lambda$ and $\beta < \gamma$, then there is $\xi, i(\gamma) \leq \xi < \kappa$ such that $\beta \in Lim(C_{\gamma\xi})$.*
6. *There is no closed unbounded subset C of λ with the property that, for all $\gamma \in Lim(C)$, there is $\xi < \kappa$ such that $C_{\gamma\xi} = C \cap \gamma$.*

The following was proved by P. Lücke and S. Müller [1]:

Theorem 4.2 *Let $\langle C_{\gamma\xi} \mid \gamma < \lambda, i(\gamma) \leq \xi < \kappa \rangle$ be a sequence which witnesses $\square^{ind}(\lambda, \kappa)$. Let*

$$j(\langle C_{\gamma\xi} \mid \gamma < \lambda, i(\gamma) \leq \xi < \kappa \rangle) = \langle C'_{\gamma\xi} \mid \gamma < \lambda, i(\gamma) \leq \xi < j(\kappa) \rangle$$

be its image in the ultrapower M. Then $C' = \cup_{\alpha \in Lim \cap \lambda} C'_{j(\alpha)\kappa}$ is fresh subset of λ over M and for every $\alpha \in Lim \cap \lambda$, $C' \cap j(\alpha) = C'_{j(\alpha)\kappa}$.

We will force $\square^{\text{ind}}(\lambda, \kappa)$ by approximations of size $< \lambda$.
Namely, let $\mathcal{P}(\square^{\text{ind}}(\lambda, \kappa))$ be the set of all sequences $\langle C_{\gamma\xi} \mid \gamma < \alpha + 1, i(\gamma) \leq \xi < \kappa \rangle$ which satisfy the conditions (1)-(5) of Definition 4.1, for some $\alpha < \lambda$. The forcing order on $\mathcal{P}(\square^{\text{ind}}(\lambda, \kappa))$ is end-extension.

The next lemmas are easy.

Lemma 4.3 *The forcing $\mathcal{P}(\square^{ind}(\lambda, \kappa))$ is $< \lambda-$strategically closed.*

Lemma 4.4 *Let $H \subseteq \mathcal{P}(\square^{ind}(\lambda,\kappa))$ be generic and let $\langle C_{\gamma\xi} \mid \gamma < \lambda, i(\gamma) \leq \xi < \kappa\rangle$ be the matrix generated by H.*
Then $\langle C_{\gamma\xi} \mid \gamma < \lambda, i(\gamma) \leq \xi < \kappa\rangle$ satisfies (6) of Definition 4.1.

Proof. Just otherwise let a club $C \subseteq \lambda$ be a tread. Work in V and, using the strategic closure, decide everything up to some $\alpha < \lambda$ of countable cofinality. Then, define $C_{\alpha\xi}$'s which disagree with $C \cap \alpha$.
□

Force Cohen functions $\langle r_\alpha \mid \alpha < \lambda\rangle$ over $V^{\mathcal{P}(\square^{\text{ind}}(\lambda,\kappa))}$. Let $\langle r'_\alpha \mid \alpha < \lambda\rangle$ be the corresponding Cohen's in M and let $C' \subseteq \lambda$ be a fresh set constructed as in 4.2 from such generic $\square^{\text{ind}}(\lambda,\kappa)$ matrix.
Use $\langle r'_\alpha \mid \alpha \in C'\rangle$ to define a new Cohen function r.
Let $\beta < \lambda$. Set $r(\beta) = r_\beta(\beta)$, if $\beta \in C'$.
If $\beta \notin C'$, then let $\gamma \in C'$ be the least above β. Set $r(\beta) = r'_\gamma(\beta)$.

Let us argue that such r is a Cohen generic over $M[\langle r'_i \mid i < \lambda\rangle]$.
Suppose otherwise. Then there is a dense open $D \subseteq Cohen(\lambda)$ in $V[G]$ such that $r \restriction \beta \notin j'(D)$, for every $\beta < \lambda$.

Let $\underset{\sim}{D}$ and $\underset{\sim}{r}$ be $Cohen(\lambda)-$names of D and r respectively. We assume for simplicity that the empty condition forces this.

For every $\alpha < \lambda$, pick some $d_\alpha \in j'(D)$ such that $d_\alpha \geq r \restriction j(\alpha) + 1$.
Then, clearly, $d_\alpha \in M$, since G' does not add new bounded subsets of λ. Fix a function $f_\alpha : \kappa \to D$ which represents d_α.
We would like to change one value of some of f_α's.
Thus, suppose that $\text{cof}(\alpha) = \tau, \kappa > \tau > \aleph_0$ and assume that there is a club $C_\alpha \subseteq \alpha$ such that for every $\tilde{\nu}, \tilde{\rho} \in C_\alpha$, if $\tilde{\nu} \leq \tilde{\rho}$, then

$$f_{\tilde{\nu}}(\tau) \restriction \tilde{\nu} + 1 = f_{\tilde{\rho}}(\tau) \restriction \tilde{\nu} + 1.$$

In this case we set $f_\alpha(\tau)$ to be an element of D stronger than $\bigcup_{\tilde{\nu} \in C_\alpha}(f_{\tilde{\nu}}(\tau) \restriction \tilde{\nu} + 1)$.

For every $\alpha < \lambda$, pick $p_\alpha \in G$ which decides $f_\alpha(\tau)$ and forces that it is in D, for every $\tau < \kappa$.

Define

$$h(\alpha) = \max(\sup(\bigcup_{\nu<\kappa} \text{dom}(f_\alpha(\nu)), \text{dom}(p_\alpha) + 1)).$$

So, $h : \lambda \to \lambda$.

Work in V. Pick an increasing continuous sequence $\langle N_\zeta \mid \zeta \leq \kappa\rangle$ such that, for every $\zeta < \kappa$,

1. $N_\zeta \preceq H_\chi$, for some χ large enough,

2. $\langle N_\mu \mid \mu \leq \zeta \rangle \in N_{\zeta+1}$,

3. $|N_\zeta| < \lambda$,

4. $N_\zeta \cap \lambda$ is an ordinal.

Set $\alpha_\zeta = N_\zeta \cap \lambda$.

Now, using N_ζ's, we construct an increasing continuous sequence of conditions. Describe a typical successor stage. So, we deal with $\zeta < \kappa$, $\langle C_{\alpha_\zeta \xi} \mid \xi < \kappa \rangle$ is defined in $N_{\zeta+1}$. Set $i(\alpha_{\zeta+1} = 0, \alpha_\zeta \in \mathrm{Lim}(C_{\alpha_{\zeta+1}\eta'})$, for every $\eta' < \kappa$, $\min(C_{\alpha_{\zeta+1}\zeta} \setminus \alpha_\zeta + 1) > h(\alpha_\zeta)$ and, in addition, $r_{\eta_\zeta} \restriction (\alpha_\zeta, h(\alpha_\zeta)) \geq f(\zeta) \setminus \alpha_\zeta$, where η_ζ denotes $\min(C_{\alpha_{\zeta+1}\zeta} \setminus \alpha_\zeta + 1)$.
Note that then by (4) of Definition 4.1, $C_{\alpha_\zeta \eta'} = C_{\alpha_{\zeta+1}\eta'} \cap \alpha_\zeta$.

Using density arguments and $< \lambda-$strategic closure of the forcing, we can pick from the generic object (which adds $\square^{\mathrm{ind}}(\lambda, \kappa)$ and Cohens) a sequence as defined above.
Let $\langle \alpha_\zeta \mid \zeta \leq \kappa \rangle$ be such a sequence.

Lemma 4.5 *In the ultrapower, we will have* $\min(C'_{j(\alpha_\kappa)\kappa} \setminus \alpha^* + 1) > j'(h)(\alpha^*)$.

Proof. Let $\tau < \kappa$. Then $\min(C_{\alpha_{\tau+1}\tau} \setminus \alpha_\tau + 1) > h(\alpha_\tau)$. Also, $\alpha_\tau \in \mathrm{Lim}(C_{\alpha_{\tau+1}0})$. Then, by induction, for every $\xi < \rho < \kappa$, $\alpha_\xi \in \mathrm{Lim}(C_{\alpha_\rho 0})$. So, for every $\eta < \kappa$, $\alpha_\xi \in \mathrm{Lim}(C_{\alpha_\rho \eta})$. Hence, $C_{\alpha_\rho \eta} \cap \alpha_\xi = C_{\alpha_\xi \eta}$.
So, for every regular $\rho, \tau < \rho \leq \kappa$, $\min(C_{\alpha_\rho \tau} \setminus \alpha_\tau + 1) > h(\alpha_\tau)$.
In particular, $\min(C_{\alpha_\kappa \tau} \setminus \alpha_\tau + 1) > h(\alpha_\tau)$.
Now the conclusion follows.
$\square$

Now we proceed as in the first section, only using $\langle r_{\eta_\zeta} \mid \zeta \in \mathrm{Lim}(\kappa) \rangle$.
Then, in M', $r'_{\eta_{\alpha_\kappa}} \restriction (\alpha^*, j'(h)(\alpha^*)) \geq j(f)(\kappa) \setminus \alpha^*$. By the definition of r, then $r \restriction \eta_{\alpha_\kappa} \in j'(D)$, as forced by $\langle r_{\eta_\zeta} \restriction (\alpha_\zeta, h(\alpha_\zeta)) \mid \zeta \in \mathrm{Lim}(\kappa) \rangle$ over V.
Contradiction.

5 Adding many Cohen functions to λ

As in the previous section, we can use the above to get more Cohen functions over M. Thus, suppose that for some regular $\mu > \lambda$, instead of $\lambda-$many Cohen functions $\langle r_i \mid i < \lambda \rangle$ we add $\mu-$many, i.e. if we force with $Cohen(\lambda, \mu)$ instead of $Cohen(\lambda, \lambda)$. Then, using similar ideas, it is possible to build in $V[\langle r_i \mid i < \mu\rangle]$ $\mu-$many Cohen functions mutually generic over $M[\langle r'_i \mid i < \mu\rangle]$.

The proof is similar to those of 3.9. However there are some additional points that will be address below.

We describe two different ways for doing this.

5.1 First construction

Iterate the forcing $\mathcal{P}(\Box^{\rm ind}(\tau,\kappa))$, for every regular $\tau, \lambda \leq \tau \leq \mu$, with Easton support. Then we force with $Cohen(\lambda,\mu)$. Finally, split μ into disjoint intervals of length λ and proceed as in Section 2.

Let us address cofinality κ cases. The difference form Section 2 is that there the closure of the ultrapower under $\kappa-$sequences was used. Here, if $\lambda > \kappa^+$, it is not closed under $< \lambda-$sequences.

Deal with a situation of Lemma 3.6. Let us build a scale $\langle f_i \mid i < \eta^+\rangle$ be a scale in $\prod_{\nu<\kappa} h(\nu)$ using $\Box^{\rm ind}(\eta^+,\kappa)$. Namely, for each $\alpha < \eta^+, \kappa < {\rm cof}(\alpha) < \eta$ and $\tau < \kappa$, let

$$f_\alpha(\tau) = \bigcup_{\beta \in C_{\eta^+,\alpha\tau}} f_\beta(\tau).$$

Then, in the ultrapower,

$$j(f_\alpha)(\kappa) = \bigcup_{\beta \in C'_{j(\eta^+),j(\alpha)\kappa}} f'_\beta(\kappa),$$

where C', f' stand for the images of the corresponding sequences under the ultrapower embedding j.

Now we can use the fact that $C'_{j(\eta^+),j(\alpha)\kappa}$'s cohere and each initial segment of $\bigcup_{\alpha<\eta^+} C'_{j(\eta^+),j(\alpha)\kappa}$ belongs to the ultrapower.

We obtain the following analog of 3.9:

Theorem 5.1 *Suppose that $\kappa < \mu$ are measurable cardinals, $\lambda, \kappa < \lambda < \mu$, is a regular cardinal . Assume GCH. Then in a generic extension with iteration of $\mathcal{P}(\Box^{ind}(\tau,\kappa))$, for every regular $\tau, \lambda \leq \tau \leq \mu$, followed by $Cohen(\lambda,\mu^+)$ there will be a μ^+-Cohen functions over the ultrapower.*

The disadvantage of the present approach is that if μ was a measurable in V, then it will not be such after the iteration the indexed squares. Let us use an other preparation forcing in order to overcome this obstacle.

5.2 Second construction

Here we describe the construction that allows to preserve a measurability. Concentrate mostly on new points.

First we force with $\mathcal{P}(\square^{\rm ind}(\lambda,\kappa))$. Let $\langle C_{\lambda\alpha\tau} \mid \alpha \in {\rm Lim}(\lambda), \tau < \kappa\rangle$ be a generic. Denote $V[\langle C_{\lambda\alpha\tau} \mid \alpha \in {\rm Lim}(\lambda), \tau < \kappa\rangle]$ by V^*. Set $A_\lambda = \bigcup_{\alpha\in{\rm Lim}(\lambda)} C'_{\lambda j(\alpha)\kappa}$.
Now, let us deal with λ^+. Define the forcing Q_{λ^+} in V^*. It is supposed to add a club Z_{λ^+} to λ^+ and $\square_{\lambda^{++}}-$sequence for points in Z_{λ^+} of cofinalities $\leq \lambda$.

So, let Q_{λ^+} consists of approximations of such Z_{λ^+} and such square sequences.
Formally:
$p \in Q_{\lambda^+}$ iff

1. ${\rm dom}(p)$ is a closed subset of λ^+ of cardinality $< \lambda^+$ which consists of limit ordinals and has a maximal element,

2. for every $\alpha \in {\rm dom}(p)$ of cofinality $\leq \lambda$ which is a limit point of ${\rm dom}(p)$, $p(\alpha)$ is a club of α of an order type $\leq \lambda$,

3. if $\alpha \in {\rm dom}(p)$ of cofinality $\leq \lambda$ and β is a limit point of $p(\alpha)$, then ${\rm cof}(\beta) < \lambda, \beta \in {\rm dom}(p)$ and $p(\beta) = p(\alpha) \cap \beta$.

Q_{λ^+} is ordered by the end-extension order.

The following lemmas are standard:

Lemma 5.2 *The forcing Q_{λ^+} is $< \lambda-$closed.*

Lemma 5.3 *The forcing Q_{λ^+} is $\lambda + 1-$strategically closed.*

Let Z_{λ^+} be generic club added by Q_{λ^+}, i.e.,

$$Z_{\lambda^+} = \bigcup\{{\rm dom}(p) \mid p \in G(Q_{\lambda^+})\}.$$

Denote by $\langle C_\alpha^{\square_{\lambda^{++}}} \mid \alpha \in Z_{\lambda^+}, {\rm cof}(\alpha) \leq \lambda\rangle$ the square sequence added by Q_{λ^+}, i.e., $C_\alpha^{\square_{\lambda^{++}}} = p(\alpha)$, for some $p \in G(Q_{\lambda^+})$.

Let $\eta \in Z_{\lambda^+}, {\rm cof}(\eta) = \lambda$.
Denote by π_η the order isomorphism between ${\rm Lim}(\lambda)$ and ${\rm Lim}(C_\eta^{\square_{\lambda^{++}}})$.
Set $C_{\lambda^+,\eta,\pi_\eta(\alpha),\tau} = \pi_\eta{}'' C_{\lambda\alpha\tau}$, for every $\alpha \in {\rm Lim}(\lambda), \tau < \kappa$.
Note, that due to coherency, $C_{\lambda^+,\eta,\pi_\eta(\alpha),\tau}$ depends on $\pi_\eta(\alpha)$ rather than on η.

Now turn to the ultrapower. Let Z'_{λ^+} denotes the image of Z_{λ^+},
$\langle C'^{\Box_{\lambda^{++}}}_{\alpha} \mid \alpha \in Z'_{\lambda^+}, \mathrm{cof}(\alpha) \leq \lambda\rangle$ the image of $\langle C^{\Box_{\lambda^{++}}}_{\alpha} \mid \alpha \in Z_{\lambda^+}, \mathrm{cof}(\alpha) \leq \lambda\rangle$ and
$\langle C'_{\lambda^+,\eta,\pi_\eta(\alpha),\tau} \mid \eta \in Z'_{\lambda^+}, \mathrm{cof}(\eta) = \lambda, \alpha \in \mathrm{Lim}(\lambda), \tau < j(\kappa)\rangle$
the image of $\langle C_{\lambda^+,\eta,\pi_\eta(\alpha),\tau} \mid \eta \in Z_{\lambda^+}, \mathrm{cof}(\eta) = \lambda, \alpha \in \mathrm{Lim}(\lambda), \tau < \kappa\rangle$.

Set

$$A_{\lambda^+} = (Z'_{\lambda^+} \cap Cof(\geq \lambda)) \cup \bigcup\{C'_{\lambda^+,\eta,\pi_\eta(j(\alpha)),\kappa} \mid \eta \in Z'_{\lambda^+}, \mathrm{cof}(\eta) = \lambda, \alpha \in \mathrm{Lim}(\lambda)\}.$$

If η has a pre-image, then $\bigcup\{C'_{\lambda^+,\eta,\pi_\eta(j(\alpha)),\kappa} \mid \alpha \in \mathrm{Lim}(\lambda)\}$ will be fresh.
Let us argue that this remains true also in case when η has no pre-image.
So, fix such η and let $f_\eta : \kappa \to \lambda^+$ be a function which represents η in the ultrapower M. We can assume that for every $\xi < \kappa$, $\mathrm{cof}(f_\eta(\xi)) = \lambda$. Let $\beta < \alpha < \lambda$ be limit ordinals. Then there is $i^* < \kappa$ such that $\beta \in C_{\lambda\alpha i^*}$, and so, for every $i, i^* \leq i < \kappa$, $C_{\lambda\beta i} = C_{\lambda\alpha i} \cap \beta$.
Then, for every $\xi < \kappa$, we will have $C_{\lambda^+\pi_{f_\eta(\xi)}(\beta)i} = C_{\lambda\pi_{f_\eta(\xi)}(\alpha)i} \cap \pi_{f_\eta(\xi)}(\beta)$, whenever $i^* \leq i < \kappa$.
Hence, in the ultrapower,
$C_{\lambda^+\pi_\eta(j(\beta))i} = C_{\lambda\pi_\eta(j(\alpha))i} \cap \pi_\eta(j(\beta))$, whenever $i^* \leq i < j(\kappa)$.
In particular, $C_{\lambda^+\pi_\eta(j(\beta))\kappa} = C_{\lambda\pi_\eta(j(\alpha))\kappa} \cap \pi_\eta(j(\beta))$.

We use A_{λ^+} in order to define Cohen functions over the ultrapower.

Now given $a \subseteq \lambda^+, a \in M$ and $M \models |a| < \lambda$. Pick a function $f_a : \kappa \to \mathcal{P}_\lambda(\lambda^+)$ which represents a. We look at the configuration of $f_a(\nu)$ inside Z_{λ^+} and then, if needed inside $\langle C^{\Box_{\lambda^{++}}}_{\alpha} \mid \alpha \in Z_{\lambda^+}, \mathrm{cof}(\alpha) \leq \lambda\rangle$, for every $\nu < \kappa$. Then the function $\nu \mapsto$ the configuration of $f_a(\nu)$ will represent the configuration of a in the ultrapower, which what is needed in order to argue that restrictions of Cohen functions to the coordinates in a are in the ultrapower.

Continue to cardinals $> \lambda^+$ in a similar fashion.

Deal with λ^{++}. Work in $V^*[G(Q_{\lambda^+}]$.

Definition 5.4 *$p \in Q_{\lambda^{++}}$ iff*

1. $\mathrm{dom}(p)$ *is a closed subset of λ^{++} of cardinality $< \lambda^{++}$ which consists of limit ordinals and has a maximal element,*
2. *for every $\alpha \in \mathrm{dom}(p)$ of cofinality $\leq \lambda$ which is a limit point of* $\mathrm{dom}(p)$, *$p(\alpha)$ is a club of α of an order type $\leq \lambda$,*
3. *if $\alpha \in \mathrm{dom}(p)$ of cofinality $\leq \lambda$ and β is a limit point of $p(\alpha)$, then* $\mathrm{cof}(\beta) < \lambda, \beta \in \mathrm{dom}(p)$ *and $p(\beta) = p(\alpha) \cap \beta$.*

$Q_{\lambda^{++}}$ is ordered by the end-extension order.

The following lemma is obvious:

Lemma 5.5 *The forcing Q_{λ^+} is $< \lambda-$closed.*

The next two lemmas are proved similar to 5.3.

Lemma 5.6 *The forcing $Q_{\lambda^{++}}$ is λ^+-distributive.*

Proof.

Let $\underset{\sim}{f}$ be a $Q_{\lambda^{++}}-$name of a function from λ^+ to ordinals. We need to show that f is in $V^*[G(Q_{\lambda^+}]$.

Fix in V an increasing continuous sequence $\langle N_\xi \mid \xi \leq \lambda^+ \rangle$ of elementary submodels of H_χ, for some χ large enough such that

1. $|N_\xi| = \lambda^+$,
2. $N_\xi \cap \lambda^{++}$ is an ordinal,
3. $\langle N_\zeta \mid \zeta \leq \xi \rangle \in N_{\xi+1}$,
4. $Q_{\lambda^{++}}, \underset{\sim}{f}$ etc. are in N_0.

Denote $N_\xi \cap \lambda^{++}$ by δ_ξ.

Define now an increasing sequence of conditions $\langle p_i \mid i \in Z_{\lambda^+} \rangle$.

Let i_0 be the first element of Z_{λ^+}. Set $p'_{i_0} = \{\delta_{i_0}\}$. Then inside $N_{i_0+1}[G(Q_{\lambda^+}]$ find $q \geq p'_{i_0}$ which decides $\underset{\sim}{f}(0)$. Extend its domain by adding $\{\delta_{i_0+1}\}$. Set p_{i_0} to be such condition. Proceed further in a similar fashion at each successor stage.

If i is a limit point of Z_{λ^+} and $\langle p_{i'} \mid i' \in Z_{\lambda^+} \cap i \rangle$ is defined, then set $q' = \bigcup_{i'<i} p_{i'}$. Extend its domain by adding $\{\delta_i\}$. Set $q(\delta_i) = \{\delta_{i'} \mid i' \in Z_{\lambda^+} \cap i\}$.

Then extend q to q'' inside N_{i+1} such that q'' decides $\underset{\sim}{f}(i^*)$, where i is the i^* element of Z_{λ^+}. Add $\{\delta_{i+1}\}$ to its domain. Let p_i be such condition.

Now set $p' = \bigcup_{i \in Z_{\lambda^+}} p_i$. to its domain. Define p by setting $\mathrm{dom}(p) = \mathrm{dom}(p') \cup \{\delta_\lambda\}$, $p \restriction \mathrm{dom}(p') = p'$ and $p(\delta_\lambda) = \{\delta_i \mid i \in Z_{\lambda^+}\}$.

□

Now we would like to iterate this type forcing notions for all regular cardinals $\eta, \lambda < \eta \leq \mu$.

Define (over V^*) an Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha < \mu+1, \beta \leq \mu\rangle$ as follows. Let Q_β be trivial unless $\lambda < \beta \leq \mu$ is a regular cardinal. If $\beta, \lambda < \beta \leq \mu$ is a regular cardinal, then define Q_β similar to $Q_{\lambda^+}, Q_{\lambda^{++}}$ above.

Let Z_γ be generic club added by Q_γ, $\gamma < \beta$.

Definition 5.7 *$p \in Q_\beta$ iff*

1. $\mathrm{dom}(p)$ *is a closed subset of β of cardinality $< \beta$ with a maximal element,*
2. *for every limit point α of* $\mathrm{dom}(p)$ *of cofinality $\leq \lambda$, $p(\alpha)$ is a club of α of an order type $\leq \lambda$,*
3. *if $\alpha \in$* $\mathrm{dom}(p)$ *of cofinality $\leq \lambda$ and β is a limit point of $p(\alpha)$, then* $\mathrm{cof}(\beta) < \lambda, \beta \in$ $\mathrm{dom}(p)$ *and $p(\beta) = p(\alpha) \cap \beta$.*

Q_β is ordered by the end-extension order.

Require also the following:

if $p = \langle p_\beta \mid \beta < \alpha\rangle \in P_\alpha$, then for every $\beta \in supp(p)$, $p \restriction \beta$ decides $\max(p_\beta)$.

The following lemma is an analog of 5.6:

Lemma 5.8 *Let $\alpha \leq \mu$ be a regular cardinal. Then the forcing $P_{\mu+1}/\underset{\sim}{G}(P_\alpha)$ $\alpha-$distributive.*

Proof. The proof basically repeats those of 5.6 with obvious adaptations, but there is one new point due to the iteration process.

Let $\underset{\sim}{f}$ be a $P_{\mu+1}/\underset{\sim}{G}(P_\alpha)-$name of a function from α to ordinals and suppose that the weakest condition forces this. We need to show that f is in $V_0[G(P_\alpha)]$. Denote by Z_α the generic club in α added by $G(P_\alpha)$.

Fix in V an increasing continuous sequence $\langle \vec{N}_\xi \mid \xi \leq \alpha\rangle$ such that, for every $\xi < \alpha$, the following hold:

1. $\vec{N}_\xi = \langle N_{\xi,\gamma} \mid \gamma \in (\alpha, \mu] \cap \mathrm{Reg}\rangle$ is an increasing sequence of elementary submodels of H_χ, for some χ large enough, such that, for every γ,
   (a) $|N_{\xi\gamma}| < \gamma$,
   (b) $N_{\xi\gamma} \cap \gamma$ is an ordinal,
   (c) $\langle N_{\xi\gamma'} \mid \gamma' < \gamma\rangle \in N_{\xi\gamma}$,
2. $\langle \vec{N}_\zeta \mid \zeta \leq \xi\rangle \in N_{\xi+1,\alpha}$,

3. for every γ and every limit ξ, $N_{\xi\gamma} = \bigcup_{\zeta<\xi} N_{\zeta\gamma}$,

4. P_μ, $\underset{\sim}{f}$ etc. are in $N_{0\alpha}$.

Denote $N_{\xi\gamma} \cap \gamma$ by $\delta_{\xi\gamma}$.

Define now an increasing sequence of conditions $\langle p_i = \langle p_{i\gamma} \mid \gamma \in (\alpha, \mu] \cap \mathrm{Reg}\rangle \mid i \in Z_\alpha\rangle$.

Let i_0 be the first element of Z_α. Find inside $N_{i_0\alpha^+}[G(Q_\alpha)]$ a condition q which decides $\underset{\sim}{f}(0)$. Add, for each $\gamma \in (\alpha, \mu] \cap \mathrm{Reg}$, $\delta_{i_0\gamma}$ to the $\gamma-$th coordinate of q. Note that we are in $N_{i_0+1\alpha^+}[G(Q_\alpha)]$, by Item 2 above, however some of the coordinates are outside of this model. Let p_{i_0} be such condition.

Proceed further in a similar fashion at each successor stage.
If i is a limit point of Z_λ and $\langle p_{i'} \mid i' \in Z_\lambda \cap i\rangle$ is defined, then we first define $q = \langle q_\gamma \mid \gamma \in (\alpha, \mu] \cap \mathrm{Reg}\rangle$ by setting $q_\gamma = \bigcup_{i'<i} p_{i'\gamma} \cup \{\delta_{i\gamma}\}$.
Note that such $q \in N_{i+1\alpha^+}[G(Q_\alpha)]$, by Item 2 above. Now, as at a successor stage, extend q inside $N_{i+1\alpha^+}[G(Q_\alpha)]$ to q' which decides $\underset{\sim}{f}(i^*)$, where i is the i^* element of Z_α. Add, for each $\gamma \in (\alpha, \mu] \cap \mathrm{Reg}$, $\delta_{i\gamma}$ to the $\gamma-$th coordinate of q'. Note that we are in $N_{i+1\alpha^+}[G(Q_\alpha)]$, by Item 2 above, however some of the coordinates are outside of this model. Let p_i be such condition.

Finally we define $p = \langle p_\gamma \mid \gamma \in (\alpha, \mu] \cap \mathrm{Reg}\rangle$ by setting p_γ to be $\bigcup_{i\in Z_\alpha} p_{i\gamma}$ with $\{\delta_{\alpha\gamma}\}$ added.
□

The next which lemma deals with singular α's is similar.

Lemma 5.9 *Let $\alpha \leq \mu$ be a singular cardinal.*
Then the forcing $P_{\mu+1}/\underset{\sim}{G}(P_\alpha)$ is $< \alpha-$distributive.

Lemma 5.10 *Suppose that μ was a measurable cardinal in V.*
Then it remains such after forcing with $P_{\mu+1}$.

Proof. We build a master condition sequence in μ^+-many steps similar to the argument of Lemma 5.8.

Let U_μ be a normal ultrafilter over μ in V and let $j_\mu : V \to M_\mu$ be the corresponding elementary embedding. We extend it to $j^*_\mu : V[G(P_{\mu+1})] \to M_\mu[G^*]$.
Let $G^* \cap P_{\mu+1} = G(P_{\mu+1})$. Define a master condition sequence starting with A_μ over the coordinate $j_\mu(\mu)$.
Then, we pick sequence of models
$\langle \vec{N}_\xi \mid \xi < \mu^+\rangle$ such that, for every $\xi < \mu^+$, the following hold:

1. $\vec{N}_\xi = \langle N_{\xi,\gamma} \mid \gamma \in (\mu, j_\mu(\mu)] \cap \mathrm{Reg}^{M_\mu} \rangle \in M_\mu$ is an increasing sequence of elementary submodels of $H_\chi^{M_\mu}$, for some χ large enough, such that, for every γ,
   (a) $|N_{\xi\gamma}|^{M_\mu} < \gamma$,
   (b) $N_{\xi\gamma} \cap \gamma$ is an ordinal,
   (c) $\langle N_{\xi\gamma'} \mid \gamma' < \gamma \rangle \in N_{\xi\gamma}$,
2. $\langle \vec{N}_\zeta \mid \zeta \leq \xi \rangle \in N_{\xi+1,\alpha}$,
3. for every γ and every limit ξ, $N_{\xi\gamma} = \bigcup_{\zeta<\xi} N_{\zeta\gamma}$,
4. $P_{j_\mu(\mu+1)}$ etc. are in $N_{0\alpha}$.

Now we proceed as in 5.8, but only instead of deciding values of $\underset{\sim}{f}$, meet dense open subsets of $P_{j_\mu(\mu+1)}$ which belong to M_μ. This way G^* is build, and so, j_μ extends.
□

Now, after forcing with $P_{\mu+1}$, we proceed as in the previous subsection.

References

[1] P. Lücke and S. Müller, Closure properties of measurable ultrapowers