Violating the Singular Cardinals Hypothesis Without Large Cardinals*

by Moti Gitik and Peter Koepke

Version of August 20, 2010

Abstract

We extend a transitive model V of $\mathrm{ZFC} + \mathrm{GCH}$ cardinal preservingly to a model N of ZF + "GCH holds below $\aleph_\omega$" + "there is a surjection from the power set of $\aleph_\omega$ onto λ" where λ is an arbitrarily high fixed cardinal in V. The construction can roughly be described as follows: add $\aleph_{n+1}$ many Cohen subsets of $\aleph_{n+1}$ for every $n < \omega$, and adjoin λ many subsets of $\aleph_\omega$ which are unions of ω-sequences of those Cohen subsets; then let N be a choiceless submodel generated by equivalence classes of the λ subsets of $\aleph_\omega$ modulo an appropriate equivalence relation.

In [1], Arthur Apter and the second author constructed a model of ZF + "GCH holds below $\aleph_\omega$" + "there is a surjection from $[\aleph_\omega]^\omega$ onto λ" where λ is an arbitrarily high fixed cardinal in the ground model V. This amounts to a strong *surjective* violation of the *singular cardinals hypothesis* SCH. The construction assumed a measurable cardinal in the ground model. It was also shown in [1] that a measurable cardinal in some inner model is necessary for that combinatorial property, using the Dodd-Jensen covering theorem [2].

The first author then noted that one can work without measurable cardinals if one considers surjections from $\mathcal{P}(\aleph_\omega)$ onto λ instead of surjections from $[\aleph_\omega]^\omega$ onto λ:

Theorem 1. *Let V be any ground model of* $\mathrm{ZFC} + \mathrm{GCH}$ *and let λ be some cardinal in V. Then there is a cardinal preserving model $N \supseteq V$ of the theory* ZF + "GCH *holds below $\aleph_\omega$*" + *"there is a surjection from $\mathcal{P}(\aleph_\omega)$ onto λ".*

Note that in the presence of the axiom of choice (AC) the latter theory for $\lambda \geqslant \aleph_{\omega+2}$ has large consistency strength and implies the existence of measurable cardinals of high Mitchell orders in some inner model (see [3] by the first author). The pcf-theory of Saharon Shelah [4] shows that the situation for $\lambda \geqslant \aleph_{\omega_4}$ is incompatible with AC. Hence Theorem 1 yields a choiceless violation of pcf-theory without the use of large cardinals.

*. The second author was partly supported by ISF Grant 234/08.

The forcing

Fix a ground model V of ZFC + GCH and let λ be some regular cardinal in V. We first present two building blocks of our construction. The forcing $P_0 = (P_0, \supseteq, \emptyset)$ adjoins one COHEN subset of $\aleph_{n+1}$ for every $n < \omega$.

$$P_0 = \{ p \mid \exists (\delta_n)_{n<\omega} \ (\forall n < \omega : \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p : \bigcup_{n<\omega} [\aleph_n, \delta_n) \to 2)\}.$$

A standard product analysis proves that forcing with P_0 does not collapse cardinals. Adjoining one COHEN subset of $\aleph_{n+1}$ for every $n < \omega$ is equivalent to adjoining $\aleph_{n+1}$-many by the following "two-dimensional" forcing $(P_*, \supseteq, \emptyset)$:

$$P_* = \{ p_* \mid \exists (\delta_n)_{n<\omega} \ (\forall n < \omega : \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p_* : \bigcup_{n<\omega} [\aleph_n, \delta_n)^2 \to 2)\}.$$

For $p_* \in P_*$ and $\xi \in [\aleph_0, \aleph_\omega)$ let $p_*(\xi) = \{(\beta, p_*(\xi, \zeta)) | (\xi, \zeta) \in \operatorname{dom}(p_*)\}$ be the ξ-th section of p_*.

The forcing employed in the subsequent construction is a kind of finite support product of λ copies of P_0 where the factors are eventually coupled via P_*.

Definition 2. *Define the forcing $(P, \leqslant_P, \emptyset)$ by:*

$$\begin{aligned} P = \{(p_*, (a_i, p_i)_{i<\lambda}) \ \mid \ & \exists (\delta_n)_{n<\omega} \exists D \in [\lambda]^{<\omega} (\forall n < \omega : \delta_n \in [\aleph_n, \aleph_{n+1}), \\ & p_* : \textstyle\bigcup_{n<\omega} [\aleph_n, \delta_n)^2 \to 2, \\ & \forall i \in D : p_i : \textstyle\bigcup_{n<\omega} [\aleph_n, \delta_n) \to 2 \wedge p_i \neq \emptyset, \\ & \forall i \in D : a_i \in [\aleph_\omega \setminus \aleph_0]^{<\omega} \wedge \forall n < \omega : \operatorname{card}(a_i \cap [\aleph_n, \aleph_{n+1})) \leqslant 1, \\ & \forall i \notin D \, (a_i = p_i = \emptyset))\}. \end{aligned}$$

If $p = (p_*, (a_i, p_i)_{i<\lambda}) \in P$ then $p \in P_*$ and $p_i \in P_0$, with all but finitely many p_i being $\emptyset$. Extending p_i is controlled by linking ordinals *$\xi \in a_i$. More specifically extending p_i in the interval $[\aleph_n, \aleph_{n+1})$ is controlled by $\xi \in a_i \cap [\aleph_n, \aleph_{n+1})$ if that intersection is nonempty. Let* $\operatorname{supp}(p) = \{i < \lambda | p_i \neq \emptyset\}$ *be the* support *of $p = (p_*, (a_i, p_i)_{i<\lambda})$, i.e., the set D in the definition of P. P is partially ordered by*

$$p' = (p'_*, (a'_i, p'_i)_{i<\lambda}) \leqslant_P (p_*, (a_i, p_i)_{i<\lambda}) = p$$

iff

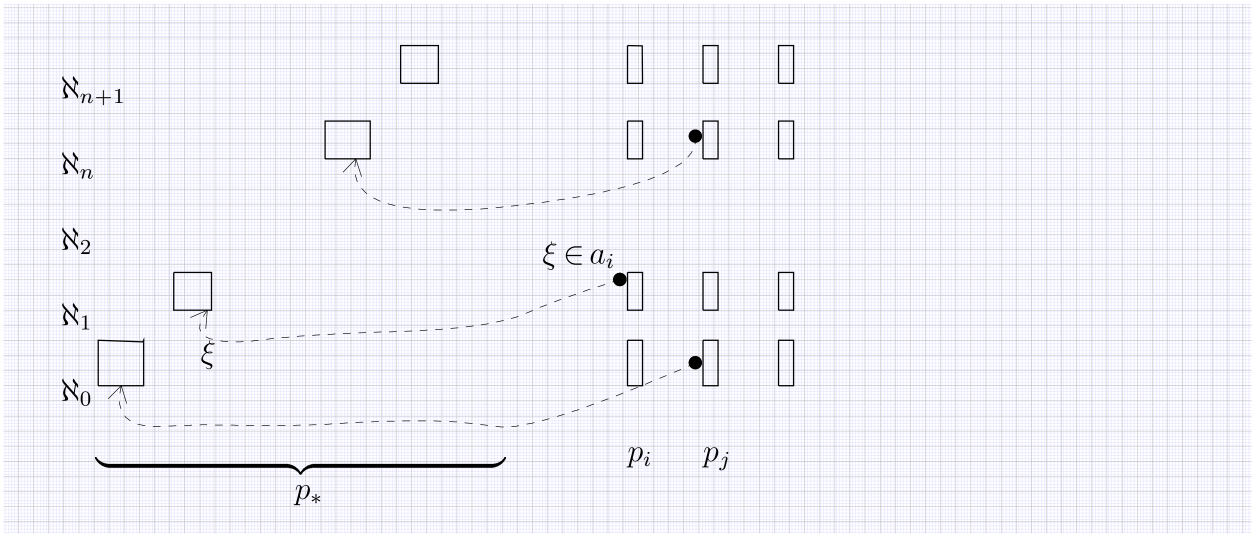
a) $p'_* \supseteq p_*, \forall i < \lambda \, (a'_i \supseteq a_i \wedge p'_i \supseteq p_i)$,

b) (Linking property) $\forall i < \lambda \forall n < \omega \forall \xi \in a_i \cap [\aleph_n, \aleph_{n+1}) \forall \zeta \in \operatorname{dom}(p'_i \setminus p_i) \cap [\aleph_n, \aleph_{n+1})$: $p'_i(\zeta) = p'_*(\xi)(\zeta)$, *and*

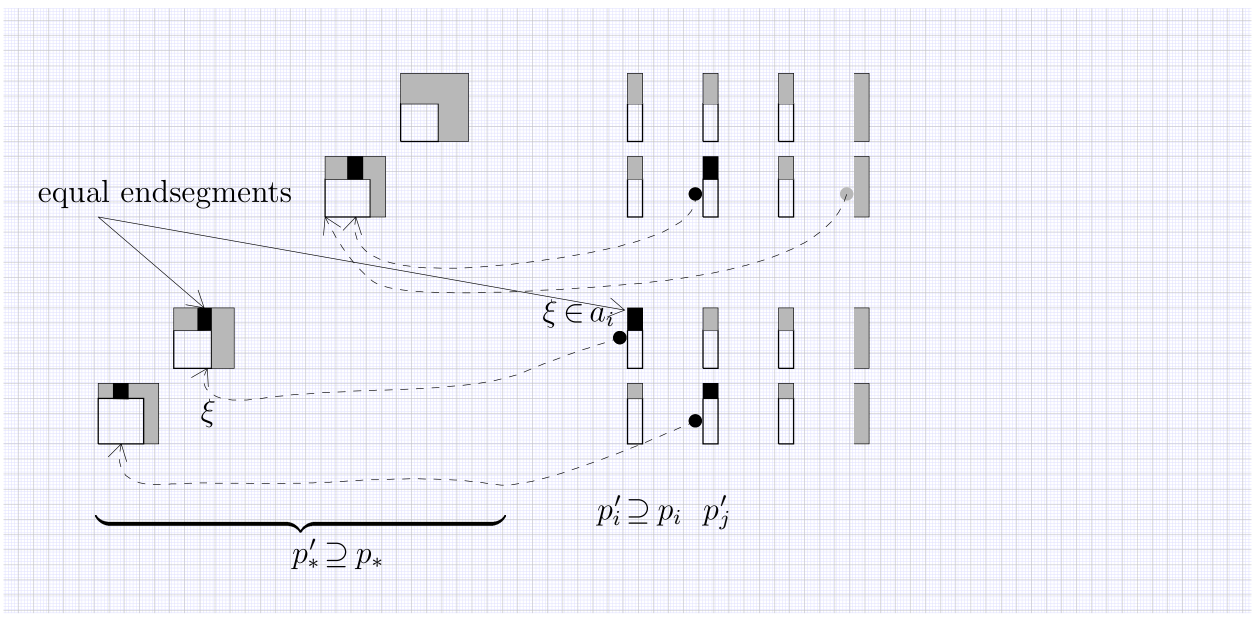
c) (Independence property) $\forall j \in \operatorname{supp}(p) : (a'_j \setminus a_j) \cap \bigcup_{i \in \operatorname{supp}(p), i \neq j} a'_i = \emptyset$.

$1 = (\emptyset, (\emptyset, \emptyset)_{i<\lambda})$ *is the maximal element of P.*

One may picture a condition $p \in P$ as



and an extension $(p'_*, (a'_i, p'_i)_{i<\lambda}) \leqslant_P (p_*, (a_i, p_i)_{i<\lambda})$ as



The gray areas indicate new 0-1-values in the extension $p' \leqslant_P p$, and the black areas indicate equality of new values forced by linking ordinals ξ.

Let G be a V-generic filter for P. Several generic objects can be extracted from G. It is easy to see that the set

$$G_* = \{p_* \in P_* \mid (p_*, (a_i, p_i)_{i<\lambda}) \in G\}$$

is V-generic for the partial order P_*. Set $A_* = \bigcup G_*$: $\bigcup_{n<\omega} [\aleph_n, \aleph_{n+1})^2 \to 2$. For $\xi \in [\aleph_n, \aleph_{n+1})$ let

$$A_*(\xi) = \{(\zeta, A_*(\xi, \zeta)) | \zeta \in [\aleph_n, \aleph_{n+1})\} \colon [\aleph_n, \aleph_{n+1}) \to 2$$

be the (characteristic function of the) ξ-th new COHEN subset of $\aleph_{n+1}$ in the generic extension.

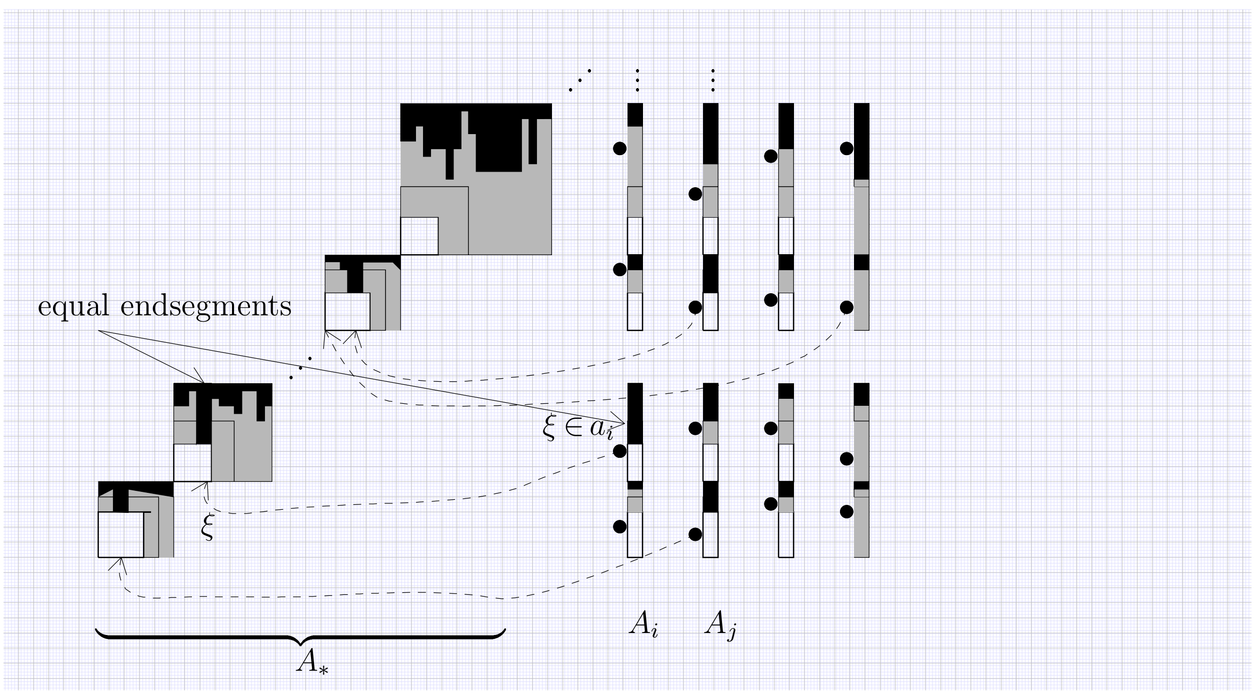
For $i<\lambda$ let

$$A_i = \bigcup \{p_i \,|\, (p_*, (a_j, p_j)_{j<\lambda}) \in G\} \colon [\aleph_0, \aleph_\omega) \to 2$$

be the (characteristic function of the) i-th subset of $\aleph_\omega$ adjoined by the forcing P. A_i is V-generic for the forcing P_0.

By the linking property $b)$ of Definition 2, on a final segment, the characteristic functions $A_i \upharpoonright [\aleph_n, \aleph_{n+1})$ will be equal to some $A_*(\xi)$. The independence property $c)$ ensures that sets $A_i, A_j \subseteq \aleph_\omega$ with $i \neq j$ correspond to eventually disjoint, "parallel" paths through the forcing P_*.

The generic filter and the extracted generic objects may be pictured as follows. Black colour indicates agreement between parts of the A_i and of A_*; for each $i<\lambda$, some endsegment of $A_i \cap \aleph_{n+1}$ occurs as an endsegment of some vertical cut in $A_* \cap \aleph_{n+1}^2$.



Lemma 3. *P satisfies the $\aleph_{\omega+2}$-chain condition.*

Proof. Let $\{(p_*^j, (a_i^j, p_i^j)_{i<\lambda}) | j < \aleph_{\omega+2}\} \subseteq P$. We shall show that two elements of the sequence are compatible. Since

$$\mathrm{card}(P_*) = \mathrm{card}(P_0) \leqslant 2^{\aleph_\omega} = \aleph_{\omega+1}$$

we may assume that there is $p_* \in P_*$ such that $\forall j < \aleph_{\omega+2}$: $p_*^j = p_*$. We may assume that the supports $\operatorname{supp}((p_*, (a_i^j, p_i^j)_{i<\lambda})) \subseteq \lambda$ form a Δ-system with a finite kernel $I \subseteq \lambda$. Finally we may assume that there are $(a_i, p_i)_{i\in I}$ such that $\forall j < \aleph_{\omega+2} \forall i \in I$: $(a_i^j, p_i^j) = (a_i, p_i)$. Then $(p_*^0, (a_i^0, p_i^0)_{i<\lambda})$ and $(p_*, (a_i^1, p_i^1)_{i<\lambda})$ are compatible since

$$(p_*, (a_i^0 \cup a_i^1, p_i^0 \cup p_i^1)_{i<\lambda}) \leqslant_P (p_*^0, (a_i^0, p_i^0)_{i<\lambda})$$

and

$$(p_*, (a_i^0 \cup a_i^1, p_i^0 \cup p_i^1)_{i<\lambda}) \leqslant_P (p_*^1, (a_i^1, p_i^1)_{i<\lambda}).$$

□

By Lemma 3, cardinals $\geqslant \aleph_{\omega+2}^V$ are absolute between V and $V[G]$.

Fuzzifying the A_i

We want to construct a model which contains all the A_i and a map which maps every A_i to its index i. An *injective* map $\lambda \rightarrowtail \mathcal{P}(\aleph_\omega)$ for some high λ would imply large consistency strength (see [1]). To disallow such maps, the A_i are replaced by their equivalence classes modulo an appropriate equivalence relation.

The *exclusive or* function $\oplus: 2 \times 2 \to 2$ is defined by

$$a \oplus b = 0 \text{ iff } a = b.$$

Obviously, $(a \oplus b) \oplus (b \oplus c) = a \oplus c$. For functions

$$A, A': \operatorname{dom}(A) = \operatorname{dom}(A') \to 2$$

define the *pointwise exclusive or* $A \oplus A': \operatorname{dom}(A) \to 2$ by

$$(A \oplus A')(\xi) = A(\xi) \oplus A'(\xi).$$

For functions $A, A': (\aleph_\omega \setminus \aleph_0) \to 2$ define an equivalence relation $\sim$ by

$$A \sim A' \text{ iff } \exists n < \omega ((A \oplus A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V).$$

This relation is clearly reflexive and symmetric. We show transitivity. Consider $A \sim A' \sim A''$. Choose $n < \omega$ such that

$$(A \oplus A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V$$

and

$$(A' \oplus A'') \restriction \aleph_{n+1} \in V[G_*] \wedge (A' \oplus A'') \restriction [\aleph_{n+1}, \aleph_\omega) \in V.$$

Then

$$(A \oplus A'') \restriction \aleph_{n+1} = ((A \oplus A') \restriction \aleph_{n+1} \oplus (A' \oplus A'') \restriction \aleph_{n+1}) \in V[G_*]$$

and

$$(A \oplus A'') \restriction [\aleph_{n+1}, \aleph_\omega) = ((A \oplus A') \restriction [\aleph_{n+1}, \aleph_\omega) \oplus (A' \oplus A'') \restriction [\aleph_{n+1}, \aleph_\omega)) \in V.$$

Hence $A \sim A''$.

For $A: (\aleph_\omega \setminus \aleph_0) \to 2$ define the $\sim$-equivalence class $\tilde{A} = \{A' | A' \sim A\}$.

The symmetric submodel

Our final model will be a model generated by the following parameters and their constituents

- $T_* = \mathcal{P}(<\kappa)^{V[A_*]}$, setting $\kappa = \aleph_\omega^V$;
- $\vec{A} = (\tilde{A}_i \,|\, i < \lambda)$.

The model

$$N = \mathrm{HOD}^{V[G]}(V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i<\lambda} \tilde{A}_i)$$

consists of all sets which, in $V[G]$ are hereditarily definable from parameters in the transitive closure of $V \cup \{T_*, \vec{A}\}$. This model is *symmetric* in the sense that it is generated from parameters which are invariant under certain (partial) isomorphisms of the forcing P.

Lemma 4. *N is a model of* ZF*, and there is a surjection $f: \mathcal{P}(\kappa) \to \lambda$ in N.*

Proof. Note that for every $i < \lambda$: $A_i \in N$.
(1) Let $i < j < \lambda$. Then $A_i \nsim A_j$.
Proof. Assume instead that $A_i \sim A_j$. Then take $n < \omega$ such that $v = (A_i \oplus A_j) \restriction [\aleph_{n+1}, \aleph_\omega) \in V$. The set

$$D = \{(p_*, (a_k, p_k)_{k<\lambda}) | \exists \xi \in [\aleph_{n+1}, \aleph_\omega) (\xi \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi))\} \in V$$

is readily seen to be dense in P. Take $(p_*, (a_k, p_k)_{k<\lambda}) \in D \cap G$. Take $\xi \in [\aleph_{n+1}, \aleph_\omega)$ such that

$$\xi \in \mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi)).$$

Since $p_i \subseteq A_i$ and $p_j \subseteq A_j$ we have $v(\xi) \neq A_i(\xi) \oplus A_j(\xi)$ and $v \neq (A_i \oplus A_j) \upharpoonright [\aleph_{n+1}, \aleph_\omega)$. Contradiction. $qed(1)$

Thus

$$f(z) = \begin{cases} i, \text{ if } z \in \tilde{A}_i\,; \\ 0, \text{ else}; \end{cases}$$

is a well-defined surjection $f \colon \mathcal{P}(\kappa) \to \lambda$, and f is definable in N from the parameters κ and $\vec{A}$. □

The main theorem will be established by showing that, in N, the situation below κ is largely as in V, in particular $\kappa = \aleph_\omega^N$. This requires an analysis of sets of ordinals in N.

Lemma 5. *Every set $X \in N$ is definable in $V[G]$ in the following form: there are an $\in$-formula φ, $x \in V$, $n < \omega$, and $i_0, \ldots, i_{l-1} < \lambda$ such that*

$$X = \{u \in V[G] \,|\, V[G] \vDash \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}})\}.$$

Proof. By the original definition, every set in N is definable in $V[G]$ from finitely many parameters in

$$V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i<\lambda} \tilde{A}_i\,.$$

To reduce the class of defining parameters to

$$V \cup \{T_*, \vec{A}\} \cup \{A_* \upharpoonright (\aleph_{n+1}^V)^2 \,|\, n < \omega\} \cup \{A_i | i < \lambda\}$$

observe:

- Let $x \in T_*$ be a bounded subset of $\aleph_\omega^V$ with $x \in V[A_*]$. A standard product analysis of the generic extension $V[G_*] = V[A_*]$ of V yields that $x \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$ for some $n < \omega$.

- Let $y \in \tilde{A}_i$. Then $y \sim A_i$, i.e.,

$$(y \oplus A_i) \upharpoonright \aleph_{m+1} \in V[G_*] \wedge (y \oplus A_i) \upharpoonright [\aleph_{m+1}, \aleph_\omega) \in V$$

for some $m<\omega$. Let $z=(y\oplus A_i)\upharpoonright\aleph_{m+1}\in V[A_*]$. By the previous argument $z\in V[A_*\upharpoonright(\aleph_{n+1}^V)^2]$ for some $n<\omega$. Let $z'=(y\oplus A_i)\upharpoonright[\aleph_{m+1},\aleph_\omega)\in V$. Then

$$\begin{aligned} y &= (y\upharpoonright\aleph_{m+1})\cup(y\upharpoonright[\aleph_{m+1},\aleph_\omega)) \\ &= ((z\oplus A_i)\upharpoonright\aleph_{m+1})\cup((z'\oplus A_i)\upharpoonright[\aleph_{m+1},\aleph_\omega)) \\ &\in V[A_*\upharpoonright(\aleph_{n+1}^V)^2, A_i]. \end{aligned}$$

Finitely many parameters of the form $A_*\upharpoonright(\aleph_{n+1}^V)^2$ can then be incorporated into a single such parameter taking a sufficiently high $n<\omega$. □

Approximating N

Concerning sets of ordinals, the model N can be approximated by "mild" generic extensions of the ground model. Note that many set theoretic notions only refer to ordinals and sets of ordinals.

Lemma 6. *Let $X\in N$ and $X\subseteq\mathrm{Ord}$. Then there are $n<\omega$ and $i_0,\ldots,i_{l-1}<\lambda$ such that*

$$X\in V[A_*\upharpoonright(\aleph_{n+1}^V)^2, A_{i_0},\ldots,A_{i_{l-1}}].$$

Proof. By Lemma 5 take an $\in$-formula φ, $x\in V$, $n<\omega$, and $i_0,\ldots,i_{l-1}<\lambda$ such that

$$X=\{u\in\mathrm{Ord}\,|\,V[G]\vDash\varphi(u,x,T_*,\vec{A},A_*\upharpoonright(\aleph_{n+1}^V)^2,A_{i_0},\ldots,A_{i_{l-1}})\}.$$

By taking n sufficiently large, we may assume that

$$\forall j<k<l\,\forall m\in[n,\omega)\forall\delta\in[\aleph_m,\aleph_{m+1})\colon A_{i_j}\upharpoonright[\delta,\aleph_{m+1})\neq A_{i_k}\upharpoonright[\delta,\aleph_{m+1}).$$

For $j<l$ set

$$a_{i_j}^*=\{\xi\,|\,\exists m\leqslant n\exists\delta\in[\aleph_m,\aleph_{m+1})\colon A_{i_j}\upharpoonright[\delta,\aleph_{m+1})=A_*(\xi)\upharpoonright[\delta,\aleph_{m+1})\}$$

where $A_*(\xi)=\{(\zeta,A_*(\xi,\zeta))|(\xi,\zeta)\in\mathrm{dom}(A_*)\}$. By the properties of Q, $a_{i_j}^*\subseteq\aleph_{n+1}$ is finite and $\forall m\leqslant n\colon\mathrm{card}(a_{i_j}^*\cap[\aleph_m,\aleph_{m+1}))=1$.

Now define

$$\begin{aligned} X' = \{\,u\in\mathrm{Ord}\;|\; & \text{there is } p=(p_*,(a_i,p_i)_{i<\lambda})\in P \text{ such that} \\ & p_*\upharpoonright(\aleph_{n+1}^V)^2\subseteq A_*\upharpoonright(\aleph_{n+1}^V)^2, \\ & a_{i_0}\supseteq a_{i_0}^*,\ldots,a_{i_{l-1}}\supseteq a_{i_{l-1}}^*, \\ & p_{i_0}\subseteq A_{i_0},\ldots,p_{i_{l-1}}\subseteq A_{i_{l-1}},\ \text{and} \\ & p\Vdash\varphi(\check{u},\check{x},\sigma,\tau,\dot{A}\upharpoonright(\check{\aleph}_{n+1})^2,\dot{A}_{i_0},\ldots,\dot{A}_{i_{l-1}})\}, \end{aligned}$$

where $\sigma, \tau, \dot{A}, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}$ are canonical names for $T_*, \vec{A}, A_*, A_{i_0}, \ldots, A_{i_{l-1}}$ resp.

Then $X' \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$.

(1) $X \subseteq X'$.
Proof. Consider $u \in X$. Take $p = (p_*, (a_i, p_i)_{i<\lambda}) \in G$ such that

$$p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}).$$

Then $p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$ and $p_{i_0} \subseteq A_{i_0}, \ldots, p_{i_{l-1}} \subseteq A_{i_{l-1}}$. Using a density argument we may also assume that $\mathrm{card}(a_{i_0} \cap \aleph_{n+1}) = \ldots = \mathrm{card}(a_{i_{l-1}} \cap \aleph_{n+1}) = n$. Then $a_{i_0} \supseteq a_{i_0}^*, \ldots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*$. Thus $u \in X'$. *qed*(1)

The converse direction, $X' \subseteq X$, is more involved and uses an isomorphism argument. Suppose for a contradiction that there were $u \in X' \setminus X$. Then take a condition $p = (p_*, (a_i, p_i)_{i<\lambda}) \in P$ as in the definition of X', i.e.,

(2) $p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$,

(3) $a_{i_0} \supseteq a_{i_0}^*, \ldots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*$,

(4) $p_{i_0} \subseteq A_{i_0}, \ldots, p_{i_{l-1}} \subseteq A_{i_{l-1}}$, and

(5) $p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}})\}$.

By $u \notin X$ take $p' = (p'_*, (a'_i, p'_i)_{i<\lambda}) \in G$ such that

(6) $p' \Vdash \neg\varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}})$.

By genericity we may assume that

(7) $p'_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$

(8) $a'_{i_0} \supseteq a_{i_0}^*, \ldots, a'_{i_{l-1}} \supseteq a_{i_{l-1}}^*$, and

(9) $p'_{i_0} \subseteq A_{i_0}, \ldots, p'_{i_{l-1}} \subseteq A_{i_{l-1}}$.

By strengthening the conditions we can arrange that p and p' have similar "shapes" whilst preserving the above conditions (2) to (9):

(10) ensure that $\mathrm{supp}(p) = \mathrm{supp}(p')$; choose some $\aleph_{m+1}$ such that $\forall i \in \mathrm{supp}(p)(a_i \subseteq \aleph_{m+1} \wedge a'_i \subseteq \aleph_{m+1})$;

(11) extend the a_i and a'_i such that

$$\forall i \in \mathrm{supp}(p) \forall k \leqslant m \colon \mathrm{card}(a_i \cap [\aleph_k, \aleph_{k+1})) = \mathrm{card}(a'_i \cap [\aleph_k, \aleph_{k+1})) = 1;$$

(12) also extend the conditions so that they involve the same "linking" ordinals, possibly at different positions within the conditions:

$$\bigcup_{i<\lambda} a_i = \bigcup_{i<\lambda} a'_i$$

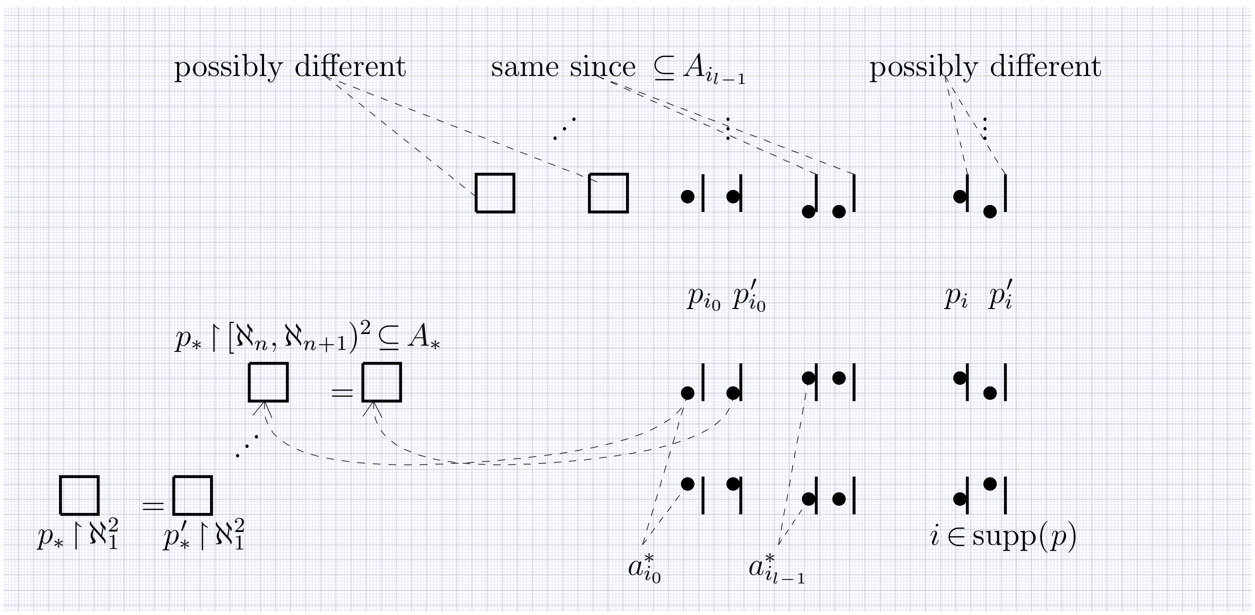
(13) extend the p_* and p_i's in p and p' resp. so that for some sequence $(\delta_k | k < \omega)$:

$$\operatorname{dom}(p_*) = \operatorname{dom}(p'_*) = \bigcup_{k<\omega} [\aleph_k, \delta_k)^2$$

and

$$\forall i \in \operatorname{supp}(p) \colon \operatorname{dom}(p_i) = \operatorname{dom}(p'_i) = \bigcup_{k<\omega} [\aleph_k, \delta_k).$$

The following picture tries to capture some aspects of the shape similarity between p and p'; corresponding components of p and p' are drawn side by side



Now define a map

$$\pi \colon (P \restriction p, \leqslant_P) \to (P \restriction p', \leqslant_P),$$

where the restricted partial orders are defined as $P \restriction p = \{q \in P \mid q \leqslant_P p\}$ and $P \restriction p' = \{q' \in P \mid q' \leqslant_P p'\}$. For $q = (q_*, (b_i, q_i)_{i<\lambda}) \leqslant_P (p_*, (a_i, p_i)_{i<\lambda}) = p$ define $\pi(p) = \pi(q_*, (b_i, q_i)_{i<\lambda})) = (q'_*, (b'_i, q'_i)_{i<\lambda})$ by the following three conditions:

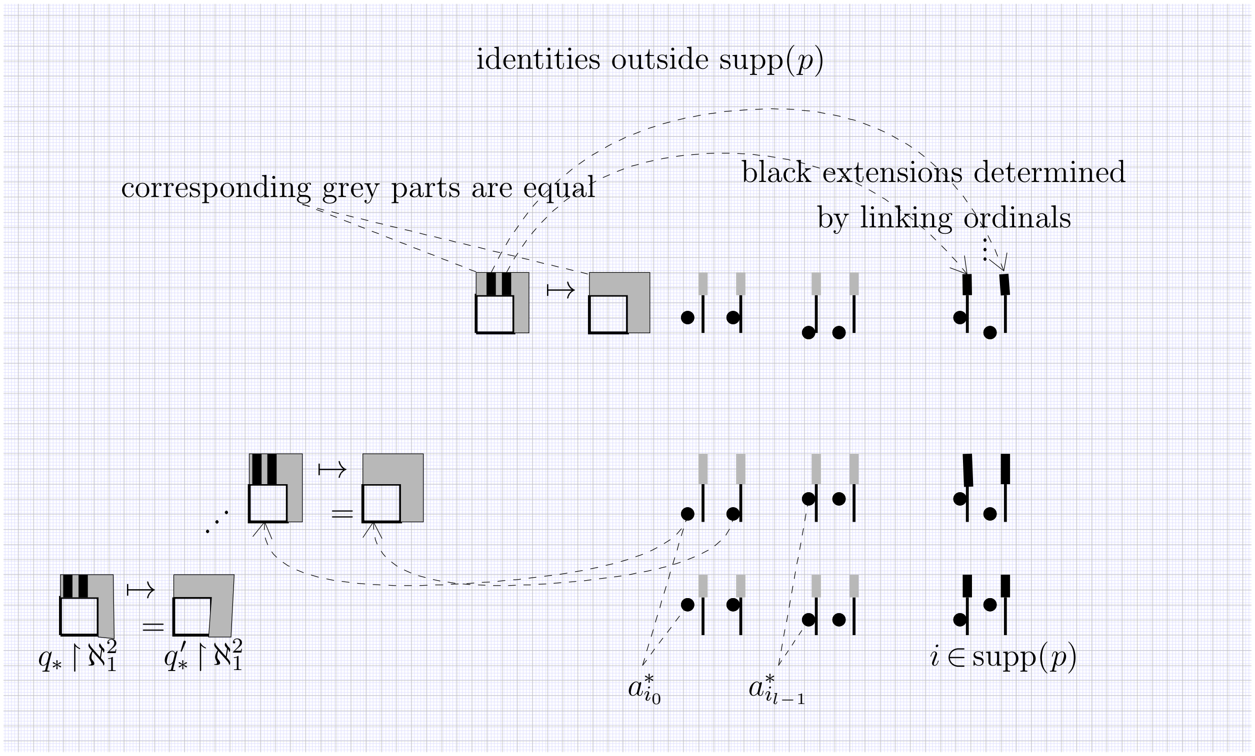
(14) $q'_* = (q_* \setminus p_*) \cup p'_*$; note that this is a legitimate function since $\operatorname{dom}(p_*) = \operatorname{dom}(p'_*)$;

(15) for $i < \lambda$ let $b'_i = (b_i \setminus a_i) \cup a'_i$; so if $i \in \operatorname{supp}(p)$, the $m+1$ ordinals in a_i are substituted by the $m+1$ ordinals in a'_i; if $i \notin \operatorname{supp}(p)$, we have $b'_i = b_i$;

(16) for $i \in \lambda \setminus \operatorname{supp}(q)$ let $q'_i = \emptyset$, and for $i \in \operatorname{supp}(q)$ define $q'_i \colon \operatorname{dom}(q_i) \to 2$ by

$$q'_i(\zeta) = \begin{cases} p'_i(\zeta), \text{ if } \zeta \in \operatorname{dom}(p_i); \\ q_*(\xi', \zeta), \text{ if } \zeta \notin \operatorname{dom}(p_i) \wedge \exists k < \omega \colon \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi'\}; \\ q_i(\zeta), \text{ if } \zeta \notin \operatorname{dom}(p_i) \wedge \exists k < \omega \colon \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset. \end{cases}$$

Here is a picture of (some features of) π.



We verify that $\pi\colon (P \restriction p, \leqslant_P) \to (P \restriction p', \leqslant_P)$ is an *isomorphism*.

(17) $(q'_*, (b'_i, q'_i)_{i<\lambda}) \in P$, since it has the same structure as $(q_*, (b_i, q_i)_{i<\lambda})$, with some function values altered.

(18) $(q'_*, (b'_i, q'_i)_{i<\lambda}) \leqslant_P (p'_*, (a'_i, p'_i)_{i<\lambda})$.
Proof. $q'_* \supseteq p'_*$ since $q'_* = (q_* \setminus p_*) \cup p'_*$, see (14). Similarly we get $b'_i \supseteq a'_i$ and $q'_i \supseteq p'_i$. To check the linking property (Definition 2, b)), consider $i < \lambda$, $n < \omega$, and $\xi' \in a'_i \cap [\aleph_n, \aleph_{n+1})$. For $\zeta \in \mathrm{dom}(q'_i \setminus p'_i)$ we have

$$q'_i(\zeta) = q_*(\xi', \zeta) = q'_*(\xi', \zeta).$$

Finally we have to show the independence property (Definition 2, c)) within the linking ordinals. Consider $j \in \mathrm{supp}(p') = \mathrm{supp}(p)$. We claim that $(b'_j \setminus a'_j) \cap \bigcup_{i \in \mathrm{supp}(p'), i \neq j} b'_i = \emptyset$. Assume for a contradiction that $\xi' \in (b'_j \setminus a'_j) \cap b'_i$ for some $i \in \mathrm{supp}(p')$, $i \neq j$. Then $\xi' \in (b_j \setminus a_j) \cap ((b_i \setminus a_i) \cup a'_i)$. The case $\xi' \in (b_j \setminus a_j) \cap (b_i \setminus a_i)$ is impossible by the independence property in $q \leqslant_P p$. And

$$(b_j \setminus a_j) \cap \bigcup_{i \in \mathrm{supp}(p'), i \neq j} a'_i = (b_j \setminus a_j) \cap \bigcup_{i \in \mathrm{supp}(p), i \neq j} a_i = \emptyset$$

by the independence property in $q \leqslant_P p$ and by (12). *qed*(18)

(19) π is order-preserving.
Proof. Consider

$$r=(r_*,(c_i,r_i)_{i<\lambda})\leqslant_P q=(q_*,(b_i,q_i)_{i<\lambda})\leqslant_P p=(p_*,(a_i,p_i)_{i<\lambda})$$

and $\pi(r)=r'=(r'_*,\,(c'_i,\,r'_i)_{i<\lambda})$, $\pi(q)=q'=(q'_*,\,(b'_i,\,q'_i)_{i<\lambda})$. We show that $r'\leqslant_P q'$. Concerning the inclusions:

- $r'_*=(r_*\setminus p_*)\cup p'_*\supseteq(q_*\setminus p_*)\cup p'_*=q'_*$;
- $c'_i=(c_i\setminus a_i)\cup a'_i\supseteq(b_i\setminus a_i)\cup a'_i=b'_i$;
- if $i\in\lambda\setminus\mathrm{supp}(q)$, then $q'_i=\emptyset$ and hence $q'_i\subseteq r'_i$. If $i\in\mathrm{supp}(q)$ then $i\in\mathrm{supp}(r)$, and $\mathrm{dom}(r'_i)=\mathrm{dom}(r_i)$ and $\mathrm{dom}(q'_i)=\mathrm{dom}(q_i)$. So we have

  $$\mathrm{dom}(p_i)=\mathrm{dom}(p'_i)\subseteq\mathrm{dom}(q'_i)\subseteq\mathrm{dom}(r'_i).$$

  For $\zeta\in\mathrm{dom}(q'_i)$ we have to show that $q'_i(\zeta)=r'_i(\zeta)$. In case $\zeta\in\mathrm{dom}(p_i)$ we have

  $$q'_i(\zeta)=p'_i(\zeta)=r'_i(\zeta).$$

  In case $\zeta\notin\mathrm{dom}(p_i)\wedge\exists k<\omega\colon\zeta\in[\aleph_k,\aleph_{k+1})\wedge a'_i\cap[\aleph_k,\aleph_{k+1})=\{\xi'\}$ we have

  $$q'_i(\zeta)=q_*(\xi',\zeta)=r_*(\xi',\zeta)=r'_i(\zeta).$$

  In case $\zeta\in\mathrm{dom}(p_i)\wedge\exists k<\omega\colon\zeta\in[\aleph_k,\aleph_{k+1})\wedge a'_i\cap[\aleph_k,\aleph_{k+1})=\emptyset$ we have

  $$q'_i(\zeta)=q_i(\zeta)=r_i(\zeta)=r'_i(\zeta).$$

For the linking property consider $i<\lambda$, $n<\omega$, and $\xi'\in b'_i\cap[\aleph_n,\aleph_{n+1})$. We have to check that $\forall\zeta\in\mathrm{dom}(r'_i\setminus q'_i)\cap[\aleph_n,\aleph_{n+1})\colon r'_i(\zeta)=r'_*(\xi,\zeta)$. So consider $\zeta\in\mathrm{dom}(r'_i\setminus q'_i)\cap[\aleph_n,\aleph_{n+1})$. Note that $b'_i=(b_i\setminus a_i)\cup a'_i$. In case $\xi'\in a'_i$ we get:

$$r'_i(\zeta)=r_*(\xi',\zeta)=r'_*(\xi',\zeta).$$

If $\xi'\in b_i\setminus a_i$, $\xi'\notin a'_i$ and so $a'_i\cap[\aleph_n,\aleph_{n+1})=\emptyset$. Hence

$$r'_i(\zeta)=r_i(\zeta)=r_*(\xi',\zeta)=r'_*(\xi',\zeta).$$

For the linking property consider $j\in\mathrm{supp}(q')$. We have to show that

$$(c'_j\setminus b'_j)\cap\bigcup_{i\in\mathrm{supp}(q'),i\neq j}c'_i=\emptyset.$$

Suppose for a contradiction that $\xi' \in (c_j' \setminus b_j') \cap \bigcup_{i \in \mathrm{supp}(q'), i \neq j} c_i'$. Then $\xi' \in c_j' \setminus b_j' = c_j \setminus b_j$. Take $i \in \mathrm{supp}(q')$, $i \neq j$ such that $\xi' \in c_i'$. If $i \in \mathrm{supp}(p')$ this contradicts the property $r' \leqslant_P p'$. So $i \in \mathrm{supp}(q') \setminus \mathrm{supp}(p')$. Then $c_i' = c_i$ and $\xi' \in (c_j \setminus b_j) \cap c_i$. But this contradicts the independence property for $r \leqslant_P q$. *qed*(19)

The definition of the map π only uses properties of p and p' which are the same for both of p and p'. So we can similarly define a map

$$\pi' : (P \upharpoonright p', \leqslant_P) \to (P \upharpoonright p, \leqslant_P),$$

where for $q' = (q_*', (b_i', q_i')_{i<\lambda}) \leqslant_P (p_*', (a_i', p_i')_{i<\lambda}) = p'$ the image $\pi'(q') = (q_*, (b_i, q_i)_{i<\lambda})$ is defined by

(20) $q_* = (q_*' \setminus p_*') \cup p_*$;

(21) for $i < \lambda$ let $b_i = (b_i' \setminus a_i') \cup a_i$;

(22) for $i \in \lambda \setminus \mathrm{supp}(q')$ let $q_i = \emptyset$, and for $i \in \mathrm{supp}(q')$ define $q_i : \mathrm{dom}(q_i') \to 2$ by

$$q_i(\zeta) = \begin{cases} p_i(\zeta), \text{ if } \zeta \in \mathrm{dom}(p_i'), \\ q_*'(\xi, \zeta), \text{ if } \zeta \notin \mathrm{dom}(p_i') \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi\}, \\ q_i'(\zeta), \text{ if } \zeta \notin \mathrm{dom}(p_i') \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset. \end{cases}$$

The maps π and π' are inverses:

(23) $\pi' \circ \pi : (P \upharpoonright p, \leqslant_P) \to (P \upharpoonright p, \leqslant_P)$ is the identity on $(P \upharpoonright p, \leqslant_P)$.
Proof. Let $(q_*, (b_i, q_i)_{i<\lambda}) \leqslant_P p = (p_*, (a_i, p_i)_{i<\lambda})$ and let $\pi(q_*, (b_i, q_i)_{i<\lambda}) = (q_*', (b_i', q_i')_{i<\lambda})$. Concerning the first component,

$$q_* \xrightarrow{\pi} (q_* \setminus p_*) \cup p_*' \xrightarrow{\pi'} (((q_* \setminus p_*) \cup p_*') \setminus p_*') \cup p_* = q_*.$$

For $i < \lambda$,

$$b_i \xrightarrow{\pi} (b_i \setminus a_i) \cup a_i' \xrightarrow{\pi'} (((b_i \setminus a_i) \cup a_i') \setminus a_i') \cup a_i = b_i.$$

For $i \in \lambda \setminus \mathrm{supp}(q)$, $q_i = q_i' = \emptyset$ and so

$$q_i \xrightarrow{\pi} q_i' \xrightarrow{\pi'} q_i.$$

Now consider $i \in \mathrm{supp}(q) = \mathrm{supp}(q')$. Then $\mathrm{dom}(q_i) = \mathrm{dom}(q_i')$. Let $\zeta \in \mathrm{dom}(q_i)$. In case $\zeta \in \mathrm{dom}(p_i) = \mathrm{dom}(p_i')$ we have

$$q_i(\zeta) = p_i(\zeta) \xrightarrow{\pi} p_i'(\zeta) = q_i'(\zeta) \xrightarrow{\pi'} p_i(\zeta) = q_i(\zeta).$$

In case $\zeta \notin \mathrm{dom}(p_i) \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi\}$ let $a_i' \cap [\aleph_k, \aleph_{k+1}) = \{\xi'\}$. Then $q_i(\zeta) = q_*(\xi, \zeta)$ and $q_i'(\zeta) = q_*'(\xi', \zeta)$. Then

$$q_i(\zeta) \xrightarrow{\pi} q_*'(\xi', \zeta) \xrightarrow{\pi'} q_*(\xi, \zeta) = q_i(\zeta).$$

Finally, if $\zeta \notin \mathrm{dom}(p_i) \wedge \exists k < \omega \colon \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset$ then $q_i(\zeta) = q_i(\zeta)$ and so

$$q_i(\zeta) \stackrel{\pi}{\longmapsto} q_i'(\zeta) \stackrel{\pi'}{\longmapsto} q_i(\zeta).$$

Thus

$$p \stackrel{\pi}{\longmapsto} p' \stackrel{\pi'}{\longmapsto} p.$$

qed(23)

Similarly,

(24) $\pi \circ \pi' \colon (P' \upharpoonright p', \leqslant_P) \to (P' \upharpoonright p', \leqslant_P)$ is the identity on $(P' \upharpoonright p', \leqslant_P)$.

Hence $\pi \colon (P \upharpoonright p, \leqslant_P) \to (P \upharpoonright p', \leqslant_P)$ is an isomorphism. Before we apply π to generic filters and objects defined from them, we note some properties of π.

(25) Let $q = (q_*, (b_i, q_i)_{i<\lambda}) \leqslant_P p$ and $\pi(p) = (q_*', (b_i', q_i')_{i<\lambda})$. Then $q_*' \upharpoonright (\aleph_{n+1}^V)^2 = q_* \upharpoonright (\aleph_{n+1}^V)^2$, and $q_{i_0}' = q_{i_0}, \ldots, q_{i_{l-1}}' = q_{i_{l-1}}$.

Now let H_0 be a V-generic filter for $(P \upharpoonright p, \leqslant_P)$ with $p \in H_0$. Then

$$H = \{r \in P \mid \exists q \in H_0 : q \leqslant_P r\}$$

is a V-generic filter for P with $p \in H$.

Moreover, $H_0' = \pi[H_0]$ is a V-generic filter for $(P \upharpoonright p', \leqslant_P)$ with $p' \in H_0'$ and

$$H' = \{r \in P \mid \exists q \in H_0' : q \leqslant_P r\}$$

is a V-generic filter for P with $p' \in H'$.

(26) $V[H] = V[H']$ since the generic filters can be defined from each other using the isomorphism $\pi \in V$.

Now define the parameters used in the definition of the model N from the generic filters H and H':

$$H_* = \{q_* \in P_* \mid (q_*, (b_i, q_i)_{i<\lambda}) \in H\}, T_* = \sigma^H, \vec{A} = \tau^H, A_* = \dot{A}^H, \text{ and } A_i = \dot{A}_i^H \text{ for } i < \lambda$$

and

$$H_*' = \{q_* \in P_* \mid (q_*, (b_i, q_i)_{i<\lambda}) \in H'\}, T_*' = \sigma^{H'}, \vec{A}' = \tau^{H'}, A_*' = \dot{A}^{H'}, \text{ and } A_i' = \dot{A}_i^{H'} \text{ for } i < \lambda.$$

where $\sigma, \tau, \dot{A}, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}$ are the canonical names for $T_*, \vec{A}, A_*, A_i$ resp. used before. The symbols G_*, $\vec{A}$, ... are only used within the current proof, they do not conflict with their use before and after this proof.

(27) $V[H_*] = V[H_*']$.
Proof. Since H_* is V-generic for P_* and $p_* \in P_*$, $H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}$ is, over the ground model V, equidefinable with H_*. Hence

$$\begin{array}{rcl} V[H_*] & = & V[H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}] \\ & \supseteq & V[\{(q_* \setminus p_*) \cup p_*' \mid q_* \in H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}] \\ & = & V[H_*' \cap \{q_* \in P_* \mid q_* \supseteq p_*'\}] \\ & = & V[H_*'] \end{array}$$

qed(27)

This implies

(28) $T_* = T_*'$.

Let $A_* = \bigcup H_*$ and $A_*' = \bigcup H_*'$.

(29) $A_* \upharpoonright (\aleph_{n+1}^V)^2 = A_*' \upharpoonright (\aleph_{n+1}^V)^2$.
Proof. Note that the map π is the identity on the $*$-component below $\aleph_{m+1}^V$. *qed*(29)

(30) For $i < \lambda$: $A_i \sim A_i'$.
Proof. Recall $A_i = \bigcup \{q_i \mid (q_*, (b_j, q_j)_{j<\lambda}) \in H\}$: $[\aleph_0, \aleph_\omega^V) \to 2$. Since the map π maps the set b_i of linking ordinals to $(b_i \setminus a_i) \cup a_i'$ the linking ordinals in the relevant sets b_i are equal to the linking ordinals in the sets b_i' with possibly finitely many exceptions. This means that the characteristic functions A_i and A_i' will be equal above p_i and p_i' respectively in all cardinal intervals $[\aleph_k, \aleph_{k+1})$ with $k > m$). In other words,

$$(A_i \oplus A_i') \upharpoonright [\aleph_{m+1}^V, \aleph_\omega^V) \in V.$$

The functions $A_i \upharpoonright \aleph_{m+1}^V$ and $A_i' \upharpoonright \aleph_{m+1}^V$ are determined in the cardinal intervals $[\aleph_k^V, \aleph_{k+1}^V)$ for $k \leqslant m$ by $p_i \upharpoonright [\aleph_k^V, \aleph_{k+1}^V)$ and $p_i' \upharpoonright [\aleph_k^V, \aleph_{k+1}^V)$ and some cuts $A_*(\xi)$ and $A_*(\xi')$ respectively. Hence $A_i \upharpoonright [\aleph_k^V, \aleph_{k+1}^V), A_i' \upharpoonright [\aleph_k^V, \aleph_{k+1}^V) \in V[A_* \upharpoonright (\aleph_{m+1}^V)^2] = V[A_*' \upharpoonright (\aleph_{m+1}^V)^2]$. Thus

$$(A_i \oplus A_i') \upharpoonright \aleph_{m+1}^V \in V[H_*] \text{ and } (A_i \oplus A_i') \upharpoonright [\aleph_{m+1}^V, \aleph_\omega^V) \in V,$$

i.e., $A_i \sim A_i'$. *qed*(30)

This implies immediately that the sequences of equivalence classes agree in both models:

(31) $\vec{A} = \vec{A}'$.

(32) $A_{i_0} = A_{i_0}', \ldots, A_{i_{l-1}} = A_{i_{l-1}}'$.
Proof. Note that the isomorphism π is the identity at the indices $i_0, \ldots, i_{l-1}$. *qed*(32)

Since $p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}})\}$ and $p \in H$ we have

$$V[H] \vDash \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}).$$

Since $p' \Vdash \neg\varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \restriction (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \ldots, \dot{A}_{i_{l-1}}) \}$ and $p' \in H'$ we have

$$V[H'] \vDash \neg\varphi(u, x, T'_*, \vec{A}', A'_* \restriction P_* \restriction (\aleph_{n+1}^V)^2, A'_{i_0}, \ldots, A'_{i_{l-1}}).$$

But the various equalities proved above imply

$$V[H] \vDash \neg\varphi(u, x, T_*, \vec{A}, A_* \restriction P_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}),$$

which is the desired contradiction. □

Wrapping up

We show that the approximation models are mild generic extensions of V.

Lemma 7. *Let $n < \omega$ and $i_0, \ldots, i_{l-1} < \lambda$. Then cardinals are absolute between V and $V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$.*

Proof. Take $p^0 = (p_*^0, (a_i^0, p_i^0)_{i<\lambda}) \in G$ such that $\{i_0, \ldots, i_{l-1}\} \subseteq \mathrm{supp}(p^0)$. Since the models $V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}]$ are monotonely growing with n we may assume that n is large enough such that

$$\forall i \in \mathrm{supp}(q) \forall \xi \in a_i^0 \colon \xi \in \aleph_{n+1}.$$

Since every $A_{i_j} \cap \aleph_{n+1}^V$ can be computed from $A^* \restriction (\aleph_{n+1}^V)^2$, we have

$$V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \ldots, A_{i_{l-1}}] = V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0} \restriction [\aleph_{n+1}^V, \aleph_\omega^V), \ldots, A_{i_{l-1}} \restriction [\aleph_{n+1}^V, \aleph_\omega^V)].$$

Let $P' = (P', \supseteq, \emptyset)$ be the forcing

$$P' = \{r \mid \exists (\delta_m)_{n<m<\omega} (\forall m \in (n, \omega) \colon \delta_m \in [\aleph_m^V, \aleph_{m+1}^V) \wedge r \colon \bigcup_{n<m<\omega} [\aleph_m^V, \delta_m) \to 2)\},$$

which adjoins Cohen subsets to the $\aleph_m$'s with $m > n$.

(2) $(A_{i_0} \restriction [\aleph_{n+1}^V, \aleph_\omega^V), \ldots, A_{i_{l-1}} \restriction [\aleph_{n+1}^V, \aleph_\omega^V))$ is V-generic for $(P')^l = \underbrace{P' \times \ldots \times P'}_{l \text{ times}}$.

Proof. Let $D \subseteq (P')^l$ be dense open, $D \in V$. We have to show that D is met by $(A_{i_0} \restriction [\aleph_{n+1}^V, \aleph_\omega^V), \ldots, A_{i_{l-1}} \restriction [\aleph_{n+1}^V, \aleph_\omega^V))$. Let

$$D' = \{(p_*, (a_i, p_i)_{i<\lambda}) \in Q \mid (p_{i_0} \restriction [\aleph_{n+1}^V, \aleph_\omega^V), \ldots, p_{i_{l-1}} \restriction [\aleph_{n+1}^V, \aleph_\omega^V)) \in D\}.$$

This set is dense in P below p^0: consider $p^1=(p^1_*,(a^1_i,p^1_i)_{i<\lambda})\leqslant_P(p^0_*,(a^0_i,p^0_i)_{i<\lambda})=p^0$. Take $(\delta_m)_{n<m<\omega}$ such that

$$p^1_*\restriction[\aleph^V_{n+1},\aleph^V_\omega)^2\colon\bigcup_{n<m<\omega}[\aleph^V_m,\delta_m)^2\rightarrow 2.$$

Take $p_{i_0},\ldots,p_{i_{l-1}}$ such that

$$(p_{i_0}\restriction[\aleph^V_{n+1},\aleph^V_\omega),\ldots,p_{i_{l-1}}\restriction[\aleph^V_{n+1},\aleph^V_\omega))\in D,$$

and such that $p_{i_0},\ldots,p_{i_{l-1}}$ have the same domains. Through some ordinals in $a^1_{i_0},\ldots,a^1_{i_{l-1}}$, the choice of $p_{i_0},\ldots,p_{i_{l-1}}$ determines some values of p_* by the linking property b) of Definition 2:

$$\forall i<\lambda\forall m<\omega\forall\xi\in a_i\cap[\aleph_m,\aleph_{m+1})\forall\zeta\in\mathrm{dom}(p_i\setminus p^1_i)\cap[\aleph_m,\aleph_{m+1})\colon p_i(\zeta)=p_*(\xi)(\zeta).$$

The independence property implies that the linking sets $a^1_{i_0},\ldots,a^1_{i_{l-1}}$ are pairwise disjoint *above* $\aleph^V_{n+1}$, i.e., the sets

$$a^1_{i_0}\cap[\aleph^V_{n+1},\aleph^V_\omega),\ldots,a^1_{i_{l-1}}\cap[\aleph^V_{n+1},\aleph^V_\omega)$$

are pairwise disjoint. So the linking requirements can be satisfied simultaneously. Then we can amend the definition of the other components of $p\leqslant p^1$ and obtain $p\in D'$.

By the genericity of G take $(p_*,(a_i,p_i)_{i<\lambda})\in D'\cap G$. Then

$$(p_{i_0}\restriction[\aleph^V_{n+1},\aleph^V_\omega),\ldots,p_{i_{l-1}}\restriction[\aleph^V_{n+1},\aleph^V_\omega))\in D$$

with

$$p_{i_0}\restriction[\aleph^V_{n+1},\aleph^V_\omega)\subseteq A_{i_0}\restriction[\aleph^V_{n+1},\aleph^V_\omega),\ldots,p_{i_{l-1}}\restriction[\aleph^V_{n+1},\aleph^V_\omega))\subseteq A_{i_{l-1}}\restriction[\aleph^V_{n+1},\aleph^V_\omega),$$

as required. $qed(2)$

The forcing $(P')^l$ is $<\aleph_{n+2}$-closed. $A_*\restriction(\aleph^V_{n+1})^2$ is V-generic for the forcing

$$P_*\restriction(\aleph^V_{n+1})^2=\{r\restriction(\aleph^V_{n+1})^2\,|\,r\in P_*\}.$$

By the GCH in V, $\mathrm{card}(P_*\restriction(\aleph^V_{n+1})^2)=\aleph_{n+1}$. Hence every dense subset of $P_*\restriction(\aleph^V_{n+1})^2$ which is in $V[A_{i_0}\restriction[\aleph^V_{n+1},\aleph^V_\omega),\ldots,A_{i_{l-1}}\restriction[\aleph^V_{n+1},\aleph^V_\omega)]$ is already an element of V. Thus $A_*\restriction(\aleph^V_{n+1})^2$ is $V[A_{i_0}\restriction[\aleph^V_{n+1},\aleph^V_\omega),\ldots,A_{i_{l-1}}\restriction[\aleph^V_{n+1},\aleph^V_\omega)]$-generic for $P^*\restriction(\aleph^V_{n+1})^2$.

This means that

$$V[A_*\restriction(\aleph^V_{n+1})^2,A_{i_0}\restriction[\aleph^V_{n+1},\aleph^V_\omega),\ldots,A_{i_{l-1}}\restriction[\aleph^V_{n+1},\aleph^V_\omega)]$$

is equal to the two-stage iteration

$$V[A_{i_0}\restriction[\aleph_{n+1}^V,\aleph_\omega^V),\ldots,A_{i_{l-1}}\restriction[\aleph_{n+1}^V,\aleph_\omega^V)][A_*\restriction(\aleph_{n+1}^V)^2],$$

which does not destroy cardinals. $\square$

Lemma 8. *Cardinals are absolute between N and V, and in particular $\kappa=\aleph_\omega^V=\aleph_\omega^N$.*

Proof. If not, then there is a function $f\in N$ which collapses a cardinal in V. By Lemma 6, f is an element of some model $V[A_*\restriction(\aleph_{n+1}^V)^2, A_{i_0},\ldots,A_{i_{l-1}}]$ as above. But this contradicts Lemma 7. $\square$

Lemma 9. GCH *holds in N below $\aleph_\omega$.*

Proof. If $X\subseteq\aleph_n$ and $X\in N$ then X is an element of some model $V[A_*\restriction(\aleph_{n+1}^V)^2, A_{i_0},\ldots, A_{i_{l-1}}]$ as above. Since $A_{i_0},\ldots,A_{i_{l-1}}$ do not adjoin new subsets of $\aleph_n$ we have that

$$X\in V[A_*\restriction(\aleph_{n+1}^V)^2].$$

Hence $\mathcal{P}(\aleph_n^V)\cap N\in V[A_*\restriction(\aleph_{n+1}^V)^2]$. GCH holds in $V[A_*\restriction(\aleph_{n+1}^V)^2]$. Hence there is a bijection $\mathcal{P}(\aleph_n^V)\cap N\leftrightarrow\aleph_{n+1}^V$ in $V[A_*\restriction(\aleph_{n+1}^V)^2]$ and hence in N. $\square$

Discussion and Remarks

The above construction straightforwardly generalises to other cardinals κ of cofinality ω. In that extension, cardinals $\leqslant\kappa$ are preserved, GCH holds below κ, and there is a surjection from $\mathcal{P}(\kappa)$ onto some arbitrarily high cardinal λ. To work with singular cardinals κ of *uncountable* cofinality, finiteness properties in the construction have to be replaced by the property of being of cardinality $<\mathrm{cof}(\kappa)$. This yields results like the following choiceless violation of Silver's theorem [6].

Theorem 10. *Let V be any ground model of* ZFC + GCH *and let λ be some cardinal in V. Then there is a cardinal preserving model $N\supseteq V$ of the theory* ZF + "GCH *holds below $\aleph_{\omega_1}$" + "there is a surjection from $\mathcal{P}(\aleph_{\omega_1})$ onto λ". Moreover, the axiom of dependent choices* DC *holds in N.*

Note that in [5], Saharon Shelah studied uncountably singular cardinal arithmetic under DC, without assuming AC. The "local" GCH below $\aleph_{\omega_1}$ in the conclusion of the above Theorem cannot be changed to the property $\mathrm{card}(\bigcup_{\alpha<\aleph_{\omega_1}}\mathcal{P}(\alpha))=\aleph_{\omega_1}$ since Theorem 4.6 of [5] basically implies that then $\mathcal{P}(\aleph_{\omega_1})$ would be wellorderable of ordertype $\geqslant\lambda$. By results of [1] an *injective* failure of SCH with big λ has high consistency strength. But here we are working without assuming any large cardinals.

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