On the Splitting Number at Regular Cardinals

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Abstract

Let κ,λ be regular uncountable cardinals such that $\kappa^+ < \lambda$. We construct a generic extension with $s(\kappa) = \lambda$ starting from a ground model in which $o(\kappa) = \lambda$ and prove that assuming $\neg 0^{\P}$, $s(\kappa) = \lambda$ implies that $o(\kappa) \geq \lambda$ in the core model.

1 Introduction

The splitting number is a cardinal invariant mostly known for its continuum version $s = s(\aleph_0)$. Generalizations of this invariant to regular uncountable cardinals have been studied mainly by S. Kamo, T. Miyamoto, M. Motoyoshi, T. Suzuki [3] and J. Zapletal [2].

For a cardinal κ and two sets $a, x \in [\kappa]^\kappa$ we say x splits a if both $a \setminus x$ and $a \cap x$ have cardinality κ. A family of sets $F \subset [\kappa]^\kappa$ is a splitting family if for all $a \in [\kappa]^\kappa$ there exists some $x \in F$ which splits a. The splitting number $s(\kappa)$ is the minimal cardinality of a splitting family $F \subset [\kappa]^\kappa$.

M. Motoyoshi showed that for a regular uncountable cardinal κ, $s(\kappa) \geq \kappa$ if and only if κ is inaccessible, and T. Suzuki [3] proved that $s(\kappa) \geq \kappa^+$ if and only if κ is weakly compact. S. Kamo and T. Miyamoto independently showed how to force $s(\kappa) \geq \kappa^{++}$ from the assumption of a $2^\kappa-$supercompact cardinal κ. J. Zapletal [2] proved that $s(\kappa) = \kappa^{++}$ implies there exists an inner model with a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.

The question of whether the lower bound of Zapletal can be improved remained open.[1] The purpose of the present paper is to answer it negatively and to show the following:

[1]The second author likes to thank J. Cummings and S. Friedman for stating to him this question.

Theorem 1. *Let κ,λ be regular uncountable cardinals such that $\kappa^+ < \lambda$. $s(\kappa) = \lambda$ is equiconsistent to the existence of a measurable cardinal κ with $o(\kappa) = \lambda$.*

The following explains some ideas behind the forcing construction. The basic construction of S. Kamo, as sketched in [2], starts with a supercompact cardinal κ and uses a $\kappa-$support iteration $\mathbb{P}_{\kappa^{++}} = \{\mathbb{P}_\alpha, Q_\alpha, \mid \alpha < \kappa^{++}\}$ of generalized Mathias forcings $Q_\alpha = \mathbb{P}(U_\alpha)$ which adds a generating set $k_\alpha \subset \kappa$ to the $V^{\mathbb{P}_\alpha}$ measure U_α on κ, i.e. $k_\alpha \subset^* x$ for every set $x \in U_\alpha$. This iteration satisfies $\kappa^+ - c.c.$ So every family $F \subset [\kappa]^\kappa$ in $V^{\mathbb{P}_{\kappa^{++}}}$ belongs already to $V^{\mathbb{P}_\alpha}$, for some $\alpha < \kappa^{++}$. The fact that the rest of the iteration $\mathbb{P}_{\kappa^{++}}/\mathbb{P}_\alpha$ adds an $U_\alpha-$generating set, implies that no family $F \subset [\kappa]^\kappa$ in $V^{\mathbb{P}_\alpha}$ is a splitting family in the final model $V^{\mathbb{P}_{\kappa^{++}}}$.

In our situation, we do not have a supercompact cardinal, and so it is unclear how to use generalized Mathias forcings. However there is a natural replacement, the Radin forcing, which also produces generating sets. Iteration of Radin forcing is problematic, but in many cases it is possible to avoid it. So, suppose that κ is a measurable of the Mitchell order $o(\kappa) \geq \kappa^{++}$, which necessary by the result of Zapletal [2]. Let $\vec{U} = \langle U_\alpha \mid \alpha < \kappa^{++} \rangle$ be a witnessing sequence of measures over κ. The sequence is long enough to have repeat points (we assume GCH). Consider first applying Radin forcing with $\vec{U}$. This forcing is equivalent to the Radin forcing with the initial segment of $\vec{U}$, up to its first repeat point. This implies that 2^κ will remain κ^+, and hence $s(\kappa)$ will not increase.

The next attemp will be to use the extender based Radin forcing, in order to blow up simultaneously the power of κ. But how do we do this only with measures? It is possible to assume initially a bit more like $\mathcal{P}^2(\kappa)-$hyper-measurability, and forcing with a Mitchell increasing sequence of $\kappa^{++}-$extenders. Although this version looks promising, it has some specific problems, i.e. it fails to introduce measurability of κ in some suitable intermediate extensions, which is a key of the construction (see section 2). It turns out that the solution hides in the measures but requires a modification in the point of view. The basic idea is to perceive each measure U_α as an extender. Namely, let $j_\alpha : V \to M_\alpha$ be the corresponding ultrapower embedding. Derive a $(\kappa, \kappa + \alpha + 1)-$extender E_α from j_α, i.e.
$E_\alpha = \langle E_\alpha(\beta) \mid \beta \leq \kappa + \alpha \rangle$, where $X \in E_\alpha(\beta)$ iff $\beta \in j_\alpha(X)$. Note that κ is the single generator of E_α and all $E_\alpha(\beta)$'s (with $\beta \geq \kappa$) are isomorphic to

U_α. At the first glance, this replacement looks rather useless. However, it turns out that there is a crucial difference between the usual Prikry, Magidor, Radin forcing and their extender based versions. This difference can be used to create a more complex repeat point structure. From global point of view there are no repeat points in the (final) generic extension since the generic sequence added for $E_\alpha(\alpha)$ will allow us to separate U_α from the rest of the measures. On the other hand we will show that with certain restrictions, there are many subforcings which provid "local repeat points". The local repeat points will be used to extend some U_α in the generic extensions by the subforcings. The splitting number argument is completed by proving that the rest of the extender based forcing adds a generating set to the extension of U_α.
On the other hand, once one is interested in increasing the power of κ only, then the number of generators plays the crucial role. Namely, using a full extender or only its measures corresponding to the generators in the extender based Prikry forcing has the same effect on the power of κ.

Let us show, for example, that the Prikry forcing and its extender based variation are not the same. Let U be a normal measure over κ, $j_U : V \to M_U$ the corresponding elementary embedding. Define a (κ, κ^+)–extender $E = \langle E(\beta) \mid \beta < \kappa^+ \rangle$ derived from j_U:

$$X \in E(\beta) \text{ iff } \beta \in j_U(X).$$

Clearly, U and E have the same elementary embedding and the same ultrapower. However the Prikry forcing $\mathcal{P}_U$ and the extender based Prikry forcing $\mathcal{P}_E$ are not the same. Obviously, $\mathcal{P}_U$ is a natural subforcing of $\mathcal{P}_E$, but the last forcing is richer. Thus, let $\langle t_\beta \mid \beta < \kappa^+ \rangle$ be generic sequences added by a generic $G(\mathcal{P}_E)$ of $\mathcal{P}_E$, i.e. t_β is a generic ω–sequence for $E(\kappa, \beta)$. Consider

$$A := \{\beta < \kappa^+ \mid t_\beta(0) \neq t_\kappa(0)\}.$$

Then, obviously, $A \notin V$. However, for every $\alpha < \kappa^+$, $A \cap \alpha \in V$. This implies that A is not in a Prikry extension, since by [8] such extension cannot add new fresh subsets to κ^+.

So, in general, a restriction of an extender to the supremum of its generators may produce a weaker forcing than the forcing with the full extender.

Carmi Merimovich in [1] introduced a very general setting for dealing with the extender based Magidor and Radin forcings. The forcing used here will fit nicely his frame.

We assume a familiarity with Merimovich paper [1] and will follow his notation.

2 Forcing $s(\kappa) = \lambda$ from $o(\kappa) = \lambda$

Let κ, λ be regular cardinal such that $\kappa^+ < \lambda$ and $o(\kappa) = \lambda$.
Fix a Mitchell increasing sequence of extenders $\overline{E} = \langle E_\alpha \mid \alpha < \lambda \rangle$ be such that for every $\alpha < \lambda$

1. $\overline{E} \restriction \alpha \in \mathrm{Ult}(V, E_\alpha)$,
   where $\overline{E} \restriction \alpha = \langle E_\beta \mid \beta < \alpha \rangle$,

2. E_α is a $(\kappa, \kappa + \sigma(E_\alpha))$–extender, for some $\sigma(E_\alpha), \alpha < \sigma(E_\alpha) < \lambda$.

The first non-trivial case is $\lambda = \kappa^{++}$. We fix a Mitchell increasing sequence of measures $\langle U_\alpha \mid \alpha < \kappa^{++} \rangle$ on κ and derive extenders E_α from the ultrapower embeddings $j_\alpha : V \to M_\alpha$ by U_α's. The simplest is to take $\sigma(E_\alpha) = \alpha + 1$.

Following [1], we denote the Magidor-Radin extender based forcing associated with $\overline{E}$ by $\mathbb{P}_{\overline{E},\lambda}$.

We will argue that $V^{\mathbb{P}_{\overline{E},\lambda}}$ satisfies $s(\kappa) = \lambda$. The outline of this argument is similar to the construction of S. Kamo, as sketched in [2]. Every family $F \subset [\kappa]^\kappa$ in a generic extension $V^{\mathbb{P}_{\overline{E},\lambda}}$ is contained in a generic extension of a sub-forcing of $\mathbb{P}'$ of $\mathbb{P}_{\overline{E},\lambda}$ for which κ is measurable in $V^{\mathbb{P}'}$, and the rest of the forcing $\mathbb{P}_{\overline{E},\lambda}/\mathbb{P}'$ adds a generating set to some $V^{\mathbb{P}'}$ measure on κ. The sub-forcings we will use are the restrictions of $\mathbb{P}_{\overline{E},\lambda}$ to some suitable models $N \prec H_\theta$ for a sufficiently large regular cardinal θ.

Let $\{\dot{X}_i \mid i < \tau\}$ be a sequence of $\tau < \lambda$ many nice $\mathbb{P}_{\overline{E},\lambda}$−names of subsets of κ. Since $\mathbb{P}_{\overline{E},\lambda}$ satisfies $\kappa^{++}.c.c$, we may find an elementary sub structure $N \prec H_\theta$ for some sufficiently large regular θ, which satisfy $\kappa^+ \subset N$, $\mathbb{P}_{\overline{E},\lambda}, \overline{E}, \{\dot{X}_i \mid i < \tau\} \in N$, $|N| < \lambda$, and such that every $\dot{X}_i$ is a $\mathbb{P}_{\overline{E},\lambda} \cap N$ name. The key of the argument is that it is possible for $N \cap \mathbb{P}_{\overline{E},\lambda}$ to be a sub-forcing of $\mathbb{P}_{\overline{E},\lambda}$ by which κ remains measurable and the complement forcing $\mathbb{P}_{\overline{E},\lambda}/(N \cap \mathbb{P}_{\overline{E},\lambda})$ adds a generating set to some measure on κ. More precisely, assuming that $N \cap \lambda = \delta < \lambda$, we prove that $N \cap \mathbb{P}_{\overline{E},\lambda}$ is isomorphic to the extender based poset $\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ associated to the restricted sequences $\overline{E} \restriction \delta = \{E_\alpha \mid \alpha < \delta\}$. We then apply a repeat point argument to prove

that the $E_\delta-$normal measure $E_\delta(\kappa)$ extends to a measure U_δ in $V^{N\cap\mathbb{P}_{\overline{E},\lambda}}$, and prove that the completion poset adds a generating set k_δ to this measure.

2.1 The sub-forcing $N \cap \mathbb{P}_{\overline{E},\lambda}$

Let us fix some sufficiently large regular cardinal θ such that $\mathbb{P}_{\overline{E},\lambda} \in H_\theta$. Throughout this section we shall consider elementary substructures $N \prec H_\theta$ which satisfy:

- $|N| < \lambda$,
- $\mathbb{P}_{\overline{E},\lambda} \in N$,
- ${}^\kappa N \subset N$,
- $N \cap \lambda = \delta \in \lambda$.

Lemma 2. *the poset $N \cap \mathbb{P}_{\overline{E},\lambda}$ is a sub-forcing of $\mathbb{P}_{\overline{E},\lambda}$, i.e. the inclusion map of $N \cap \mathbb{P}_{\overline{E},\lambda}$ in $\mathbb{P}_{\overline{E},\lambda}$ is a complete embedding.*

Proof. It is clear that for $p, q \in N \cap \mathbb{P}_{\overline{E},\lambda}$ then $p \leq_{\mathbb{P}_{\overline{E},\lambda}} q$ if and only if $p \leq_{N\cap\mathbb{P}_{\overline{E},\lambda}} q$. Since $\mathbb{P}_{\overline{E},\lambda} \in N$, it is also clear that p, q are incompatible in $\mathbb{P}_{\overline{E},\lambda}$ if and only if they are incompatible in $N\cap\mathbb{P}_{\overline{E},\lambda}$. For every $p \in \mathbb{P}_{\overline{E},\lambda}$ define $q = p \restriction N \in \mathbb{P}_{\overline{E},\lambda} \cap N$ as follows: First consider a condition $p = p_\rightarrow = \langle f, A\rangle$ which consists only of its top part. Denote $d = dom(f)$, then A is a $d-$tree. For every $\nu \in OB(d)$ the restricted function $\nu \restriction N$ belongs to $OB(d \cap N)$. Define a $d \cap N-$ tree, $A \restriction N = \{\langle \nu_0 \restriction N, ..., \nu_n \restriction N\rangle \mid \langle \nu_0, ..., \nu_n\rangle \in A\}$. Now set $p \restriction N = \langle f \restriction N, A \restriction N\rangle$.

For a general condition $p = p_\leftarrow{}^\frown p_\rightarrow \in \mathbb{P}_{\overline{E},\lambda}$, then set $p \restriction N = p_\leftarrow{}^\frown(p_\rightarrow \restriction N)$. It is straight forward to verify that for every $p \in \mathbb{P}_{\overline{E},\lambda}$, if $q' \in N$ is an extension of $p \restriction N$ then q' is compatible with p. Therefore $N \cap \mathbb{P}_{\overline{E},\lambda} \subset \mathbb{P}_{\overline{E},\lambda}$ is a sub-forcing. □

Therefore, for every $V-$generic set $G \subset \mathbb{P}_{\overline{E},\lambda}$ then $G \cap N$ is $V-$generic for the poset $N \cap \mathbb{P}_{\overline{E},\lambda}$.

We would now like to show that $N\cap\mathbb{P}_{\overline{E},\lambda}$ is isomorphic to the poset $\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ which is the Magidor-Radin forcing associated with the restricted sequence $\overline{E} \restriction \delta = \{E_\alpha \mid \alpha < \delta\}$. Writing that $\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ is the forcing associated with $\overline{E} \restriction \delta$ entails a nontrivial statement that $\delta = \sup_{\alpha<\delta} j_{E_\alpha}(\kappa)$. [2] Our assumption that

[2] see section 4 in [1].

$\alpha > \sigma(E_\alpha)$ for every $\alpha < \lambda$ implies that $\delta \geq \sup_{\alpha<\delta} j_{E_\alpha}(\kappa)$. Also, for every $\alpha < \lambda$ we assume $\sigma(E_\alpha) < \lambda$. Since λ is a regular cardinal then $j_{E_\alpha}(\kappa) < \lambda$, and if $\alpha < \delta = N \cap \lambda$ then $E_\alpha \in N$, so $j_{E_\alpha}(\kappa) \in N \cap \lambda = \delta$.

Another important consequences of $\delta = N \cap \lambda$ is the fact it is a local repeat point.

Definition 3. ρ is a local repeat point of $\overline{E}$ if for every $x \in [\rho]^{\leq\kappa}$, letting $d = \{\overline{\alpha} \in \mathfrak{D} \mid \alpha \in x\}$ then

$$\bigcap_{\alpha<\rho} E_\alpha(d) = \bigcap_{\alpha<o(\overline{E})} E_\alpha(d).$$

Lemma 4. *$\delta = N \cap \lambda$ is a local repeat point.*

Proof. Take $x \in [\delta]^{\leq\kappa}$ and let $d = \{\overline{\alpha} \in \mathfrak{D} \mid \alpha \in x\}$. Then $d \in N$ since ${}^{\kappa}N \subset N$ so $\langle E_\alpha(d) \mid \alpha < \lambda\rangle \in N$ as well. Each $E_\alpha(d)$ measures V_κ, and since $\kappa^+ < \lambda$ and λ is regular then there exists some $\tau < \lambda$ such that

$$\bigcap_{\alpha<\tau} E_\alpha(d) = \bigcap_{\alpha<o(\overline{E})} E_\alpha(d).$$

The elementarity of N implies that there exists such $\tau < \lambda$ in N. Hence $\tau < \delta$ and the result follows. □

Proposition 5. *$N \cap \mathbb{P}_{\overline{E},\lambda}$ is isomorphic to $\mathbb{P}_{\overline{E}\restriction\delta,\delta}$.*

Proof. We shall compare the structures of conditions $p \in N \cap \mathbb{P}_{\overline{E},\lambda}$ with conditions $q \in \mathbb{P}_{\overline{E}\restriction\delta,\delta}$.

By making a small abuse of notation, let us denote the set $\{\overline{\alpha} \in \mathfrak{D} \mid \alpha < \delta\}$[3] by $\mathfrak{D} \cap \delta$. Then $dom(f^{p\rightarrow}) \subset \mathfrak{D} \cap \delta$ for every $p \in N \cap \mathbb{P}_{\overline{E},\lambda}$. We claim that the property $dom(f^{p\rightarrow}) \subset \mathfrak{D} \cap \delta$ actually characterizes the conditions of $\mathbb{P}_{\overline{E},\lambda}$ which belong to N. First note that for any $p \in \mathbb{P}_{\overline{E},\lambda}$ then $p_\leftarrow \in V_\kappa \subset N$. Second, since $|f^{p\rightarrow}| \leq \kappa$, $rng(f^{p\rightarrow}) \subset V_\kappa$, and ${}^{\kappa}N \subset N$, we get that $dom(f^{p\rightarrow}) \subset \mathfrak{D} \cap \delta$ implies $f^{p\rightarrow} \in N$. Finally, if $f^{p\rightarrow} \in N$ then every $\nu \in OB(dom(f^{p\rightarrow}))$ must be a member of N as well and as $|A^{p\rightarrow}| = \kappa$ we find that $A^{p\rightarrow} \in N$.

Let us now consider this with the structure of conditions $q \in \mathbb{P}_{\overline{E}\restriction\delta,\delta}$. First note that the extender sequences

$$\overline{\alpha} = \langle\alpha\rangle^\frown\langle E_\xi \mid \xi < o(\overline{E}), \alpha < j_{E_\xi}(\kappa)\rangle$$

[3] see [1] 4.2 for the definition of $\mathfrak{D}$

appearing in the components of conditions $p \in \mathbb{P}_{\overline{E},\lambda}$ [4] are now replaced by a shorter extender sequences

$$\overline{\alpha} \restriction \delta = \langle \alpha \rangle^\frown \langle E_\xi \mid \xi < \delta, \alpha < j_{E_\xi}(\kappa) \rangle.$$

So the base set used in domain of functions appearing in $\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ is

$$\mathfrak{D} \restriction \delta = \{\overline{\alpha} \restriction \delta \mid \alpha < \delta\},$$

functions $f' \in \mathbb{P}^*_{\overline{E}\restriction\delta,\delta}$ have domain $d' \in [\mathfrak{D} \restriction \delta]^{\leq\kappa}$, $f' : d' \to \mathfrak{R}^{<\omega}$. Objects $\nu' \in OB(d')$, measures $E_\xi(d')$, $\xi < o(\overline{E} \restriction \delta) = \delta\}$, and $d'-$trees $T \subset [OB(d')]^{<\omega}$ are the appropriate variants of the cut down sequence $\overline{E} \restriction \delta$. For every $d \subset [\mathfrak{D}]^{\leq\kappa}$, let $d \restriction \delta = \{\overline{\alpha} \restriction \delta \mid \overline{\alpha} \in d\}$. We conclude that by replacing the sequences $\overline{\alpha} \restriction \delta \in \mathfrak{D} \restriction \delta$ with $\overline{\alpha} \in \mathfrak{D} \cap \delta$, we can therefore construct a simple translation map T such that

- for $d \in [\mathfrak{D} \cap \delta]^{<\kappa}$, T maps function $f' : d \restriction \delta \to \mathfrak{R}^{<\omega}$ to functions $f : d \to \mathfrak{R}^{<\omega}$, by replacing every $\overline{\alpha} \restriction \delta \in dom(f')$ with $\overline{\alpha} \in dom(f)$.
- T maps objects $\nu' \in OB(d \restriction \delta)$ to objects $\nu \in OB(d)$, by replacing every $\overline{\alpha} \restriction \delta \in dom(\nu')$ with $\overline{\alpha} \in dom(\nu)$.
- By extending T hereditarily, T maps $d \restriction \delta-$trees $A' \subset [OB(d \restriction \delta)]^{<\omega}$ to trees $A \subset [OB(d)]^{<\omega}$.

T is clearly a bijection. We claim that T maps $d \restriction \delta-$trees to $d-$trees. Following the definition of the measures $E_\alpha(d)$, it is clear that for every $\alpha < \delta$, T maps $E_\alpha(d \restriction \delta)$ sets to $E_\alpha(d)$ sets, this implies that sets

$$X \in E(d \restriction \delta) = \bigcap_{\alpha<\delta} E_\alpha(d \restriction \delta)$$

are mapped to sets

$$T(X) \in \bigcap_{\alpha<\delta} E_\alpha(d).$$

which by lemma 4 are members of $E(d)$. It is clear that in $Y \in E(d)$ then $T^{-1}(Y) \in \bigcap_{\alpha<\delta} E_\alpha(d \restriction \delta)$. It follows that we can extend T to a bijection from $\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ to $N \cap \mathbb{P}_{\overline{E},\lambda}$: First, for $q_\rightarrow = \langle f', A' \rangle \in \mathbb{P}^{\rightarrow}_{\overline{E}\restriction\delta,\delta}$ let $T(q_\rightarrow) = \langle T(f'), T(A') \rangle$. Then for general $q = q_\leftarrow{}^\frown q_\rightarrow \in \mathbb{P}_{\overline{E}\restriction\delta,\delta}$ set $T(q) = q_\leftarrow{}^\frown T(q_\rightarrow)$. Obviously T respects $\leq, \leq^*$ so T is an isomorphisms of Prikry type forcings. □

[4] these include $d = dom(f^{p_\rightarrow})$ and $dom(\nu)$ for $\nu \in OB(d)$

Our next goal is to show that the normal ultrafilter $E_\delta(\kappa) = \{X \subset \kappa \mid \kappa \in j_{E_\delta}(X)\}$ extends to a normal ultrafilter in a generic extension by $N \cap \mathbb{P}_{\overline{E},\lambda}$. We will apply a variant of the repeat point arguments in section 5 of [1] to our situation in which δ is a local repeat point of $\overline{E}$. Let $j_{E_\delta} : V \to M_\delta \cong Ult(V, E_\delta)$ be the E_δ induced ultrapower. Since $\overline{E}$ is Michell increasing sequence of extenders then $\overline{E} \restriction \delta$ is the sequence of extenders which appears on κ in M_δ. Moreover $\overline{E} \restriction \delta$ is used to generate the measure $E_\delta(d)$ for every $d \in [\mathfrak{D}]^{\leq \kappa}$. More precisely [1] defines

$$mc_\delta(d) = \{\langle j_{E_\delta}(\overline{\alpha}), R_\delta(\overline{\alpha})\rangle \mid \overline{\alpha} \in d, \alpha < j_{E_\delta}(\kappa)\}$$

where $R_\delta(\overline{\alpha})$ corresponds to an end segment of $\overline{\alpha}$ and is given by

$$R_\delta(\overline{\alpha}) = \langle \alpha \rangle^\frown \{E_\tau \mid \tau < \delta, \alpha < j_{E_\tau}(\kappa)\}.$$

The measure $E_\delta(d)$ is defined by $X \in E_\delta(d)$ if and only if $mc_\delta(d) \in j_{E_\delta}(X)$.

Let $p \in \mathbb{P}_{\overline{E},\lambda}$ and consider the end extension $j_{E_\delta}(p)_{\langle mc_\delta(d)}$ of $j_{E_\delta}(p)$ in $j_{E_\delta}(\mathbb{P}_{\overline{E},\lambda})$. As $R_\delta(\overline{\kappa}) = \langle \kappa \rangle^\frown \overline{E} \restriction \delta$ it follows that

$$j_{E_\delta}(p)_{\langle mc_\delta(d)\rangle \leftarrow} \in \mathbb{P}_{\overline{E}\restriction\delta,\delta}.^5$$

Now assuming that $p \in N \cap \mathbb{P}_{\overline{E},\lambda}$, it is straight forward to verify that

$$T(j_{E_\delta}(p)_{\langle mc_\delta(d)\rangle \leftarrow}) = p.$$

We are now ready to prove that $E_\delta(\kappa)$ extends in a generic extension by $N \cap \mathbb{P}_{\overline{E},\lambda}$. We first need the following preliminary lemma.

Lemma 6. *Let $p \in N \cap \mathbb{P}_{\overline{E},\lambda}$ and $\dot{X}$ such that $p \Vdash \dot{X} \subset \kappa$, then there exists an extension $q \leq^* p$ such that*

$$j_{E_\delta}(q)_{mc_\delta(q_\rightarrow)} \parallel \check{\kappa} \in j_{E_\delta}(\dot{X}).$$

Proof. Choose an elementary sub model $N^* \prec N$ such that $|N^*| = \kappa$, ${}^{<\kappa}N^* \subset N^*$, $N^* \cap \kappa^+ \in \kappa^+$, and $\dot{X}, p, N \cap \mathbb{P}_{\overline{E},\lambda} \in N^*$. The collection of all $\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ dense sets in N^* has cardinality κ. Using the fact $\leq^*$ for $\mathbb{P}^*_{\overline{E},\lambda}$ is κ^+-closed we can construct a direct extension $f^* \leq^* f^p$, and an f^*-tree A^* such that

- $p^* = \langle f^*, A^*\rangle \leq^* p_\rightarrow$,

[5] We use the fact that $\mathbb{P}^{M_\delta}_{\overline{E}\restriction\delta,\delta} = \mathbb{P}_{\overline{E}\restriction\delta,\delta}$

- for every $\langle\nu\rangle \in A^*$ and $D \in N$, if D is dense open (in $\mathbb{P}^*_{\overline{E},\lambda}$) below $f^p_{\vec{\nu}\restriction dom(f^p)}$ then $f^*_{\vec{\nu}} \in D$.

Denote $dom(f)$ by d and $\langle f^*, A^*\rangle$ by p^*. The construction of p^* can be carried inside N so we may assume $p^* \in N$. For every $\langle\nu\rangle \in A^*$ let

$$D_{\langle\nu\rangle} = \{g \leq^* f_{\langle\nu\restriction d\rangle} \mid \exists q_0, q_1, B, \text{ s.t. } q_1 \leq^* (p_{\langle\nu\rangle_\leftarrow})_\rightarrow, \\ q_0 \leq^* p^*_\leftarrow, \text{ and } q_0{}^\frown q_1{}^\frown\langle g, B\rangle \parallel \nu(\check{\kappa})_0 \in \dot{X}\}.$$

Then $D_{\langle\nu\rangle}$ belongs to N^* since $p^*_{\langle\nu\rangle_\leftarrow} \in V_\kappa \subset N$ and is dense open (in $\mathbb{P}^*_{\overline{E},\lambda}$) below $f_{\langle\nu\restriction d\rangle}$ by Prikry condition. Hence $f^*_{\langle\nu\rangle} \in D_{\langle\nu\rangle}$. Denote the components q_0, q_1, B which witness $f^*_{\langle\nu\rangle} \in D_{\langle\nu\rangle}$ by $q_0(\nu), q_1(\nu), B(\nu)$ respectively. Since $q_0(\nu) \leq p_\leftarrow$ then there exists some fixed q^*_0 such that the set $\{\nu \in Lev_0(A^*) \mid q^*_0 = q_0(\nu)\} \in E_\delta(f^*)$.

Next define

$$q'_1 = [q_1(\nu)]_{E_\delta(f^*)} = j_{E_\delta}(q_1)(mc_\delta(f^*)),$$

then $q'_1 \leq^* j_{E_\delta}(p^*)_{\langle mc_\delta(f^*)\rangle_\leftarrow}$. Setting $q^*_1 = T(q'_1)$ then $q^*_1 \in N \cap \mathbb{P}_{\overline{E},\lambda}$ is a direct extension of $p^*_\rightarrow$. Define $q^* = q^*_0{}^\frown q^*_1$, then $q^* \leq^* p$. We need to reduce the tree $A^{q^*_1}$. Let q be a direct extension of q^* such that for $A^{q\rightarrow}_{\langle\nu\rangle} \restriction dom(f^*) \subset B(\nu \restriction dom(f^*))$ for every $\nu \in Lev_0(A^{q\rightarrow})$. Since q'_1 was defined via the $E_\delta(f^*)$ ultrapower, then there exists a subset $Y \subset Lev_0(A^{q\rightarrow})$, $Y \in E_\delta(f^q)$ such that for all $\nu \in Y$,

$$q_{\langle\nu\rangle} \leq^* q_0(\nu)^\frown q_1(\nu)^\frown\langle f^*_{\langle\nu\restriction dom(f^*)\rangle}, B(\nu \restriction dom(f^*))\rangle$$

for every $\nu \in Y$. [6] Therefore

$$j_{E_\delta}(q)_{mc_\delta(q_\rightarrow)} \parallel \check{\kappa} \in j_{E_\delta}(\dot{X}).$$

□

We are now ready to define the extension $E_\delta(\kappa)$. Let $G_N \subset N \cap \mathbb{P}_{\overline{E},\lambda}$ be $V-$generic filter.

Definition 7. In $V[G_N]$ define $U_\delta \subset \mathcal{P}(\kappa)$ as follows: For every $X \subset \kappa$ in $V[G_N]$, $X \in U_\delta$ if and only if there exists some $p \in G_N$ such that

$$j_{E_\delta}(p)_{\langle mc_\delta(p_\rightarrow)\rangle} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{X}).$$

[6] Note that we cannot in general take $Y = Lev_0(A^{q\rightarrow})$, since q'_1 was defined by the $E_\delta(f^*)$ ultrapower, so the identification of $q_\rightarrow\langle\nu\rangle_\leftarrow$ with $q_1(\nu)$ may not hold for every $\nu \in Lev_0$ but only on some $E_\delta(f^*)$ set.

Note that $p \Vdash \dot{X} \in \dot{U}_\delta$
does not necessary imply that $j_{E_\delta}(p)_{\langle mc_\delta(p_\rightarrow)\rangle} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{X})$.

Proposition 8. *U_δ is a κ complete normal ultrafilter on κ in $V[G_N]$. Furthermore it extends $E_\delta(\kappa)$.*

Proof. We start by verifying that every $p \in N \cap \mathbb{P}_{\overline{E},\lambda}$ such that $p \Vdash \dot{X} \in \dot{U}_\delta$, has a direct extension $p^* \leq^* p$ such that $p^*_\leftarrow = p_\leftarrow$ and

$$j_{E_\delta}(p^*)_{\langle mc_\delta(p^*_\rightarrow)\rangle} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{X}).$$

First note that $p \Vdash \dot{X} \in \dot{U}_\delta$ and $j_{E_\delta}(p)_{\langle mc_\delta(p)\rangle} \parallel \check{\kappa} \in j_{E_\delta}(\dot{X})$ then $j_{E_\delta}(p)_{\langle mc_\delta(p)\rangle}$ must force "$\check{\kappa} \in j_{E_\delta}(\dot{X})$". Note that the number of possible extensions of $p_\leftarrow$ is less than κ. Let $\langle r_i \mid i < \tau\rangle$ be an enumeration these extensions, then we can construct an $\leq^*$-decreasing sequence of extensions $\langle t_i \mid i < \tau\rangle$ stronger than $p_\rightarrow$ such that for each $i < \tau$, if there exists some $t \leq^* t_{i\rightarrow}$ such that for $p' = r_i{}^\frown t$,

$$j_{E_\delta}(p')_{\langle mc_\delta(p')\rangle} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{X})$$

then t_{i+1} is one of such t. Otherwise, $t_{i+1} = t_i$. Let p^* be an extension of p such that $p^*_\leftarrow = p_\leftarrow$ and $p^*_\rightarrow$ is a common direct extension of all $\langle t_i \mid i < \tau\rangle$. It is easily seen that p^* meets our requirements, by Lemma 6.
We can now show that U_δ is a normal $\kappa-$complete ultrafilter on κ. U_δ clearly extends $E_\delta(\kappa)$. If $X \in U_\delta$ and $X \subset Y \subset \kappa$, then for some suitable names $\dot{X}, \dot{Y}$ there exists some $p \in G_N$ such that $p \Vdash \dot{X} \subset \dot{Y}$ and

$$j_{E_\delta}(p)_{\langle mc_\delta(p_\rightarrow)\rangle} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{X}).$$

Hence

$$j_{E_\delta}(p)_{\langle mc_\delta(p_\rightarrow)\rangle} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{Y}).$$

Next, let $p \in G_N$ be a condition forcing "$\langle \dot{X}_i \mid i < \kappa\rangle \subset \dot{U}_\delta$". Using the observation in the beginning of this proof, we can construct a decreasing sequence of direct extensions below $p_\rightarrow$ such that

$$j_{E_\delta}(p_i)_{mc_\delta(p_i)} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{X}).$$

Now set $f^* = \bigcup_{i<\kappa} f^{p_i}$ and construct an f^*-tree A^*: First set

$$Lev_0(A^*) = \{\nu \in OB(f^*) \mid \forall i < \nu(\overline{\kappa})_0 \quad \nu \restriction dom(f^{p_i}) \in A^{p_i\rightarrow}\},$$

then, for every $i < \kappa$ define

$$A_i = \{\vec{\nu} \in OB(f^*)^{<\omega} \mid \vec{\nu} \restriction dom(f^{p_i}) \in A^{p_i \to}\}$$

and set

$$A^*_{\langle \nu \rangle} = \bigcap_{i < \nu(\overline{\kappa})_0} A_i.$$

Then $p^* = p_{\leftarrow}{}^\frown \langle f^*, A^* \rangle$ force $\Delta_{i<\kappa} X_i \in U_\delta$. □

Remark 9. *We point out the differences between the local repeat point of $\overline{E}$ and (global) repeat point as defined in section 5 of [1]. If $\delta < o(\overline{E})$ is a (global) repeat point of $\overline{E}$ then all the normal measures $\{E_\gamma(\kappa) \mid \delta \leq \gamma < o(\overline{E})\}$ extend to measures in $\mathbb{P}_{\overline{E},\lambda}-$generic extension. Here in the local case, the argument is given to sub-forcing extensions $V^{\mathbb{P}_{\overline{E}\restriction\delta,\delta}}$, and $E_\delta(\kappa)$ is the only normal measure between those appearing in $\overline{E}$ which extends. For suitable $\delta' > \delta$ (that is $\delta' = N' \cap \mathbb{P}_{\overline{E},\lambda}$ for an appropriate structure N') one needs to force with $\mathbb{P}_{\overline{E}\restriction\delta',\delta'}/\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ over $V^{\mathbb{P}_{\overline{E}\restriction\delta,\delta}}$ in order to extend $E_{\delta'}(\kappa)$. In the next section we shall prove that forcing with $\mathbb{P}_{\overline{E}\restriction\delta',\delta'}/\mathbb{P}_{\overline{E}\restriction\delta,\delta}$ over $V^{\mathbb{P}_{\overline{E}\restriction\delta,\delta}}$ adds a generating set k_δ to the extension U_δ of $E_\delta(\kappa)$. We can therefore show that non of the measures $\{E_\alpha(\kappa) \mid \alpha < o(\overline{E})\}$ extends in the final model $V^{\mathbb{P}_{\overline{E},\lambda}}$. Therefore the forcing $\mathbb{P}_{\overline{E},\lambda}$ can be thought of as an iteration of adding generating sets to measures. As we will see in the next section, the generating sets k_δ can be seen to come from differences between generic Magidor-Radin clubs. Under this interpretation $\mathbb{P}_{\overline{E},\lambda}$ "hides" an iteration of club shooting.*

2.2 Adding a generating set to U_δ

We now address the forcing $\mathbb{P}_{\overline{E},\lambda}/N \cap \mathbb{P}_{\overline{E},\lambda}$. We shall prove that this forcing adds a generating set k_δ to the $V^{N\cap\mathbb{P}_{\overline{E},\lambda}}$ measure U_δ defined above. We need to add preliminary notations in order to define and work with the sets k_δ.

Let $G \subset \mathbb{P}_{\overline{E},\lambda}$ be a $V-$generic set. For every $\alpha < \lambda$ the following sets where defined in [1]:

- $G^\alpha = \bigcup\{f^{p\to}(\overline{\alpha}) \mid p \in G, \overline{\alpha} \in dom(f^{p\to})\}$,
- $C^\alpha = \{\overline{\nu}_0 \mid \overline{\nu} \in G^{\overline{\alpha}}\} \subseteq \kappa$.

C^κ is the well known Radin club associated with the Mitchell increasing sequence of measures $\{E_\beta(\kappa) \mid \beta < \lambda\}$. The other generic sets C^α, $\kappa < \alpha < \lambda$

are not clubs and correspond to $\{E_\beta(\alpha) \mid \beta < \lambda, \alpha < length(E_\beta)\}$. They can be associated with a filtration of C^κ by clubs.

Definition 1. Let $\tau \in C^\kappa$ be an ordinal in the generic Magidor-Radin club. Then there are $p \in G$ and $\nu \in Lev_0(A^{p\to})$ such that $\tau = \nu(\overline{\kappa})_0$ and $p_{\langle\nu\rangle} \in G$. Define

- $o^G(\tau) = o(\nu(\overline{\kappa}))$, for any (every) p, ν as above.
  Recall that $\nu(\overline{\kappa}) = \langle\tau, e_0, ..., e_\xi, ...\rangle$ $(\xi < \mu)$ and μ is called $o(\nu(\overline{\kappa}))$, i.e. the length of the sequence of extenders on τ.

- If $\alpha < \lambda$ is an ordinal for which $\overline{\alpha} \in dom(\nu)$ for some p, ν as above, then define $f_\alpha(\tau) = \nu(\overline{\alpha})_0$.
  I.e. it is the ordinal which corresponds to α (C_α) over the level τ.

Definition 2. For every $\alpha < \lambda$ set

$$k_\alpha = \{\tau \in C^{\overline{\kappa}} \mid \tau \in dom(f_\alpha) \text{ and } o^G(\tau) = f_\alpha(\tau)\}.$$

Remark 10. *Let $p \in \mathbb{P}_{\overline{E},\lambda}$ and $\alpha < \lambda$ for which $\overline{\alpha} \in dom(f^{p\to})$. By examining the definition of [1] the following can be easily verified:*

1. $\{\langle\nu\rangle \in A^{p\to} \mid \overline{\alpha} \in dom(\nu)\} \in E_\gamma(p_\to)$ *for every* $\gamma \geq \alpha$.

2. *Let*

   $$W_\alpha = \{\nu \mid \nu \in OB(d) \text{ for some } d, \overline{\alpha} \in dom(\nu), \text{ and } \nu(\overline{\alpha})_0 = o(\nu(\overline{\kappa}))\},$$

   then

   $$Lev_0(A^{p\to}) \cap W_\alpha \in E_\alpha(f^{p\to}) \setminus \bigcap_{\beta\neq\alpha} E_\beta(f^{p\to})^7.$$

   .

3. $p \Vdash \dot{k_\alpha} \subset^* \{\nu(\overline{\kappa})_0 \mid \nu \in Lev_0(A^{p\to}) \cap W_\alpha\}$. [8]

Let $N \prec H_\theta$, and $\delta = N \cap \lambda$ be as in the previous section. By lemma 2, the set $G_N = G \cap N$ is a $V-$generic for $N \cap \mathbb{P}_{\overline{E},\lambda}$. Let U_δ be the measure on κ extending $E_\delta(\kappa)$ in $V[G_N]$. We claim

[7] see also [1], definition 4.9 and $X_=, X_>, X_<$

[8] Here $Y \subset^* Y'$ if $Y' \setminus Y$ is bounded in κ.

Proposition 11. *k_δ is a $U_\delta-$generating set, i.e. for every $X \subset \kappa$ in $V[G_N]$ if $X \in U_\delta$ then $k_\delta \subset^* X$ in $V[G]$.*

Proof. Let $\dot{X}$ be a name for X in $V[G_N]$ and let $p \in G_N$ such that $p \Vdash_{\mathbb{P}_{\overline{E},\lambda} \cap N^*} \dot{X} \in \dot{U}_\delta$. Let us show that p has an extension $q \leq p$, $q \Vdash_{\mathbb{P}_{\overline{E},\lambda}} \dot{k}_\delta \subset^* \dot{X}$. By 5.8 in [1] or by the argument of Proposition 8, we may assume

$$j_{E_\delta}(p)_{\langle mc_\delta(p_\rightarrow)\rangle} \Vdash \check{\kappa} \in j_{E_\delta}(\dot{X}).$$

Define

$$X' = \{\nu \in Lev_0(A^{p_\rightarrow}) \mid p_{\langle\nu\rangle} \Vdash \nu(\check{\overline{\kappa}})_0 \in \dot{X}\}.$$

Then $X' \in E_\delta(p_\rightarrow)$. Let p^* be a direct extension of p in $\mathbb{P}_{\overline{E},\lambda}$ obtained by adding $\overline{\delta}$ to $dom(f^{p_\rightarrow})$.
Set

$$X^* = \{\nu \in Lev_0(A^{p^*_\rightarrow}) \mid p^*_{\langle\nu\rangle} \Vdash \nu(\check{\overline{\kappa}})_0 \in \dot{X}\}.$$

Clearly $X^* \in E_\delta(p^*_\rightarrow)$. Let p^{**} be the strong Prikry extension of p^* obtained by reducing $Lev_0(A^{p^*})$ to $\big(Lev_0(A^{p^*}) \setminus W_\delta\big) \cup \big(Lev_0(A^{p^*}) \cap W_\delta \cap X^*\big)$. We claim that $p^{**} \Vdash \dot{k}_\delta \subset^* \dot{X}$. Since

$$p^{**} \Vdash \dot{k_\delta} \subset^* \{\nu(\overline{\kappa})_0 \mid \nu \in Lev_0(A^{p_\rightarrow}) \cap W_\delta\}$$

it is sufficient to consider elements $\nu(\overline{\kappa})_0$ for $\nu \in Lev_0(A^{p^{**}})$. Let $q \geq p^{**}$ such that $q \Vdash \nu(\overline{\kappa})_0 \in \dot{k_\delta}$, then $q \Vdash p^{**}_{\langle\nu\rangle} \in \dot{G}$, but $p^{**}_{\langle\nu\rangle} \Vdash \nu(\overline{\kappa})_0 \in \dot{X}$ as $\nu \in X^*$. □

We can now deduce the main result of this section.

Theorem 12. *For every $V-$generic filter $G \subset \mathbb{P}_{\overline{E},\lambda}$, κ is a regular and $s(\kappa) = \lambda$ in $V[G]$.*

Proof. The fact κ remains regular and $2^\kappa \leq \lambda$ in $V[G]$ is obtained in [1]. Let $F \subset [\kappa]^\kappa$ be a family of size $|F| < \lambda$. Since no cardinals are collapsed in $\mathbb{P}_{\overline{E},\lambda}$ we can form a sequence of names $\{\dot{X}_i \mid i < \eta\}$ for some $\eta < \lambda$ which are interpreted as F in $V[G]$. Take an elementary submodel $N \prec H_\theta$ as described at the beginning of this section, such that $\{\dot{X}_i \mid i < \eta\} \subset N$, and denote $\delta = N \cap \lambda$.

Since $\mathbb{P}_{\overline{E},\lambda}$ satisfies $\kappa^{++} - c.c.$ and $\kappa^+ \subset N$, we may assume that $\dot{X}_i$ are $N \cap \mathbb{P}_{\overline{E},\lambda}$ names, so $X_i \in V[G \cap N]$ for every $i < \eta$. U_δ is a measure on κ in $V[G \cap N]$ and by Proposition 11, $k_\delta \in V[G]$ is U_δ generating set. If follows that non of the sets X_i splits k_δ hence F cannot be a splitting family in $V[G]$. We conclude that $s(\kappa) = \lambda$. □

3 From $s(\kappa) = \lambda$ to $o(\kappa) = \lambda$

We argue now that the initial assumption $o(\kappa) = \lambda$ of our previous construction is the optimal one. This generalizes the argument of [2] which proves $s(\kappa) = \kappa^{++}$ implies that there exists an inner model with a measurable cardinal α such that $o(\alpha) = \alpha^{++}$. The argument of [2] relies on the structure of the core model K below $o(\alpha) = \alpha^{++}$ which is a core model for sequence of measures. The fact all extenders are equivalent to measures is required in [2] to argue that when U is the normal measures derived from an elementary embedding $i : K \to K^*$, then U does not belong to K^*. Unfortunately, this property fails in core models for sequences of extenders. The argument suggested here appeals to the ideas of [7].

Theorem 13. *Assume $\neg 0^{\P}$. Then $s(\kappa) = \lambda$ in V implies that in the core model $\mathcal{K}$, $o(\kappa) \geq \lambda$.*

Proof. Suppose otherwise then the core model $K = J^E$. Since $o^K(\kappa) < \lambda$, there exists some $\eta < \lambda$ such that J^E_η is a mouse which includes all extenders on the sequence E with critical point κ. Let us say that an iteration $i : J^E_\eta \to Y$ is mild in κ if $\kappa = crit(i)$ and no extender of $E \restriction \eta$ is used more than ω_1 many times along this iteration. Since λ is a regular cardinal then the set

$$\{i(\kappa) \mid i : J^E_\eta \to Y, \text{ is mild in } \kappa\}$$

is bounded by some $\tau < \lambda$. Since κ us inaccessible, we can choose a sufficiently large regular cardinal $\theta > \lambda$, and find $N \prec H_\theta$ which satisfies:

1. $\tau + 1 \subset N$,
2. $|N| < \lambda$,
3. ${}^{<\kappa}N \subset N$,
4. $J^E_\eta \subset N$,
5. $\lambda \in N$.

Let N_0 be the transitive collapse of N, then all but the last properties are valid in N_0, and

$$N_0 \models \kappa \text{ is inaccessible and } 2^\kappa > \tau.$$

Since $|\mathcal{P}(\kappa) \cap N_0| < \lambda$, this set cannot be a splitting family. Let $a \in \mathcal{P}(\kappa)$ be a witness, i.e. either $|a \setminus x| < \kappa$ or $|a \cap x| < \kappa$ for every $x \in \mathcal{P}(\kappa) \cap N_0$. The induced $U_a = \{x \in \mathcal{P}(\kappa) \cap N_0 \mid |a \setminus x| < \kappa\}$ is a κ complete nonprincipal N_0-ultrafilter. Hence the structure $Ult(N_0, U_a) = (N_0 \cap {}^{\kappa}N_0)/U_a$ is well founded. Denote its transitive collapse by N^*. N_0 satisfies sufficient fraction of set theory to apply Łoś theorem and obtain an elementary embedding $i^* : N_0 \to N^*$. We conclude

1. $V_\kappa^K = V_\kappa \cap J_\eta^E \subset N^*$,
2. $N^* \models i^*(\kappa) > 2^\kappa > \tau$,
3. ${}^{<\kappa}N^* \subset N^*$.

We now appeal to inner model theory, as presented in [9], and to the proof of Theorem 1.1 in [7]. First, both N_0 and N^* satisfy sufficient fraction of set theory to define their core models $K(N_0)$, and $K(N^*)$, and prove the appropriate covering Lemma used in [7]. Let $i = i^* \restriction K(N_0)$, then the definability of the core model implies that $i : K(N_0) \to K(N)$ is elementary. Since $J_\eta^E \in N_0$, it follows that $J_\eta^E = J_\eta^{E^{N_0}}$, i.e. is an initial segment of $K(N_0)$. Furthermore, as $\kappa + 1 \subset N$, then N witness that $cp(E_\alpha) > \kappa$ for all $\alpha \geq \eta$. Hence, the same is true in $K(N_0)$.

We claim that there exists a normal iteration $\pi : K(N_0) \to K'$ of $K(N_0)$, obtained by ultrapowers via extenders which originate in J_η^E, and that $i = k \circ \pi$, where $k : K' \to K(N^*)$ satisfies $cp(k) > \pi(\kappa)$. For this, consider the coiteration of $K(N_0)$ with $K(N^*)$. Let us denote $K(N_0)$ by K^0, $K(N^*)$ by K^*, their coiterands by K_i^0 and K_i^* respectively, and their iteration maps by $\pi_{i,j}^{K_0}$ and $\pi_{i,j}^{K^*}$, for $i < j$ below the length of the coiteration. $J_\eta^E = K^0 \restriction \eta$ is an initial segment of the core model K (i.e. it is incompressible). The arguments of sections 7.4,8.3 of [9] imply that K^* does not move along in an initial segement of the coiteration, as long as the first point of disagreement between K_i^0 and K^* is below $\pi_{0,i}^{K_0}(\eta)$. It follows that if θ be the first index i of the coiteration in which K_i^0 and K_i^* agree above $\pi_i^{K^0}(\kappa) + 1$, then $K_i^* = K^*$. We set $K' = K_\theta^0$ and $\pi = \pi_\theta^{K_0} : K(N_0) \to K'$.

Hence, every $x \in K'$ is of the form $\pi(f)(\xi_1, .., \xi_n)$, where $n < \omega$, $f : \kappa^n \to K(N_0)$ is a function in $K(N_0)$, and $\xi_1, .., \xi_n \leq \pi(\kappa)$. We then define $k : K' \to K(N^*)$ by sending $\pi(f)(\xi_1, .., \xi_n)$ as above, to $i(f)(\xi_1, .., \xi_n)$. It is standard to verify that k is well define, elementary, and $cp(k) > \pi(\kappa)$. Finally, as ${}^{<\kappa}N^* \subset N^*$, then we can apply the proof of Theorem 1.1 in [7]

to N^* with respect to $K(N^*) \restriction \pi(\kappa) = K' \restriction \pi(\kappa)$, and π. We conclude that $\pi \restriction J^E_\eta : J^E_\eta \to \pi(J^E_\eta)$ is mild. However this is absurd as $cp(k) > \pi(\kappa)$ so

$$\pi(\kappa) = k \circ \pi(\kappa) = i(\kappa) = i^*(\kappa) > \tau,$$

contradicting the choice of τ. □

Open Questions

Let us conclude with some questions:

Question 1. What is the consistency strength of κ is a measurable and $s(\kappa) = \kappa^{++}$?

In the model of Kamo, κ remains a measurable (and even a supercompact).

Question 2. Is it possible to have GCH below κ and $s(\kappa) = \kappa^{+3}$?

Note that in our model for $s(\kappa) = \kappa^{+3}$, $2^\alpha = \alpha^{++}$ holds on a club below κ.

Question 3. Is it possible $s(\kappa) = \lambda$ for a singular λ?

Note that this is known for $\kappa = \aleph_0$.

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