SUSTAINABLE SOLUTIONS FOR DYNAMIC BARGAINING PROBLEMS

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This paper investigates the dynamic bargaining problem in which players have to agree on a time path of actions. The axiomatic approach for such problems may yield a solution which is not sustainable. For every given partition of the time interval however, we can construct a sustainable solution.

1. Introduction

By referring to the Bargaining Problem the economic literature means a situation in which a group of individuals seek to reach an agreement on a partition of a pie. Following Nash (1950), the two players bargaining problem is described using only two components \((S, d)\) when \(d\) is a point in the plane which can be interpreted as the outcome when no agreement is reached and \(S\) is a compact convex subset of the plane which contains \(d\) and describes the set of all feasible utility payoffs that can be achieved by cooperation. Consequently two different bargaining problems that can be characterized by the same \((S, d)\) will yield the same solution [see for example Nash (1950), Kalai and Smorodinsky (1975), and Kalai (1977)].

One approach to solve the bargaining problem is the axiomatic approach [for survey see Roth (1979)] in which the bargaining process is not specified and the assumptions are directly made on the solution itself. A different way of discussing the bargaining problem is the strategic approach in which the bargaining process itself is discussed [see Rubinstein (1982)].

In daily bargaining we deal quite often with a situation in which the
players have to agree on a path of actions which takes place over time. (Players in this case can also receive their payoff along time.) As an example we can think of a firm in which activities are controlled by a group of individuals each wanting to maximize a different objective function (profit, sales, growth and so on). The firm has to decide on a production path for the next several periods. Since a production path that maximizes simultaneously profits sales and growth is not likely to exist, the path will be decided by bargaining between the executives. Given the objective functions of the executives and the set of all possible production paths, the problem can again be characterized by the two components \((S, d)\). We denote the one stage bargaining problem (partitions of a pie) as the static problem and the problem discussed above (bargaining on a path) as the dynamic bargaining problem. As we show in this work the dynamic bargaining problem cannot be characterized just by two components. Two different problems which can be described by the same pair \((S, d)\) may have different solutions even in the axiomatic approach.

2. Formulation of the dynamic bargaining problem

Suppose that a group of players denoted by \(i = 1, \ldots, n\) have to choose a control function \(u(t)\) from a set \(U\) of admissible control functions on \([0, T]\). Let \(x(t) \in R_m\) be a vector of state variables such that

\[
\dot{x}(t) = \frac{dx}{dt} = f(x(t), u(t)), \quad x(0) = x_0. \tag{1}
\]

The payoff of the \(i\)th players is according to

\[
W_i(x_0, u(t)) = \int_0^T e^{-r}g_i(x(t), u(t))dt, \quad i = 1, \ldots, n. \tag{2}
\]

where \(x(t)\) is defined by (1), \(r\) denotes the discount rate and \(g_i \in C^2\) are all non-negative. Assuming that \(x_0\) is given, we can define for every possible control function \(u(t)\), the payoff vector \(y(x_0, u(t)) = (W_1(x_0, u(t)), \ldots, W_n(x_0, u(t)))\).

Thus the set of all feasible outcomes \(S\) is

\[
S = \{ y(x_0, u(t)) \in R^n | x(0) = x_0, u(t) \in U \}
\]

which is assumed to be a compact convex subset of \(R^n\). Let \(u^N(t)\) denote
the control function when no agreement is reached, then the threat point $d$ can be written as $d = y(x_0, u^N(t))$.

An utopia is a control function $\hat{u}(t)$ which simultaneously maximizes all the functionals $W_i$. If $\hat{u}(t)$ exists it will clearly be the solution of the bargaining problem. In general, it is very rare to find a control function which optimizes several functionals simultaneously.

Adopting the axiomatic approach, a solution is a function $\mu: B \rightarrow R^n$ such that $B$ denotes the class of all $n$-players bargaining problems and $\mu(S, d)$ is an element in $S$ that has to satisfy some given axioms. Thus for the above bargaining problem we can find a solution $y^* = \mu(S, d)$. (Different sets of axioms will yield of course different solutions.)

Let us define a function $\phi: S \rightarrow \mathcal{P}(U)$ such that for every $y \in S$, $\phi(y)$ define all the control functions in $U$, which yield the payoff $y$. Thus for the solution $y^* = \mu(S, d)$ of the bargaining problem, the players are free to choose any $u^*(t) \in \phi(y^*)$. The main problem that distinguishes the dynamic bargaining problem is whether $u^*(t)$ is sustainable or not. At every time $\tau < T$ one of the players may consider breaking the original agreement. Unlike the static problem, breaking the agreement does not bring the players to the original situation at $t = 0$. From eq. (1) we can see that at time $\tau$ the state variables have different values, namely $x^*(\tau)$, and the players have already received part of the agreed payoff [according to eq. (2)]. Thus at $t = \tau$ they are facing an entirely different situation. If for example the original solution $u^*(t)$ is such that player $i$ has received most of his planned payoff before $t = \tau$, then at time $\tau$ he can demand to change the original path $u^*(t)$, and since he does not have much to lose he will have a strong bargaining position. So if the solution of the bargaining problem is not supported by an appropriate punishment scheme the planned control function $u^*(t)$ may not be sustainable. Consequently if all $u(t) \in \phi(y^*)$ are not sustainable the players can not choose $y^*$ as a solution for the bargaining game. Since $y^*$ is the only point in $S$ that satisfied all the desired axioms, it is not always possible to use the axiomatic approach for the dynamic bargaining problem.

Considering the sustainability problem, one can think of two different definitions. In the first one players are myopic and do not consider the possibility that other players will break the agreement. The more realistic situation is that players are not myopic and consider only the agreements that will not be broken by their rivals. First lets define a sustainable path for the myopic case.

Let $U_\tau$ denote the set of all the admissible control functions $u_\tau(t)$ on $[\tau, T]$, and $x(\tau)$ the value of the state variables at time $\tau$. Let $S(\tau, x(\tau))$
= \{(y(x(\tau), u_x(t)) \in R^n | u_x(t) \in U_x)\} be the set of all feasible expected payoff from time \(\tau\) until the terminal time \(T\). We assume that for every \(\tau\) and \(x(\tau)\), \(S(\tau, x(\tau))\) is a compact convex set. Similarly let \(d(\tau, x(\tau))\) be the threat point at time \(\tau\); a sustainable path with respect to the solution \(\mu\) is a control function \(\tilde{u}(t)\) and its induced state variables path \(\tilde{x}(t)\) such that for every \(\tau \in [0, T]\), \(\tilde{u}_\tau(t) \in \phi[\mu(S(\tau, \tilde{x}(\tau)), d(\tau, \tilde{x}(\tau)))].\) A sustainable path for the non-myopic case is discussed in the next section.

3. The axiomatic approach in the dynamic bargaining problem

For simplicity let's assume that the opportunities to break the original agreement and to start a new bargaining are restricted to times \(t_j\), \(j = 1, \ldots, k\), where \(\{t_j\}\) is strictly increasing \(t_0 = 0\), \(t_k < T\). Alternatively we can think about the discrete time version of this problem. Let \(\mu\) be a solution function according to the axiomatic approach.

For every level of the state variables at \(t_k\) we can define (as in the previous section) the pair \((S(t_k, x), d(t_k, x))\) that describe the set of all feasible expected payoff from time \(t_k\) until the terminal time \(T\) and the threat point at \(t_k\) respectively, when \(x(t_k) = x\). Since \(t_k\) is the last opportunity to discuss the path, the bargaining problem at time \(t_k\) can be treated as a static problem.

Let \(\rho^k\) be a function from the state variables space to \(R^n\) such that

\[\rho^k(x) = \mu(S(t_k, x), d(t_k, x)).\]

Since \(S(t_k, x)\) is a compact convex set, \(\rho^k(\cdot)\) is well defined. Knowing \(\rho^k\) we are now able to define \(\rho^{k-1}\). For every level of state variables at time \(t_{k-1}\) denoted by \(x_{k-1}\) and for every admissible control \(u(t)\) on \([t_{k-1}, t_k]\) let \(y[x_{k-1}, u(t)]\) be a \(n\) tuple vector, such that

\[y_i(x_{k-1}, u(t)) = W_i(x_{k-1}, u(t)) + \rho^k_i(x_{k-1}), \quad \text{when}\]

\[x_{k-1} = \int_{t_{k-1}}^{t_k} f(x(t), u(t)) dt, \quad \text{and let}\]

\[S(t_{k-1}, x_{k-1}) = \{y(x_{k-1}, u(t)) \in R^n | u(t) \text{ is admissible control on} [t_{k-1}, t_k]\}.\]
Since $S(t_{k-1}, x_{t_{k-1}})$ is a compact convex subset of $R^n$, $\rho^{k-1}$ can be defined as

$$\rho^{k-1}(x_{t_{k-1}}) = \mu(S(t_{k-1}, x_{t_{k-1}}), d(t_{k-1}, x_{t_{k-1}})).$$

From the way $\rho^{k-1}$ is defined it is clear that there is a sustainable path $u(t)$ for $t \in [t_{k-1}, T]$.

In the same way for every $j = 1, \ldots, k - 1$ we can define the function $\rho^{k-j}$. Since $x_0$ is given, $\rho^0(x_0)$ defines a solution for the dynamic bargaining problem which is sustainable with respect to $\mu$ and the partition of the time interval $(t_1, \ldots, t_k)$.

Thus, although the dynamic bargaining problem cannot be characterized just by two components as the static problem. We can adopt the axiomatic approach and find a solution which will be sustainable for a given partition of the interval $[0, T]$. For every partition of the interval we will of course get a different solution. Finally, the solution we have investigated in this paper is based on the axiomatic approach. Before one tries to implement the strategic approach, it is important to note that the strategy space in the dynamic problem is much larger than in the static problem. Players can suggest different partitions of the time interval, or different punishment schemes. They can even agree just on a partial solution [a path $u(t)$ for $t \in [0, T_1]$; $T_1 < T$] and while the partial solution is executed they can continue the negotiation on the rest of the solution. Using this kind of strategy, players can avoid the problem of shrinking pie which is discussed by Rubinstein (1982).

References


