Problems with Mathematically Real Quantum Wave Functions

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abstract:

Theories of a mathematically real quantum function are discussed. The analysis, which relies on fundamental elements of quantum theories, proves the existence of a new type of contradictions in mathematically real quantum theories. The results cast doubt on merits of the following theories: The Yukawa theory of the nuclear force, the theories of the Z and the Higgs bosons, and the Majorana neutrino theory.

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Mathematically complex wave functions were the first choice of the founders of Quantum Mechanics (QM). This type of functions is mandatory for the Schroedinger and the Dirac equations, whose form is

$$i\frac{\partial \psi}{\partial t} = H\psi. \tag{1}$$

Here the Hamiltonian H is a Hermitian operator which has real eigenvalues. Furthermore, the equation holds for a massive particle and each side of (1) represents the system's energy, which means that it is a nonvanishing quantity. So, if ψ is real then the left hand side of (1) is pure imaginary whereas its right hand side is real. It follows that a real wave function cannot satisfy the Schroedinger and the Dirac equations. For this reason complex wave functions are intensively treated in QM textbooks. A different course was taken few years later, when two mathematically real quantum equations were published. They are the real version of the Klein-Gordon (KG) equation which was used by Yukawa in 1935 for a theoretical interpretation of the nuclear force (see e.g. [1], p. 211). Soon after, Majorana published a pure imaginary version of the Dirac γ matrices which cast the Dirac equation into a mathematically real form (see [2], p. 486).

The quantum equations of Yukawa and Majorana have a relativistic covariant structure. The same property holds for other real quantum equations, like those of the Higgs and the Z bosons (see e.g. [3]). General properties of real quantum wave functions are analyzed below and it is proved that they suffer from new contradictions. This work uses standard notation of relativistic expressions. Greek indices run from 0 to 3 and Latin indices run from 1 to 3. The metric is diag(1,-1,-1,-1). Units where $\hbar = c = 1$ are used.

The analysis relies on basic properties of a quantum particle which are briefly presented in the following lines. The wave nature of such a particle is considered as a primary attribute of a quantum theory. Here the de Broglie formula for the wave length of a massive particle is related to its momentum (see [4], p. 3 or [5], pp. 48, 49)

$$\lambda = 2\pi\hbar/p. \tag{2}$$

For the simplicity of the discussion, let us examine a free massive quantum particle. The phase of its wave takes the form

$$\varphi = \mathbf{k}\cdot\mathbf{x} - \omega t. \tag{3}$$

The de Broglie relationship means that the particle's energy and momentum appear as elements of its phase where ($\hbar = 1$)

$$\mathbf{k} = \mathbf{p}, \quad \omega = E. \tag{4}$$

It is interesting to note that (3) and (4) prove that the phase φ is a Lorentz scalar obtained from the contraction of the energy-momentum 4-vector $(E, \mathbf{p})$ and the space-time coordinates $(t, \mathbf{x})$ (multiplied by -1). Consequently, it takes the same form in nonrelativistic and relativistic quantum theories.

A quantum theory of a given particle must describe the time evolution of its state. This objective takes the form of a differential equation. Relying on the foregoing expressions, one finds that appropriate differential operators are related to the particle's energy and momentum. These operators take the form

$$E \to i\partial/\partial t, \quad p_x \to -i\partial/\partial x. \tag{5}$$

These relationships show the connection between dynamical quantities and differential operators.

As stated above, the quantum equation (1) proves the well known complex form of the Schroedinger and the Dirac wave function. Let us examine the possibility of describing a massive quantum particle by means of a real wave function. A simple case is that of a free particle moving along the positive x-direction. The form of the phase-dependent factor of its wave function can be written in terms of the following expressions (see [4], p. 18)

$$\cos(kx - \omega t), \ \ \sin(kx - \omega t), \ \ \exp(\pm i(kx - \omega t)). \tag{6}$$

The last expression of (6) is a complex function which depends on the energy and the momentum. Therefore, it is unsuitable for a real wave function. Evidently, the first and the second functions of (6) can be combined in this form $\sin(kx - \omega t - \delta)$, where δ is a constant number. Hence, a real wave function of a free massive particle moving along the positive x-direction is

$$\psi(t, x) = A \sin(kx - \omega t - \delta), \tag{7}$$

where A is a normalization factor.

The free quantum particle that is analyzed here is massive and it has a rest frame. In this frame the linear momentum $k = p = 0$ and the particle's wave function reduces to the form

$$\psi_{Rest}(t, x) = A \sin(-\omega t - \delta). \tag{8}$$

It follows that for every integer n, the real wave function (8) vanishes identically throughout the entire 3-dimensional space at every instant t when $\omega t + \delta = n\pi$.

The null value of a real wave function means that at the corresponding instant the particle disappears. Hence, the following results are obtained:

1. A conserved density cannot be consistently defined for a particle described by a mathematically real wave function.

2. The lack of a consistent expression for density means that a Hilbert space of quantum mechanics and its associated Fock space of quantum field theories cannot be constructed. Indeed, in quantum theories the inner product of a Hilbert space is based on a consistent expression for density.

3. Obviously, due to the absence of these spaces, operators used in mathematically real quantum theories become meaningless.

These findings prove the existence of inherent contradictions in quantum theories of a mathematically real wave function. Note that the proof takes a general form which is independent of the specific structure of any given mathematically real quantum theory. Here the following question arises: Why the Noether theorem does not provide a consistent expression for density of a particle whose quantum equation of motion takes a mathematically real form?

In the case of a Majorana particle the answer depends on the absence of an appropriate eigenfunction. Indeed, the Majorana Lagrangian density is (see [2], eq. (105))

$$\mathcal{L}_{Majorana} = \frac{1}{2}\bar{\psi}[\gamma^\mu i\partial_\mu - m]\psi, \tag{9}$$

where the four γ matrices are pure imaginary. Moreover, like the Dirac equation, the Majorana equation is a first order partial differential equation. For this reason, (7) does not solve the Majorana equation because the first derivative of the $\sin(x)$ function is the $\cos(x)$ function. Obviously, the Noether theorem applies only to cases where the function ψ of the Lagrangian density solves the quantum equation.

Another argument that disproves the Majorana theory stems from the definition of the following function of (9)

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 \tag{10}$$

(see [1], p. 24). So, if all the four Majorana γ matrices are pure imaginary then the quantum equation

$$[\gamma^\mu i\partial_\mu - m]\psi = 0. \tag{11}$$

is real and so is its solution ψ. *However*, due to the definition of (10), there is an additional γ^0 factor in the Majorana Lagrangian density. Therefore, the following contradictions hold for the Majorana theory.

The Lagrangian density and its action

$$S = \int \mathcal{L}_{Majorana} d^4x = \int \frac{1}{2}\psi^\dagger \gamma^0 [\gamma^\mu i\partial_\mu - m]\psi \, d^4x \tag{12}$$

are pure imaginary due to the additional γ^0. The same contradiction is found in the Majorana Hamiltonian, because if the Lagrangian density is pure imaginary then also the associated Hamiltonian is pure imaginary. (The mass term $m\psi^\dagger\gamma^0\psi$ provides a simple illustration of this conclusion. Thus, if γ^0 is pure imaginary then the mass term is pure imaginary.) This is a contradiction, because energy, which is an eigenvalue of the Hamiltonian, is a real quantity.

The following argument explains the origin of the problem. The Majorana γ matrices (see [2], eq. (12)) satisfy the Dirac condition where γ^0 is Hermitian and the three γ^i are anti-Hermitian. Therefore, one expects that the Majorana equation has real eigenvalues for its Hamiltonian. The pure imaginary result which is found above for the Hamiltonian eigenvalues means that eigenvalues of the Majorana Hamiltonian should be zero. This property is inconsistent with the assumed positive mass of the Majorana particle.

As far as this work is concerned, the case of a real KG equation is much simpler, because the absence of density and of a conserved 4-current for this quantum field is proved in a textbook (see [6], p. 42, eq. (12.8)).

A search of the literature provides an indirect support for the claim that there is no self-consistent expression for density of a mathematically real quantum theory. Indeed, an expression for density and its associated conserved 4-current can be found for the Schroedinger equation (see e.g. [5], pp. 53, 54) and for the Dirac equation of the electron (see e.g. [7], p. 56). By contrast, although quantum theories of a mathematically real wave function are known for eight decades, one cannot find a self-consistent expression for density in textbooks that discuss these theories.

This work analyzes a class of quantum theories whose wave function takes a mathematically real form. It proves that the corresponding parts of presently accepted physical theories contain errors. The analysis relies on well documented physical properties of quantum theories that can be found in standard textbooks. The results cast doubt on the Yukawa theory of the nuclear force, on current theories of the Z and the Higgs bosons as well as on the Majorana neutrino theory.

Error correction is an important task of every human community. This work points out several errors of presently accepted physical theories. In doing so, it aims to launch a debate about the veracity of the results obtained above. Such a debate can certainly improve the understanding of several topics of theoretical physics.

References

[1] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[2] P. B. Pal, *Am. J. Phys.*, **79**, 485 (2011).

[3] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, Mass., 1995).

[4] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955).

[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, London, 1959).

[6] V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, *Quantum Electrodynamics* (Pergamon, Oxford, 1982).

[7] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).