# Decomposition of electromagnetic fields into radiation and bound components

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The decomposition of electromagnetic fields for a system of elementary classical point charges into velocity fields and acceleration fields is suggested by the appropriate terms in the Lienard–Wiechert formulas. This paper introduces ''bound'' fields and ''radiation'' fields which are related but *not identical* to velocity and acceleration fields, respectively. It is shown how this approach can be used in the construction of an energy–momentum tensor that is free of infinite quantities and settles the 4/3 problem in the Lorentz transformation of momentum. 

## I. INTRODUCTION

The purpose of the present work is to discuss the structure of classical electrodynamics for systems of elementary classical point charges. As is well known, nature obeys the laws of classical electrodynamics only in circumstances where the classical limit of quantum mechanics holds. This requirement is not satisfied in cases where charges are too close together. Hence the discussion is restricted to the *mathematical* properties of the theory. In this sense, the notions of experiment and measurability hereafter refer to thought experiments and not to actual ones. Here, thought experiments denote hypothetical experiments with a device that is assumed to follow exactly the mathematical laws of classical electrodynamics.

Theoretical properties of classical electrodynamics depend on the structure of its charge constituents. Thus one may construct a theory for particles whose charge density is bounded. A simple example is a system which consists of particles whose charge density varies continuously within their volume and vanishes elsewhere. Particles having this property are hereafter called small charged objects, where the word ''small'' describes the object's *spatial dimensions*. Analyzing the mathematical structure of a hypothetical system of this kind, one can consider an infinitesimal volume element $\Delta V$ which contains an *infinitesimal* amount of charge $\Delta Q$. In so doing, one finds that the interaction of $\Delta Q$ with itself can be ignored because it depends on the second power of $\Delta Q$ whereas the interaction of $\Delta Q$ with the rest of the system is linear in $\Delta Q$. Another important feature of a theory of small charged objects is that it is free of infinite quantities. Thus the solutions of Maxwell equations yield retarded potentials[1,2] from which regular fields and their energy momentum tensor are derived (see Ref. 1, pp. 80–83; Ref. 2, pp. 601–608).

The phenomenon of charge quantization and the discovery of particles whose volume looks like a point motivate the introduction of elementary point charges into classical electrodynamics. One can also find theoretical reasons for doing that. Thus Landau and Lifshitz use special relativity and explain why an *elementary* classical particle should be pointlike (see Ref. 1, pp. 43–44). Rohrlich uses quantum mechanics and arrives at analogous conclusions.[3] Systems of point charges are the main topic of the present work. These particles are treated as unique entities and *not* as a limit of a sequence of small charged objects, where each of them has the same amount of charge and their spatial dimensions tend to zero.

In a system of this kind, one cannot divide space into a sum of infinitesimal volume elements $\Delta V$, each of which contains an *infinitesimal* amount of charge. This feature entails several analytical differences between systems of small charged objects and those of point charges. In the present work it is shown how pure radiation fields can be used in a classical theory of point charges which is free of expressions yielding infinite energy and momentum.

The electromagnetic fields of a point charge $q_i$ are obtained from the Lienard–Wiechert formulas (see Ref. 1, p. 162 or Ref. 2, p. 657)

$$\mathbf{E}=q_i\left[\frac{(1-v^2/c^2)(\mathbf{R}-R\mathbf{v}/c)}{(R-\mathbf{R}\cdot\mathbf{v}/c)^3}+\frac{\mathbf{R}\times\{(\mathbf{R}-R\mathbf{v}/c)\times\mathbf{a}\}}{c^2(R-\mathbf{R}\cdot\mathbf{v}/c)^3}\right], \tag{1}$$

$$\mathbf{B}=\mathbf{R}\times\mathbf{E}/R. \tag{2}$$

Here $\mathbf{R}$ denotes the displacement vector from the retarded position of the charge $q_i$ to the point where the fields are calculated; $\mathbf{v}$ and $\mathbf{a}$ denote the retarded velocity and acceleration of $q_i$, respectively.

Formula (1) serves as a basis for the decomposition of the electric field into a sum of two quantities. The first term on the right-hand side of (1) is independent of acceleration whereas the second term of this equation is linear in it. Formula (2) indicates that the same properties hold for the magnetic field. For this reason, the first terms of (1) and of (2) are called velocity fields and the second ones are called acceleration fields.

Velocity fields differ from acceleration ones in several respects, one of which is their behavior at a very large distances from $q_i$. Here one finds that velocity fields decrease like $R^{-2}$ whereas acceleration ones decrease like $R^{-1}$. This property is related to the electromagnetic energy radiated by the system.

In order to find the radiation energy, let us examine the standard expression for the energy–momentum tensor associated with these fields (see Ref. 1, p. 81; Ref. 2, p. 605).

$$T^{\mu\nu}=\frac{1}{4\pi}\left(F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta}+\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\right). \tag{3}$$

Here, the field tensor is

  

$$F^{\mu\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{4}$$

In the present work Greek indices range from 0 to 3 and Latin ones take the values 1, 2, and 3. The metric $g_{\mu\nu}$ is diagonal and its entries are $(1,-1,-1,-1)$. The symbol $_{,\nu}$ denotes the partial differentiation with respect to $x^\nu$. An upper dot denotes differentiation with respect to the particle's proper time. $v^\mu$ denotes its dimensionless 4-velocity $v^\mu = \gamma(1,\mathbf{v}/c)$ where $\gamma = (1 - v^2/c^2)^{-1/2}$. These definitions determine the quantities $a^\mu = \dot{v}^\mu$ and $\dot{a}^\mu$. The term ''energy–momentum tensor'' refers to the tensor depending on electromagnetic fields and not on other elements of the system.

Since the energy–momentum tensor (3) is a homogeneous quadratic function of the fields, only acceleration fields contribute to the energy and momentum emitted from a charge in the form of radiation. Indeed, at large distances from the system, the intensity of radiation must decrease like $R^{-2}$, whereas other terms of (3) decrease faster and their contribution can be ignored if one integrates the energy passing through a large enough spherical shell $R^2 d\Omega$. Thus, in a closed system which contains only *one* charged particle, acceleration fields and radiation fields represent the same quantity (at least in regions which are far enough from the charge). Similarly, velocity fields cannot contribute to radiation. For this reason velocity fields of a point charge are associated with bound energy. Note that bilinear terms of the energy–momentum tensor (3) which depend on velocity fields and acceleration fields decrease like $R^{-3}$ at large distances from the system. Hence, this part of the energy–momentum tensor also represents bound energy.

Acceleration is evidently a necessary condition for radiation. However, there are systems whose charges accelerate and yet no radiation is emitted.[4,5] This feature is illustrated in current-carrying superconducting rings. Here charges accelerate centripetally but no radiation is emitted.

Maxwell's equations illuminate another property of acceleration fields. Acceleration fields by themselves do not satisfy Maxwell's equations.[6] Only the sum of velocity fields and acceleration fields satisfies Maxwell's equations. As explained later, this property of acceleration fields makes them unsuitable for an important application. Hence it is desirable to define radiation fields which are consistent with Maxwell's equations. In the present work it is shown that a redefinition of the notion of radiation fields and bound fields can be applied usefully. Pure radiation fields are used in defining an energy–momentum tensor which is nonsingular.

The paper is organized as follows. In Sec. II it is shown how radiation and bound fields of a system of point charges can be defined. A nonsingular energy–momentum tensor for such a system is constructed in Sec. III. In Sec. IV it is shown how two problems are settled: the problem of the infinite energy of a point charge and that of the 4/3 factor appearing in a Lorentz transformation of the field momentum. Concluding remarks are the contents of the last section.

## II. RADIATION FIELDS OF A CLOSED SYSTEM

Consider a system which consists of a finite number of point charges. The system is examined from a frame where its center of energy is at rest. Given the motion of charges, one can use the Lienard–Wiechert formulas (1) and (2), calculate how the fields of the point charges interfere in the wave zone and find the radiation emitted from the entire system. In the present section it is shown how one can use these values and decompose the system's fields into radiation and bound components.

Radiation fields are described here as spherical waves emitted from the origin. For convenience, the origin is chosen at a point which is free of charge at all times and is not too far from the system's center of energy. Such a spatial point can always be found in a system which consists of a finite number of point charges. As a matter of fact, the mathematical formalism which yields the required fields can be readily taken from the literature (Ref. 2, pp. 739–747). The following lines explain briefly how the contents of these pages of Jackson can be adapted for the outgoing fields discussed here. The reader who wishes to find further details may consult the relevant parts of Jackson's book and the references mentioned therein.

In free space Maxwell's equations yield the homogeneous wave equation for the electric field

$$\left(\nabla^2 - \frac{1}{c^2}\frac{d^2}{dt^2}\right)\mathbf{E}(r,\theta,\phi,t)=0, \tag{5}$$

and similarly for the magnetic field. The time dependence of the fields is given as a Fourier integral

$$\mathbf{E}(r,\theta,\phi,t)=\int_{-\infty}^{\infty}\mathbf{E}(r,\theta,\phi,\omega)e^{-i\omega t}d\omega, \tag{6}$$

and similarly for the magnetic field. Performing the appropriate inverse Fourier transform, one obtains the fields spectral resolution $\mathbf{E}(r,\theta,\phi,\omega)$ and $\mathbf{B}(r,\theta,\phi,\omega)$. The application of the wave equation (5) to the right-hand side of (6) yields the Helmholtz wave equation

$$(\nabla^2+k^2)\mathbf{E}(r,\theta,\phi,\omega)=0, \tag{7}$$

where $k = \omega/c$.

Define the angular momentum operator

$$\mathbf{L}=-i\mathbf{r}\times\nabla \tag{8}$$

and the vector spherical harmonics

$$\mathbf{X}_{lm}(\theta,\phi)=\frac{1}{[l(l+1)]^{1/2}}\mathbf{L}Y_{lm}(\theta,\phi), \tag{9}$$

where $Y_{lm}(\theta,\phi)$ are the ordinary spherical harmonics. Using these quantities, one finds a solution of Maxwell equations in free space which takes the form

$$\mathbf{E}_{\rm rad}=\sum_{lm}\left[\frac{i}{k}a_E(l,m)\nabla\times h_l^{(1)}(kr)\mathbf{X}_{lm} + a_M(l,m)h_l^{(1)}(kr)\mathbf{X}_{lm}\right], \tag{10}$$

$$\mathbf{B}_{\rm rad}=\sum_{lm}\left[a_E(l,m)h_l^{(1)}(kr)\mathbf{X}_{lm} - \frac{i}{k}a_M(l,m)\nabla\times h_l^{(1)}(kr)\mathbf{X}_{lm}\right], \tag{11}$$

where $h_l^{(1)}(kr)$ denotes the spherical Hankel function of the first kind, which is associated with the *outgoing* spherical waves discussed here. The coefficients $a_E(l,m)$ and $a_M(l,m)$ are defined as follows (see Ref. 2, pp. 745–747):

$$a_E(l,m)h_l^{(1)}(kr) = \frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^*(\theta,\phi)\mathbf{r}\cdot\mathbf{E}(r,\theta,\phi,\omega)d\Omega, \tag{12}$$

$$a_M(l,m)h_l^{(1)}(kr) = \frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^*(\theta,\phi)\mathbf{r}\cdot\mathbf{B}(r,\theta,\phi,\omega)d\Omega. \tag{13}$$

The integrals (12) and (13) are performed on a spherical shell located in the wave zone. Carrying out the Fourier transform (6), one obtains the radiation fields as functions of $(t,x,y,z)$. The fields (10) and (11) are hereafter called ''radiation fields.''

This definition of radiation fields entails the corresponding definition of bound fields which satisfy the obvious requirement: The electromagnetic fields of the system is a sum of bound fields and radiation ones. Thus we have

$$\mathbf{E}_{\text{bound}}=\mathbf{E}_{\text{total}}-\mathbf{E}_{\text{rad}}, \tag{14}$$

$$\mathbf{B}_{\text{bound}}=\mathbf{B}_{\text{total}}-\mathbf{B}_{\text{rad}}. \tag{15}$$

As can be seen from the construction of radiation fields, in the wave zone bound fields of (14) and (15) decrease faster than $R^{-1}$.

It is interesting to compare the radiation fields with the acceleration fields:

(1) Radiation fields describe the actual radiation emitted from the *entire* system. Acceleration fields pertain to single particles; they are related to radiation in an indirect way which involves interference. A current loop of a superconducting material is an example of a nonradiating system whose charges accelerate.

(2) Radiation fields are referred to a specific point, which acts like a single fixed source (in the appropriate frame) whereas the number of distinct sources of acceleration fields equals the number of point charges that comprise the system. Moreover, due to their acceleration, each of these sources is not fixed in space, but moves with the corresponding point charge.

(3) Radiation fields satisfy the homogeneous Maxwell's equations at all points except the origin, where the spherical Hankel functions diverge. On the other hand, as pointed out in the Introduction, acceleration fields do not satisfy by themselves Maxwell's equations. Only the *sum* of velocity fields and acceleration fields satisfies Maxwell's equations.[6]

The last property of radiation fields is useful in the construction of the electromagnetic part of the system's energy–momentum tensor. In the following section it is shown how a natural modification of this tensor yields finite expressions for energy and momentum.

## III. ENERGY–MOMENTUM TENSOR OF ELECTROMAGNETIC FIELDS

The energy–momentum tensor $T^{\mu\nu}$ is associated with energy, momentum and stress in the fields. The entries $T^{\mu 0}$ represent the energy and momentum density (up to a factor of $c$) whereas elements of the form $T^{\mu i}$ describe energy and momentum currents. In the present section it is shown how the mathematical structure of (3) and the procedure that yields radiation fields can be used in the construction of a nonsingular energy–momentum tensor for a system of point charges.

The energy–momentum tensor is constructed here in a form that satisfies two crucial requirements:

(A) It should represent correctly the energy and momentum exchanged between charges that comprise the system.

(B) It should described correctly the radiation emitted from the system.

These requirements are hereafter denoted as (A) and (B), respectively.

For simplicity, consider a system of two point charges $q_{(1)}$ and $q_{(2)}$. Relying on the Lienard–Wiechert formulas (1) and (2), one can write the electromagnetic fields of this system as a sum of fields associated with $q_{(1)}$ and $q_{(2)}$, respectively,

$$F_{\text{total}}^{\mu\nu}=F_{(1)}^{\mu\nu}+F_{(2)}^{\mu\nu}. \tag{16}$$

Considering the energy–momentum tensor (3), let us introduce the notation

$$T^{\mu\nu}(F_{(i)}^{\lambda\rho},F_{(j)}^{\eta\theta})\equiv\frac{1}{4\pi}\left(F_{(i)}^{\mu\alpha}F_{(j)}^{\beta\nu}g_{\alpha\beta}+\frac{1}{4}F_{(i)}^{\alpha\beta}F_{(j)\alpha\beta}g^{\mu\nu}\right). \tag{17}$$

Substituting the right-hand side of (16) into (3), noting that this tensor is a second-order homogeneous function of the fields and using the notation (17), one obtains

$$\begin{aligned}T^{\mu\nu}(F_{(1)}^{\lambda\rho}+F_{(2)}^{\lambda\rho},F_{(1)}^{\eta\theta}+F_{(2)}^{\eta\theta}) &= T^{\mu\nu}(F_{(1)}^{\lambda\rho},F_{(1)}^{\eta\theta})+T^{\mu\nu}(F_{(1)}^{\lambda\rho},F_{(2)}^{\eta\theta})\\ &\quad+T^{\mu\nu}(F_{(2)}^{\lambda\rho},F_{(1)}^{\eta\theta})+T^{\mu\nu}(F_{(2)}^{\lambda\rho},F_{(2)}^{\eta\theta}).\end{aligned} \tag{18}$$

Note that the first and the last terms of (18) depend on fields of one and the same point charge whereas the other terms are bilinear functions of fields of two distinct charges.

It can be shown that there is a problem with the first and the last terms of (18). Consider for example the first term $T^{\mu\nu}(F_{(1)}^{\lambda\rho},F_{(1)}^{\eta\theta})$. As mentioned above, the entries $T^{\mu i}(F_{(1)}^{\lambda\rho},F_{(1)}^{\eta\theta})$ represent energy and momentum current of $q_{(1)}$ *itself*. In the corresponding system of small charged objects, these currents are associated with the internal stress. This is reasonable for small charged objects but is unacceptable for an *elementary* classical point charge, because, in principle, for such a particle an internal stress is *unmeasurable*. (Here, measurability is construed in the sense pointed out in the first paragraph of the Introduction.) Stress is associated with the entries $T^{kl}(F_{(1)}^{\lambda\rho},F_{(1)}^{\eta\theta})$. Covariance requirements indicate that the entire tensor $T^{\mu\nu}(F_{(1)}^{\lambda\rho},F_{(1)}^{\eta\theta})$ of a single point charge is meaningless. On the basis of this argument it is concluded that the first and the last terms of (18) should be deleted from an expression for the energy–momentum tensor of fields of two point charges.

The deletion of the first and the last terms on the right-hand side of (18) does not affect the description of the interaction between $q_{(1)}$ and $q_{(2)}$, because this quantity is bilinear in $q_{(1)}$ and $q_{(2)}$ whereas each of the deleted terms is a quadratic homogeneous function of fields of a single point charge. For this reason, requirement (A) continues to hold. On the other hand, requirement (B) is violated because the remaining terms represent only quantities which are bilinear in $q_{(1)}$ and $q_{(2)}$. Thus, since the actual radiation depends on single-particle quantities as well as on interference terms, the deletion of the first and the last terms of (18) yields an energy–momentum tensor which represents radiation incorrectly. These arguments prove that the following expression is not the required energy–momentum tensor:

$$T^{\mu\nu}\neq T^{\mu\nu}(F^{\lambda\rho}_{(1)},F^{\eta\theta}_{(2)})+T^{\mu\nu}(F^{\lambda\rho}_{(2)},F^{\eta\theta}_{(1)}). \tag{19}$$

The procedure described in the previous section can be used for curing the tensor (19) of its defects. This is done by means of pure radiation terms that are added to (19) in order to represent those of the missing single-particle radiation. Thus the acceleration fields $F^{\mu\nu}_{(1)\text{accel}}$ of $q_{(1)}$ are determined by means of the Lienard–Wiechert formulas (1) and (2). Examining the wave zone, one can use the acceleration fields of $q_{(1)}$ and construct the appropriate radiation fields $F^{\mu\nu}_{(1)\text{rad}}$ by means of the procedure described in the previous section. This process is repeated for the radiation fields of $q_{(2)}$. Thus the following energy–momentum tensor satisfies the radiation requirement (B):

$$T^{\mu\nu}_{\text{fields}}=T^{\mu\nu}(F^{\lambda\rho}_{(1)},F^{\eta\theta}_{(2)})+T^{\mu\nu}(F^{\lambda\rho}_{(2)},F^{\eta\theta}_{(1)}) +T^{\mu\nu}(F^{\lambda\rho}_{(1)\text{rad}},F^{\eta\theta}_{(1)\text{rad}})+T^{\mu\nu}(F^{\lambda\rho}_{(2)\text{rad}},F^{\eta\theta}_{(2)\text{rad}}). \tag{20}$$

The generalization of (20) to a system of $N$ point charges is straightforward:

$$T^{\mu\nu}_{\text{fields}}=\sum_{i=1}^{N}\sum_{j\neq i}^{N} T^{\mu\nu}(F^{\lambda\rho}_{(i)},F^{\eta\theta}_{(j)})+\sum_{i=1}^{N} T^{\mu\nu}(F^{\lambda\rho}_{(i)\text{rad}},F^{\eta\theta}_{(i)\text{rad}}). \tag{21}$$

It remains to prove that (20) and (21) have the required physical properties. To this end, let us compare the tensor (20) with the ordinary expression (18) (where the latter holds for small charged objects). In (20) the *full* self-interaction of each charge $T^{\mu\nu}(F^{\lambda\rho}_{(i)},F^{\eta\theta}_{(i)})$ (where $i\in\{1,2\}$) is replaced by the corresponding tensor of pure *radiation* fields $T^{\mu\nu}(F^{\lambda\rho}_{(i)\text{rad}},F^{\eta\theta}_{(i)\text{rad}})$. Hence the additional terms of (20) as well as the missing ones should be examined.

The additional terms of (20) are divergenceless at all points except the origin. This property emerges from item (3) of Sec. II, where it is pointed out that radiation fields are constructed so that they satisfy the homogeneous Maxwell's equations everywhere except the origin. Hence, carrying out a straightforward calculation (Ref. 1, pp. 82–83) one finds a vanishing 4-divergence of terms of (20) depending on radiation fields

$$T(F^{\lambda\rho}_{(i)\text{rad}},F^{\eta\theta}_{(i)\text{rad}})^{\mu\nu}_{,\nu}=F^{\mu\nu}\text{``}j\text{''}_{\nu}=0. \tag{22}$$

Here ``$j$''$_\nu$ is a null 4-current because radiation fields satisfy the *homogeneous* Maxwell's equations. It follows that the radiation terms added to the energy–momentum tensor (20) are divergenceless and exchange neither energy nor momentum with charges of the system. Hence radiation fields do not affect the motion of charges that belong to the system. (These fields, however, interact with other charges located in the wave zone.) Note that (22) depends on the fact that radiation fields satisfy the homogeneous Maxwell's equations; that is why acceleration fields cannot be used consistently for this purpose.

The origin of coordinates is the sole singular point where radiation fields diverge. This point acts like a vacuum point, which is a source of radiated energy and momentum. However, this property of (20) does not lead to a contradiction. Indeed, as pointed out in the second paragraph of Sec. II, the position of each of the system's charges never coincides with the origin. External charges, which might be affected by the emitted radiation are far away from the system and from the origin of coordinates as well. Hence, the additional terms in (20) do not yield contradictions.

Let us turn to the two single-particle terms of (18) which are omitted from (20). As is well known, these terms lead to the following features:

(a) The self-energy and momentum of the fields diverge.
(b) A self-force is exerted on a charge by its own fields.

The removal of (a) is accepted favorably because it is one of the motivations for the present work. However, result (b) should be corrected in order to restore the correspondence between a point charge and limit of a small charged object. The present work discusses the fields' sector of a system of classical point charges, whereas result (b) pertains to the particles' sector. Therefore the correction of (b) is mentioned very briefly. As shown in the literature, property (b) is accounted for if the Lorentz–Dirac (LD) equation (see Ref. 1, pp. 210–211; Ref. 3, p. 141)

$$\frac{q^2}{c^2}\,\dot{a}^{\mu}=1.5Mca^{\mu}-1.5qF^{\mu\nu}_{\text{ext}}v_{\nu}-\frac{q^2}{c^2}\,(a^{\alpha}a_{\alpha})v^{\mu} \tag{23}$$

is adopted as the equation of motion of point charges. This equation is regarded as a fundamental element of the theory and not as a result that can be deduced from the system's equations of motion. Here the terms proportional to $q^2$ are analogues of the self-force exerted on a small charged object by its own fields.

The interaction terms of the energy–momentum tensor (20) yield at the location $x^{\mu}_{(1)}$ of the first charge

$$T^{\mu\nu}_{,\nu}=-F^{\mu\nu}_{(2)}j_{\nu(1)}. \tag{24}$$

This result is obtained straightforwardly from a calculation like the one presented on pp. 82 and 83 of Ref. 1. This property is easily generalized for the case of a many-particle system. Considering the $i$th charge, one finds that the tensor (21) yields the *external* force $F^{\mu\nu}_{\text{ext}}$ associated with all other charges, except the $i$th one. This outcome is consistent with the LD equation (23), where the external field $F^{\mu\nu}_{\text{ext}}$ is used.

## IV. PROPERTIES OF THE ENERGY–MOMENTUM TENSOR OF THE FIELDS

The new energy–momentum tensor (21) was constructed in two steps. First, it is recognized that, in principle, stress effects $T^{\mu i}(F^{\lambda\rho}_{(j)},F^{\eta\theta}_{(j)})$ of the $j$th point charge on itself are unmeasurable even in a thought experiment. Hence, to maintain covariance, the entire tensor $T^{\mu\nu}(F^{\lambda\rho}_{(j)},F^{\eta\theta}_{(j)})$ is removed. The second step is the addition of tensors depending on radiation fields representing single-particle radiation in the wave zone. This step is needed in order to balance the energy–momentum of the system. Thus the principles used are measurability, covariance and energy–momentum conservation.

It is very pleasing to find that the energy–momentum (21) derived from these principles settles two problems that have haunted classical electrodynamics for a long time. One of these problems is the infinite energy associated with the fields of a single point charge. This problem simply does not arise if the tensor (21) is used. Indeed, the infinite energy emerges from terms depending on single-particle velocity fields. As explained above, single-particle terms are not included in the energy–momentum tensor (21).

Another problem is the 4/3 factor associated with the Lorentz transformation of the field momentum of a single charge. This problem is settled for small charged objects by means of a Poincaré stress required for stabilizing them (see, e.g., Ref. 7). However, this procedure cannot be applied to a point charge, because, as mentioned in the Introduction, the existence of internal stress is inconsistent with its elementary nature.

In the case of a point charge, the 4/3 problem is settled in a different way. Examining the energy–momentum tensor (21), it is realized that it *vanishes* for a system of a single motionless point charge. Therefore, the system's energy–momentum is just the particle's mechanical part

$$p^{\mu}=mcv^{\mu}, \tag{25}$$

where $m$ is the particle's rest mass. Evidently, $p^{\mu}$ of (25) is a 4-vector and all its entries transform as required. It follows that for the tensor (21), the 4/3 problem does not arise. Thus the solution of the 4/3 problem for a point charge is completely different from that of a small charged object, which involves the introduction of Poincaré stress.

## V. CONCLUDING REMARKS

Radiation and bound fields are introduced here as notions that are not identical to acceleration and velocity fields, respectively. Here, radiation fields are based on an approach which regards the system as a whole. Thus all radiation fields are seen as emitted from one fixed point (in a frame where the system's center of energy is motionless), which is chosen as the origin of spatial coordinates. There is just one singular point of the theory and that is the origin. However, no difficulty arises from this singularity, because the origin is chosen so that the position of each point charge never coincides with it. Hence, all interaction quantities are regular.

The energy–momentum tensor (21) contains interaction terms and additional terms representing the contribution of single-particle radiation fields. Its construction is based on three simple and self-evident assumptions: measurability in thought experiments, covariance and energy–momentum conservation. The outcome settles the problems of infinite energy in the fields of an elementary classical point charge and of the 4/3 factor of Lorentz transformation of momentum.

As explained in Sec. III, the LD equation (23) is an essential element required for a balance of the single-particle terms which are omitted from the energy momentum tensor (21). Hence, the formulation presented here contains all problematic aspects of this equation. A serious problem of this equation is the one-dimensional motion of a charge *attracted* to the origin by a Coulomb force. It can be shown that if this charge obeys the LD equation then the moving charge reverses its motion and recedes to infinity with a velocity that approaches the speed of light.[8,9] (In the literature discussing the LD equation, such a solution is called a runaway solution.) However, the problem of the repulsive force exerted between two charges obeying the LD equation yields physically acceptable solutions.[10] Furthermore, Rutherford scattering problems of the LD equation of a point charge attracted to the origin by a Coulomb force also yield appropriate results.[11] These calculations show that the scattered charge moves inertially as $t\rightarrow\infty$, and the entire process conserves energy. Thus these important problems provide examples illustrating the self-consistency of the theory formulated here. On the other hand, it is not clear how *instantaneous* energy balance can be defined because the two terms of the LD equation which are proportional to $q^2$ are omitted here from energy consideration.

It can be concluded that in the case of a repulsive force and that of Rutherford scattering, energy balance is restored in the final state. Indeed, in the present interpretation, the charges do not interact in the final state; their motion is inertial and their self-energy is of a purely mechanical nature. At this time, the radiated energy has already left the interaction region. Therefore, it is represented appropriately by the radiation fields described in Sec. II.

Another example illustrating the validity of the LD equation is uniform rotation of charges. For example, let external nonelectromagnetic forces maintain the uniform rotation of an insulating disk; $n$ charges are distributed evenly on the circumference of this disk. Calculations show that the three forces involved (the external nonelectromagnetic force which is exerted by the disk on the charges, the electromagnetic force exerted on each charge by all other charges, and the LD force of each charge on itself) are consistent with the multipole radiation emitted from the system.[12]

The objective of the present work is to find a formulation of classical electrodynamics for point charges which is free of infinite quantities. This assignment has also been undertaken earlier (Ref. 3, Chap. 7; Ref. 13). The results of the earlier approach are not the same as those described above. The main differences are as follows.

The earlier approach uses an action principle which depends on retarded *and* advanced potentials and fields. The LD equation does not enter explicitly into the fundamental equations of the theory and is derived from an appropriate combination of retarded and advanced fields. In systems of more than one point charge, the interactions are not mediated by fields, but by the Wheeler/Feynman action at a distance prescription.[14] The present work refrains from using advanced potentials and fields. Hence, the LD equation (23) is used explicitly. Charges interact with each other via the ordinary retarded fields. The price paid is the lack of action. Thus the energy–momentum tensor is constructed in an alternative way, as described in Sec. III. This tensor takes finite values at all points except one. As explained, charges are not affected by this singularity.

Some elements of the present work have been published earlier, where an analog of the energy–momentum tensor (21) is constructed from acceleration fields.[15] The earlier version, however, is unsatisfactory because acceleration fields do not obey the homogeneous Maxwell equations, thereby yielding an energy–momentum tensor whose 4-divergence does not vanish in the vacuum.

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