A New Quantum Paradox

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Abstract: The Hamiltonian which is derived from the Dirac Lagrangian density is used for determining the state of an electron. The fact that this Hamiltonian is free of time differential operators plays a key role in the analysis. It is proved that an application of a specific gauge transformation yields inconsistent results.

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1. Introduction

A paradox is regarded as a useful tool for finding out new properties and interrelations between elements of a theory. A physical paradox describes a hypothetical device and relevant physical laws are assumed to determine properties of the system. The outcome of a paradox is an apparent contradiction. Such a contradiction provides a motivation for a further investigation of the relevant physical laws. The following lines describe briefly two paradoxes which are used here as an illustration of this matter.

In the 1930s, Einstein, Podolsky and Rosen (EPR) described a quantum paradox of an action at a distance [1]. They used a principle which they called *physical reality* and regarded the result as an indication that quantum mechanics is not a complete theory. For this reason, EPR put forward the need for finding hidden parameters that will promote quantum mechanics to the status of a complete theory. Later Bohm and Aharonov [2, 3] and Bell [4] have added elements that were used in an experimental test of the EPR idea. The results of the experiments support the idea that there is a kind of quantum information that propagates instantaneously. Thus, the apparent EPR paradox has provided a motivation for acquiring new information on how physical processes work.

In the 1960s Shockley an James presented a paradox where a stationary system of a charge and a magnet has an electromagnetic nonzero linear momentum [5]. Soon after Coleman and Van Vleck provided a general proof showing that the system's total linear momentum must be balanced [6]. Later Comay has shown that an explicit mechanical linear momentum exists in the system. In particular, if a nonvanishing pressure gradient exists along a closed loop of current then effects related to the energy-momentum tensor yield a nonzero mechanical linear momentum [7]. This mechanical momentum balances the electromagnetic linear momentum and also shows the validity of the Coleman and Van Vleck general analysis. Thus, the Shockley and James paradox has ended up with a better understanding of elements of classical physics.

The paradox of this work is described in the second section. Expressions are written in units where $\hbar = c = 1$. The relativistic metric is diagonal and its entries are (1,-1,-1,-1). Greek indices run from 0 to 3.

2. The Paradox

The paradox described below arises from an examination of a specific gauge transformation of an electron which obeys the Dirac equation. To this end, let us examine the Lagrangian density of a Dirac electron [8, see p. 78]

$$\mathcal{L}_D = \bar{\psi}[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi, \tag{1}$$

where $A^\mu = (V, \mathbf{A})$ denote the components of the electromagnetic 4-potential [9, see p. 10] or [10, see p. 48]. Here one sees that in this equation, like in any other quantum equation, the charge interacts with the 4-potential.

It is well known that the Lagrangian density (1) is invariant under a gauge transformation $\Lambda(x)$ which is an arbitrary function of the space-time coordinates (denoted by x) [8, see p. 78] and [11, see p. 345]

$$A_\mu(x) \to A_\mu(x) + \Lambda(x)_{,\mu}\,; \qquad \psi(x) \to \exp(-ie\Lambda(x))\psi(x). \tag{2}$$

Here e is the electronic charge, which is a dimensionless Lorentz scalar in the units where $\hbar = c = 1$. Indeed, substituting (2) into (1), one finds that the contribution of the gauge 4-potential $\Lambda(x)_{,\mu}$ is canceled out by the additional terms obtained from the partial differentiation of $\exp(-ie\Lambda(x))\psi(x)$.

Let us turn to the paradox and examine a motionless electron located at the vicinity of point P in a field-free space. The Dirac Hamiltonian is used for finding the time evolution of this electron. (An expression for the Hamiltonian is also required by the Bohr correspondence principle. Here the classical limit of quantum theories should agree with classical physics. Since energy is well defined in classical theory, one requires that quantum theories should have a self-consistent expression for energy.) This Hamiltonian can be derived from the Lagrangian density (1) in the following steps.

The Hamiltonian density $\mathcal{H}$ is derived from the Lagrangian density by the following well known Legendre transformation

$$\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \dot{\psi}_i - \mathcal{L}, \tag{3}$$

where the index i runs on all functions. In the specific case of a Dirac particle one obtains from (1) and (3)

$$\mathcal{H}_D = \psi^\dagger[\boldsymbol{\alpha} \cdot (-i\nabla - e\mathbf{A}) + \beta m + eV]\psi, \tag{4}$$

which is written here in the standard notation [9]. The density of a Dirac particle is $\psi^\dagger\psi$ [9, see p. 9]. Thus, removing the density from (4), one obtains the operator form of the Dirac Hamiltonian.

$$H_D = [\boldsymbol{\alpha} \cdot (-i\nabla - e\mathbf{A}) + \beta m + eV]. \tag{5}$$

This Hamiltonian stands on the right hand side of the Dirac equation [9, see p. 11]

$$i\frac{\partial \psi}{\partial t} = H_D\psi = [\boldsymbol{\alpha} \cdot (-i\nabla - e\mathbf{A}) + \beta m + eV]\psi. \tag{6}$$

As is well known, the Dirac Hamiltonian does not contain a time derivative operator.

The Dirac equation (6) is used for finding the time evolution of the electron at the vicinity of point P. Here the field-free 4-potential is

$$A^\mu = 0. \tag{7}$$

Hence, the Dirac equation for a free electron

$$i\frac{\partial\psi}{\partial t} = [\boldsymbol{\alpha}\cdot(-i\nabla) + \beta m]\psi \tag{8}$$

determines the electronic state.

Let us examine how this system is affected by the following gauge transformation

$$\Lambda(x) = et/r. \tag{9}$$

Here e is the absolute value of the electronic charge, t is the time and r is the distance from the origin of the spatial coordinates. The gauge function (9) transforms the null 4-potential (7) and yields the following expression

$$A'_\mu = \frac{\partial(et/r)}{\partial x^\mu} = (e/r, -et\mathbf{r}/r^3). \tag{10}$$

The gauge transformation (2) also transforms the Dirac wave function. Introducing the specific gauge function (9), one finds

$$\psi'(x) = \exp(-ie^2t/r)\psi(x). \tag{11}$$

On the other hand, the new function $\psi'(x)$ of (11) must satisfy the Dirac equation (6), where the gauge terms (10) are used in the expression for the 4-potential. Here one obtains

$$\begin{aligned} i\frac{\partial\psi'}{\partial t} = H_D\psi'(x) &= [\boldsymbol{\alpha}\cdot(-i\nabla - e\mathbf{A}) + \beta m + eV]\psi'(x) \\ &= \exp(-ie^2t/r)[\boldsymbol{\alpha}\cdot(-i\nabla) + \beta m - e^2/r]\psi(x) \end{aligned} \tag{12}$$

It turns out that similarly to the case of the Dirac Lagrangian density (1), the contribution of the 3-vector part of the gauge (10) is eliminated from (12). On the other hand, the 0-component of that gauge remains as is. This outcome stems from the fact that the Dirac Hamiltonian (5) contains spatial differential operators but is free of a time differential operator.

Now, let us define the point P of the electron so that its distance from the origin of the spatial coordinates is about the Bohr radius. In this case the gauge transformation (9) produces (12), which is the equation of a bound electron in the hydrogen atom (multiplied by a phase factor) [9, see p. 52].

There is another problem with the gauge transformed function ψ' of (11). Thus, an ordinary function of a particle in a well-defined energy state takes the form

$$\psi(x) = \exp(-iEt)\chi(x, y, z). \tag{13}$$

Here the time dependence of $\psi(x)$ appears only in the phase where the energy E is a constant. On the other hand, the time dependence of the gauge transformed function ψ' of (11) shows that it is *not* an energy eigenfunction. This is a contradiction because an electron in a free space has a well defined energy.

The paradox described herein is related to two expressions that differ by a gauge transformation. Unlike a general expectation, two physically different results are obtained. The state of a free electron which is derived from (8) is inconsistent with that of the solution of (12), where the electron is bound to the Hydrogen atom. An examination of the derivation of the final form of (12) indicates that the problem stems from the fact that unlike the Dirac Lagrangian density (1), the Dirac Hamiltonian density (4) and its associated Hamiltonian (5) are independent of a time differential operator.

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