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Axiomatic Deduction of Equations of Motion in Classical Electrodynamics (*).

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Summary. — The equations of motion of a classical system of electric charges, magnetic monopoles and electromagnetic waves are derived by using five axioms. The work answers the question: what are the equations of motion of this system which can be derived from a regular Lagrangian density?

PACS. 03.30. – Special relativity.

1. – Introduction.

The formulation of classical electrodynamics of electric charges (henceforth called charges), magnetic monopoles (henceforth called monopoles) and electromagnetic waves (henceforth called waves) will use potentials, fields and currents. Subscripts $_{(e)}$, $_{(m)}$ and $_{(w)}$ will denote quantities associated with charges, monopoles and waves, respectively. Units in which the velocity of light $c = 1$ are used. Greek indices run from 0 to 3 and Latin indices run from 1 to 3. All calculations will be carried out for the vacuum and, therefore, $\boldsymbol{D}$ and $\boldsymbol{H}$ will not be used.

The work is divided into seven sections. This first section is the introduction. The second and the third sections present the theory for systems of charges and waves and for systems of monopoles and waves, respectively.

(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.

The fourth section classifies the different types of fields. The five axioms upon which the work is based are presented in the fifth section. In the sixth section, the Lagrangian, the equations of motion and the energy-momentum tensor for systems of charges, monopoles and waves are derived. The last section contains some concluding remarks.

2. – Systems without monopoles.

A theory of classical systems of charges, monopoles and waves should include classical electrodynamics of charges and waves as a subtheory. This subtheory can be neatly written in relativistic notation ((1), pp. 66-95, or (2), pp. 95-121). The electromagnetic fields are components of the tensor

$$F_{(e,w)}{}^{\mu\nu}=\begin{pmatrix}0 & E_1 & E_2 & E_3\\ -E_1 & 0 & B_3 & -B_2\\ -E_2 & -B_3 & 0 & B_1\\ -E_3 & B_2 & -B_1 & 0\end{pmatrix}_{(e,w)} . \tag{1}$$

The fields are derived from a 4-potential A^μ:

$$F_{(e,w)\mu\nu}=\partial_\mu A_{(e,w)\nu}-\partial_\nu A_{(e,w)\mu} , \tag{2}$$

The current of the charges is a 4-vector

$$J_{(e)}{}^\mu=\varrho_{(e)}(1, v_1, v_2, v_3) . \tag{3}$$

The fields satisfy the four Maxwell equations

$$\partial_\lambda F_{(e,w)_{\mu\nu}}+\partial_\mu F_{(e,w)_{\nu\lambda}}+\partial_\nu F_{(e,w)_{\lambda\mu}}=0 , \tag{4}$$

$$\partial_\nu F_{(e,w)}{}^{\mu\nu}=4\pi J_{(e)}{}^\mu , \tag{5}$$

where (4) is a compact form of the two homogeneous Maxwell equations and (5) represents the two inhomogeneous ones. The Lorentz law of force is

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau}=F_{(e,w)}{}^{\mu\nu}J_{(e)\nu} , \tag{6}$$

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(1) L. D. Landau and E. M. Lifshitz: *The Classical Theory of Fields* (Pergamon Press, London, 1962).

(2) D. E. Soper: *Classical Field Theory* (John Wiley and Sons, New York, N. Y., 1976).

where P^μ is the 4-vector of energy-momentum and τ is the invariant time.

The antisymmetry of $F_{(e,w)}{}^{\mu\nu}$, together with the assumption that the potentials are regular functions of the co-ordinates, guarantees that (4) will be satisfied. Equations (5) are the Euler-Lagrange equations derived from the electromagnetic part of the Lagrangian density of the system. This quantity is

$$L_{(e,w)} = -\frac{1}{16\pi} F_{(e,w)}{}^{\mu\nu} F_{(e,w)\mu\nu} + J_{(e)}{}^{\mu} A_{(e,w)\mu} \,. \tag{7}$$

It is seen that this Lagrangian density is a function of the potentials, their derivatives and the currents. A useful quantity is the energy-momentum tensor of the system. Its electromagnetic part is derived from (7):

$$\theta_{(e,w)}{}^{\mu\nu} = \frac{1}{4\pi}\left(F_{(e,w)}{}^{\mu}{}_{\alpha} F_{(e,w)}{}^{\nu\alpha} - \frac{1}{4} g^{\mu\nu} F_{(e,w)}{}^{\alpha\beta} F_{(e,w)\alpha\beta}\right). \tag{8}$$

For the sake of brevity, this quantity will sometimes be written by using the shorthand notation $\theta^{\mu\nu}(F, F)$.

θ^{00} is the energy density of the system and θ^{i0} is its momentum density. The 3-dimensional spatial integral of these quantities is the 4-vector of energy momentum of the system. The change in the energy-momentum density of the system is

$$\partial_\nu \theta_{(e,w)}{}^{\mu\nu} = - F_{(e,w)}{}^{\mu\nu} J_{(e)\nu} \,. \tag{9}$$

This result can be compared with (6). It shows that the sum of the energy momentum of the mechanical part and the electromagnetic part is a conserved quantity.

3. – Systems without charges.

Assume that there exists in Nature another type of source of electromagnetic fields. Particles having this type of source will behave, together with waves, analogously to our known world of charges and waves. The only difference between this world (the dual world) and our world is that charges at rest which encounter a wave accelerate in the direction of the electric field of this wave, while particles at rest having the other type of source of fields accelerate in the direction of the magnetic field of the wave. This property of particles having the other type of source justifies the term magnetic monopoles. The question of particles having both types of sources of fields is beyond the scope of this paper.

The definition of physical quantities of the dual world and their equations of motion are analogous to those of charges and waves. The following nine expressions correspond to the ones written above:

$$F_{(\mathrm{m,w})}{}^{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{pmatrix}_{(\mathrm{m,w})}, \tag{10}$$

$$F_{(\mathrm{m,w})\mu\nu} = \partial_\mu A_{(\mathrm{m,w})\nu} - \partial_\nu A_{(\mathrm{m,w})\mu}\,, \tag{11}$$

$$J_{(\mathrm{m})}{}^{\mu} = \varrho_{(\mathrm{m})}(1, v_1, v_2, v_3)\,, \tag{12}$$

$$\partial_\lambda F_{(\mathrm{m,w})\mu\nu} + \partial_\mu F_{(\mathrm{m,w})\nu\lambda} + \partial_\nu F_{(\mathrm{m,w})\lambda\mu} = 0\,, \tag{13}$$

$$\partial_\nu F_{(\mathrm{m,w})}{}^{\mu\nu} = 4\pi J_{(\mathrm{m})}{}^{\mu}\,, \tag{14}$$

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = F_{(\mathrm{m,w})}{}^{\mu\nu} J_{(\mathrm{m})\nu}\,, \tag{15}$$

$$L_{(\mathrm{m,w})} = -\frac{1}{16\pi} F_{(\mathrm{m,w})}{}^{\mu\nu} F_{(\mathrm{m,w})\mu\nu} + J_{(\mathrm{m})}{}^{\mu} A_{(\mathrm{m,w})\mu}\,, \tag{16}$$

$$\theta_{(\mathrm{m,w})}{}^{\mu\nu} = \frac{1}{4\pi}\left(F_{(\mathrm{m,w})}{}^{\mu}{}_{\alpha} F_{(\mathrm{m,w})}{}^{\nu\alpha} - \frac{1}{4} g^{\mu\nu} F_{(\mathrm{m,w})}{}^{\alpha\beta} F_{(\mathrm{m,w})\alpha\beta}\right), \tag{17}$$

$$\partial_\nu \theta_{(\mathrm{m,w})}{}^{\mu\nu} = -F_{(\mathrm{m,w})}{}^{\mu\nu} J_{(\mathrm{m})\nu}\,. \tag{18}$$

Fields of the type discussed in this section will be called magnetoelectric fields.

4. – Electromagnetic and magnetoelectric fields.

There is an apparent correspondence between (1) and (10). In both tensors the polar vectors $\boldsymbol{E}_{(\mathrm{e})}$ and $\boldsymbol{B}_{(\mathrm{m})}$ have the same indices and so do the axial vectors $\boldsymbol{B}_{(\mathrm{e})}$ and $\boldsymbol{E}_{(\mathrm{m})}$. However, while the polar fields have the same sign, the axial fields of (1) and (10) have opposite signs. Using this definition causes Maxwell's equations for the world of charges to correspond to those for the world of monopoles in two different ways. The duality of the worlds of charges and monopoles means that the homogeneous equations (4) correspond to (13) and the inhomogeneous equations (5) correspond to (14). Definition (10) makes also the left-hand sides of (4) and (14) identical and so do the left-hand sides of (5) and (13).

An important result of (10) is that all components of the energy-momentum tensor (17) are the same function of the fields as those of (8). This is essential, since the same waves are parts of the two worlds. The above-mentioned property of (8) and (17) means that energy and momentum have the same definition in the two worlds. This enables a theory that incorporates the two worlds to add the two tensors without encountering contradiction.

Charges and monopoles have different types of fields. The magnetoelectric fields of the monopoles are not the same as the electromagnetic fields of the charges. Indeed it is known that the respective fields of the two types of sources differ in parity. The magnetic field of charges is an axial vector, while the magnetic field of monopoles is a vector, and in analogous fashion for the two types of electric fields.

The electromagnetic and magnetoelectric fields satisfy four Maxwell equations. These equations can be paired so that one member of the pair is an equation for charges and the other is an equation for monopoles, and both have the same left-hand side. However, the right-hand sides of the equations are not the same in any case. Where one type satisfies a homogeneous equation, the second type satisfies an inhomogeneous one. Only the waves of the two systems are the same. They satisfy four homogeneous Maxwell equations. Therefore, the assumption that the waves emitted from charges are the same as those emitted from monopoles is self-consistent.

The fields can be classified in three distinct types: *a*) fields which satisfy four homogeneous equations in all space, *b*) fields which satisfy (4) and (5) and do not belong to the first type, *c*) fields which satisfy (13) and (14) and do not belong to the first type. Each of these types of fields is related to a different physical entity, namely the photon, the charge and the monopole, respectively.

This discussion shows that the fields of monopoles and the corresponding fields of charges have different mathematical properties. They differ in parity and they satisfy different equations, homogeneous and inhomogeneous respectively. Therefore, one cannot be sure that the fields associated with monopoles have the same properties as the respective fields of charges. The term magnetoelectric fields is introduced here in order to facilitate the analysis presented in the remainder of the paper.

The reader should realize that the notation « magnetoelectric fields » is not an introduction of an artificial entity. As an example, let us take the world of electric charges. One may denote the fields of the positive charges and those of the negative ones differently. No error has been done, since the equations of motion are linear in the fields. The expressions derived in this process will be a somewhat more complicated presentation of our known theory. By contrast, the distinction between the fields of charges and those of monopoles will prove useful and important.

Formally, if one does not distinguish between the two types of electric fields and similarly for the two types of magnetic fields, an electromagnetic-field tensor whose components are the combined electric and the combined magnetic fields can be defined. The following relation holds in this case:

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,, \tag{19}$$

where the components of F are as those of (1) and those of $\tilde{F}$ are as those of (10). $\varepsilon^{\mu\nu\alpha\beta}$ is the complete antisymmetric tensor of four indices. This definition enables us to write the extension of Maxwell equations for a world of charges, monopoles and waves in a compact form. Indeed, (4) and (13), (5) and (12) can be written as follows:

$$\partial_\nu F_{(\mathrm{e,m,w})}{}^{\mu\nu} = 4\pi J_{(\mathrm{e})}{}^{\mu}\,, \tag{20}$$

$$\partial_\nu \tilde{F}_{(\mathrm{e,m,w})}{}^{\mu\nu} = 4\pi J_{(\mathrm{m})}{}^{\mu}\,. \tag{21}$$

It will be shown that these equations are the equations of motion of the fields in a combined world of charges, monopoles and fields.

5. – Five axioms.

This work attempts to find the equations of motion of a system of charges, monopoles and electromagnetic waves which satisfy the following requirements:

a) It will reduce to the known theory of charges and waves for systems where monopoles are absent.

b) It will reduce to the dual theory for systems where charges are absent.

c) It will be derived from a Lagrangian density whose terms are regular functions of the potentials, their derivatives and the currents of the charges and of the monopoles. The terms of the Lagrangian density should be Lorentz scalars and invariant under space-time translations.

d) The equations of motion of the fields are linear (*i.e.* the principle of superposition continues to hold).

e) It is assumed that a system of one charge and one monopole, where the two particles do not move, does not change in time.

6. – Equations of motion of combined systems.

The Lagrangian density of the system will be written as a sum of terms. These terms belong to one of three groups: pure charge-wave terms, pure monopole-wave terms and charge-monopole terms. By using *a*), it is found that the first group is the expression written in (7). By using *b*), it is found that the second group is that which is written in (16). Therefore, the Lagrangian density can be put in the following form:

$$L = L_{(e,w)} + L_{(m,w)} + L' , \tag{22}$$

where $L_{(e,w)}$, $L_{(m,w)}$ are defined in (7) and (16), respectively. Using *d*) and the fact that Euler-Lagrange equations contain first-order derivatives with respect to the potentials or the fields, we see that L' should be a quadratic expression in the fields and the potentials. Therefore, it can be put in the following form:

$$L' = aF_{(e)}{}^{\mu\nu}F_{(m)\mu\nu} + bJ_{(e)}{}^{\mu}A_{(m)\mu} + dJ_{(m)}{}^{\mu}A_{(e)\mu} + fA_{(e)}{}^{\mu}A_{(m)\mu} , \tag{23}$$

where a, b, d, f are coefficients to be determined. Notice that the term $J_{(e)}{}^{\mu}J_{(m)\mu}$ is omitted. This term, when the two currents represent the same particle, is related to the mechanical term of the particle in the Lagrangian density. This term is not discussed in this work. When the two currents represent two different particles, the term is omitted, since it is assumed that the particles interact only through fields. In order to define the coefficients, let us look at a system of one charge and one monopole where both particles are at rest. In this configuration, the currents and the potentials are

$$J_{(e)}{}^{\mu} = \varrho_e(1, 0, 0, 0) , \tag{24}$$

$$J_{(m)}{}^{\mu} = \varrho_m(1, 0, 0, 0) , \tag{25}$$

$$A_{(e)}{}^{\mu} = (\Phi_{(e)}, 0, 0, 0) , \tag{26}$$

$$A_{(m)}{}^{\mu} = (\Phi_{(m)}, 0, 0, 0) . \tag{27}$$

The Lagrangian density of this system is

$$L = L_{(e)}(\mathrm{SP}) + L_{(m)}(\mathrm{SP}) + L' , \tag{28}$$

where SP denotes a Lagrangian density of a single particle and its own fields.

By using *e*), a variation of (28) with respect to the co-ordinates of the charge should vanish. This quantity is related to the change of the mechanical energy-momentum of the particle. The charge continues to stay at rest and, therefore, its energy momentum does not change. The single-particle term of (28) satisfies this requirement. Therefore, the variation of L' should vanish too. It follows that

$$be \operatorname{grad} \Phi_{(m)} \cdot \delta \boldsymbol{r} = 0 . \tag{29}$$

Since grad $\Phi_{(m)}$ does not vanish at the location of the charge, we have

$$b = 0\,. \tag{30}$$

Similarly, a variation of the co-ordinates of the monopole yields

$$d = 0\,. \tag{31}$$

A variation of the potential $A_{(e)}{}^{\mu}$ yields, in an way analogous to that which shows that (5) is derived from (7),

$$\operatorname{div} \boldsymbol{E}_{(e)} - 8\pi a \operatorname{div} \boldsymbol{B}_{(m)} = 4\pi\varrho_{(e)} + 4\pi f \Phi_{(m)}\,. \tag{32}$$

Testing (32) where there are no charges and no monopoles, we find that f vanishes. Testing (32) at the location of the monopole, we find that a vanishes. Therefore,

$$a = f = 0\,. \tag{33}$$

It follows that the electromagnetic part of the Lagrangian density is

$$L = -\frac{1}{16\pi} F_{(e,w)}{}^{\mu\nu} F_{(e,w)\mu\nu} + J_{(e)}{}^{\mu} A_{(e,w)\mu} - \\ - \frac{1}{16\pi} F_{(m,w)}{}^{\mu\nu} F_{(m,w)\mu\nu} + J_{(m)}{}^{\mu} A_{(m,w)\mu} + \frac{1}{16\pi} F_{(w)}{}^{\mu\nu} F_{(w)\mu\nu}\,, \tag{34}$$

where the last term is introduced in order that (34) reduces to (7) if there are no monopoles and to (16) if there are no charges. This term yields homogeneous equations which do not affect the results obtained. Therefore, it is ignored in the next two paragraphs.

The Lagrangian density (34) is the sum of the terms of (7) and (16), where (w) represent the combined waves. It is clear that (5) and (14) are obtained from (34). Furthermore, (4) and (13) are derived from the antisymmetry of (2) and (11) and from the regularity of the potentials. It was mentioned already that these equations can be arranged in pairs, so that the left-hand sides of the two members of a pair are identical. It follows that (20) and (21) are derived from (34).

Since (34) is a formal sum of (7) and (16), there is no change in the Lorentz law of force for charges and for monopoles. These laws are (6) and (15). In other words, the magnetoelectric fields of monopoles do not accelerate charges and the electromagnetic fields associated with charges do not accelerate monopoles. Charges and monopoles interact only through the exchange of waves.

The electromagnetic part of the energy-momentum tensor of the system is obtained from (34) in an analogous manner to that of standard textbooks ([2], pp. 120, 121), where this quantity is derived for charges and waves alone.

The result is

$$\theta_{(e,m,w)}{}^{\mu\nu} = \theta^{\mu\nu}(F_{(e,w)}, F_{(e,w)}) + \theta^{\mu\nu}(F_{(m,w)}, F_{(m,w)}) - \theta^{\mu\nu}(F_{(w)}, F_{(w)}) , \tag{35}$$

where $\theta^{\mu\nu}(F, F)$ is defined after formula (8).

Expression (35) is obtained as follows. Using the first and second terms of (34), we have the first term of (35). Repeating the process for the third and fourth terms of (34), we have the second term of (35). At this point, the contribution of the fields of the waves was counted twice. This can easily be seen as follows:

$$\theta^{\mu\nu}(F_{(e,w)}, F_{(e,w)}) = \theta^{\mu\nu}(F_{(e)} + F_{(w)}, F_{(e)} + F_{(w)}) = \\ = \theta^{\mu\nu}(F_{(e)}, F_{(e)}) + \theta^{\mu\nu}(F_{(w)}, F_{(w)}) + 2\theta^{\mu\nu}(F_{(e)}, F_{(w)}) , \tag{36}$$

$$\theta^{\mu\nu}(F_{(m,w)}, F_{(m,w)}) = \theta^{\mu\nu}(F_{(m)} + \tilde{F}_{(w)}, F_{(m)} + \tilde{F}_{(w)}) = \\ = \theta^{\mu\nu}(F_{(m)}, F_{(m)}) + \theta^{\mu\nu}(\tilde{F}_{(w)}, \tilde{F}_{(w)}) + 2\theta^{\mu\nu}(F_{(m)}, \tilde{F}_{(w)}) . \tag{37}$$

It is seen that the combination of (36) and (37) indeed counts the wave term twice. The last term of (35) guarantees the balance of the expression.

Performing the operation analogous to that of (9) and (18), it is found that the change of the energy momentum of the fields balances the mechanical energy momentum transmitted to the particles by the Lorentz force. The Lorentz force derived above and the following expression

$$\partial_\nu \theta_{(e,m,w)}{}^{\mu\nu} = - F_{(e,w)}{}^{\mu\nu} J_{(e)\nu} - F_{(m,w)}{}^{\mu\nu} J_{(m)\nu} \tag{38}$$

show that the system conserves energy and momentum.

7. – Concluding remarks.

The main result of this work is that, if one takes the five assumptions *a*)-*e*), then the extension of the equations of motion is already determined. In this case there is no freedom of choice.

The presently accepted formulation of the problem ([3]) extends the equations of motion of the fields in the same way as was obtained here, namely (20) and (21). However, the Lorentz law of force is extended differently (([3]), formula (7), and ([2]), formula (9.7.10)). It assumes that there is no difference between the electromagnetic fields of charges and the magnetoelectric fields of monopoles and that both accelerate charges (and monopoles) in the same

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([3]) P. A. M. DIRAC: *Phys. Rev.*, **74**, 817 (1948).

way. It arrives at

$$\frac{dP^\mu}{d\tau} = F_{(e,m,w)}{}^{\mu\nu} J_{(e)\nu}\,, \tag{39}$$

$$\frac{dP^\mu}{d\tau} = \tilde{F}_{(e,m,w)}{}^{\mu\nu} J_{(m)\nu}\,. \tag{40}$$

The presently accepted equations satisfy the axioms *a*), *b*), *d*) and *e*) of sect. **5**. It follows that they cannot satisfy *c*). In other words, there is no regular Lagrangian density whose Euler-Lagrange equations are the presently accepted equations of motion. This result is not new ([4]).

The approach of this work presents an answer to the following question: what are the equations of motion of a system of charges, monopoles and waves which are compatible with the simple requirements *a*), *b*), *d*) and *e*) of sect. **5** and which can be obtained from a regular Lagrangian density?

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I wish to acknowledge the hospitality of the University of Michigan where this work has been done.

([4]) F. Rohrlich: *Phys. Rev.*, **150**, 1104 (1966).

RIASSUNTO (*)

Si derivano le equazioni di moto di un sistema classico di cariche elettriche, monopoli magnetici ed onde elettromagnetiche usando cinque assiomi. Questo lavoro risponde alla domanda: quali sono le equazioni di moto di questo sistema che possono essere derivate da una regolare densità lagrangiana?

(*) *Traduzione a cura della Redazione.*

Аксиоматический вывод уравнений движения в классической электродинамике.

Резюме (*). — Используя пять аксиом, выводятся уравнения движения классической системы электрических зарядов, магнитных монополей и электромагнитных волн. Показано, что представляют собой уравнения движения этой системы, которые могут быть выведены из регулярной плотности Лагранжиана.

(*) *Переведено редакцией.*