Exposing "hidden momentum"

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It is shown that "hidden momentum" is a real physical entity found in a continuous matter under pressure when observed from an improper inertial frame. Precisely the same results are obtained from an analysis of the mechanical motion of microscopic particles that create pressure. Analogous properties hold in a pressure created by standing electromagnetic waves enclosed between two parallel mirrors. The significance of the energy-momentum tensor and its divergence is pointed out. It is also shown that in certain cases there exists an ensemble of particles where the *sum* of the energy–momentum 4-vectors of its constituents does not transform as a 4-vector. Using these results, it is shown that a stationary classical system made of a magnet and electric charges has null linear momentum.

I. INTRODUCTION

The phenomenon of a "hidden momentum" has been discussed in the literature for more than two decades.[1] It typically arises in a stationary system which contains a magnet and a system of electric charges. According to the laws of electrodynamics, such a system has a nonzero electromagnetic linear momentum $(1/4\pi c)\int \mathbf{E}\times\mathbf{B}d^3r$. Considering the fact that the system is stationary, one is obliged to identify an additional linear momentum which balances the entire system's momentum. A recent discussion[2] indicates that the problem is not yet settled. The purpose of the present work is to illuminate the phenomenon of hidden momentum. The structure and transformation laws of the energy-momentum tensor are analyzed and used in the derivation of the results. It is shown that hidden momentum is an inherent relativistic property of moving macroscopic bodies which carry pressure. An analysis of the relativistic kinematics of particles whose motion creates the pressure effect confirms this result. The same situation is found in a pressure created by free electromagnetic waves enclosed between two mirrors.

In the present work Greek indices range from 0 to 3 and Latin ones take the values 1, 2, and 3. The metric $g_{\mu\nu}$ is diagonal and its entries are $(1,-1,-1,-1)$; **i**, **j**, and **k** denote unit vectors in the x-, y- and z directions, respectively. The symbol $,\nu$ denotes a partial differentiation with respect to x^ν.

The discussion utilizes properties of the energy–momentum tensor of continuous objects $T^{\mu\nu}$. T^{00} is the energy density and T^{i0} is c times the ith component of the momentum density. Thus, energy and momentum enclosed in a volume can be obtained as appropriate integrals of $T^{\mu 0}$. [Note that the energy W and momentum **p** of a particle can be written as components of a 4-vector $P^\mu=(W/c, p_x,p_y,p_z)$. Hereafter, a statement referring to energy as a component of a 4-vector should be construed as energy divided by c.] In special relativity energy is conserved. This means that, in a closed system, $T^{\mu\nu}_{,\nu}=0$ provided this tensor represents all elements of the system. In the present work, different elements of the system are discussed *separately.* Examples where $T^{\mu\nu}_{,\nu}\neq 0$ are analyzed. It is known that in such cases one cannot be sure that the energy and momentum enclosed in a spatial volume transform as components of a 4-vector.[3] Specific devices that demonstrate this law of special relativity are described below. It is also proved that analogous properties hold for an ensemble of discrete particles enclosed in a finite volume. This outcome means that in some cases the energy and momentum obtained as integrals of corresponding densities (or as appropriate sums) do not deserve to be considered as components of a 4-vector. For this reason the terms "false 4-vector" and "false 4-momentum" are used as a notation of such quantities. In the present work false 4-vectors are written like genuine ones. Hidden momentum is the difference between the correct mechanical momentum of an object and the "naive" mechanical momentum $E\mathbf{v}/c^2$ one might ascribe to it if one thought its mechanical energy and momentum transformed as a true 4-vector. It is relativistic in nature and has to do with the internal motion of the object's constituent parts. As a matter of fact, "hidden energy" accompanies hidden momentum.

The magnets used in demonstrations aiming to show the need for a hidden momentum are classical objects where charges move in closed loops. However, for the sake of simplicity and clarity, the present work concentrates on linear motion. It is shown later that the results can be applied to the circular motion of a device in which an appropriate pressure gradient arises. In this way, the momentum problem of a classical magnet in an external electric field is settled.

The paper is organized as follows. A parallel plate capacitor is analyzed in two inertial frames. The transformation laws of its macroscopic elements are discussed in Sec. II A. The same results are obtained in Sec. II B which contains an analysis of the kinematics of microscopic particles that create pressure. In Sec. III, it is shown that analogous results are obtained in the case where the mechanical pressure is replaced by a pressure of electromagnetic waves enclosed between two parallel mirrors. An application to a classical magnet and motionless charges is described in Sec. IV. Concluding remarks are the contents of Sec. V.

II. A PARALLEL PLATE CAPACITOR

Consider a parallel plate capacitor whose plates are perpendicular to the x axis. Let d denote the distance between the plates and D their thickness. The inner surfaces of the plates are covered with positive and negative uniform surface charge density $\pm\sigma$, respectively. The attraction between the plates is compensated by the pressure of a gas which is stored between them [see Fig. 1(a)]. The electric permittivity of the gas is equal to that of the vacuum.

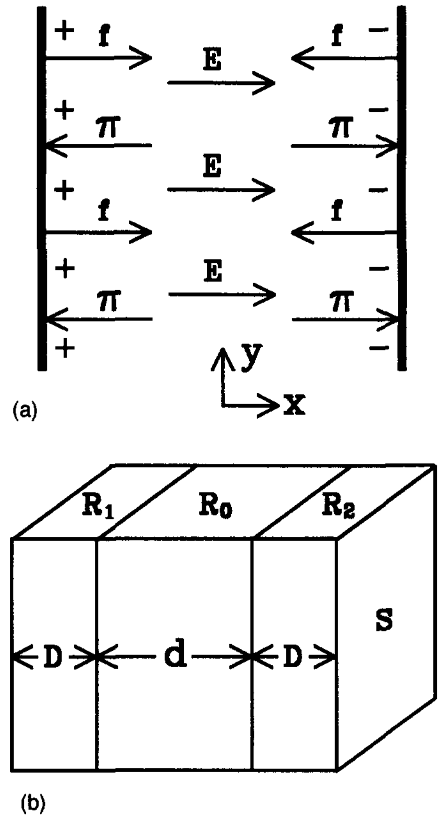

Fig. 1. (a) A cross section of the parallel plate capacitor with the (x,y) plane. The two plates are covered uniformly with charge density $\pm\sigma$, respectively. The electric field **E** exerts an attractive force **f** on each plate. This force is balanced by the pressure π. (b) The three rectangular parallelepipeds R_0, R_1, and R_2, whose lengths are d, D, and D, respectively. S denotes the area of the appropriate face. Note that here the relative size D of the plates' thickness is larger than in (a) (see the text).

The capacitor is motionless in an inertial frame Σ. Let us calculate the energy and momentum of the system. For the sake of simplicity, the discussion is restricted to the field and the gas included in a rectangular parallelepiped having two square faces (whose area is denoted by S) lying on the faces of the capacitor's plates. Let R_0 denote this rectangular parallelepiped and $V_0=Sd$ its volume in Σ. Similarly, the discussion includes also two identical rectangular parallelepipeds (named R_1 and R_2, respectively), each of which contains the material of one of the plates. These rectangular parallelepipeds have two square faces like those of R_0. Let $V_1=V_2=SD$ denote the volume of R_1 and of R_2 as measured in Σ. R_1 is placed on the left-hand side of R_0 and R_2 is on its right-hand side [see Fig. 1(b)].

A. A macroscopic analysis

In this section all elements of the capacitor are treated as macroscopic bodies. The electrostatic field between the plates is

$$\mathbf{E}=4\pi\sigma\mathbf{i}. \tag{1}$$

One-half of this quantity is associated with the positive charge located on the left-hand side plate and the second half is due to the charge on the right-hand side plate. Hence, the electrostatic attraction on the portion of the plates belonging to R_1 and R_2 is

$$\mathbf{f}=\pm 2\pi\sigma^2 S\mathbf{i}. \tag{2}$$

This force is balanced by the pressure Π

$$\Pi=2\pi\sigma^2=E^2/8\pi. \tag{3}$$

With this proviso, the entire system is stable, and neither energy nor momentum is exchanged with the environment.

The physical entities inside the rectangular parallelepipeds are the two plates, the gas, and the electric field. In the expressions below, the subscripts P, G, and F denote quantities pertinent to the plates, the gas, and the field, respectively. The calculation uses the energy–momentum tensor $T^{\mu\nu}$ of the various elements of the system. Generally, an energy–momentum tensor is a function of the position and time x^μ. The energy–momentum tensor of solids and of nonviscous fluids is[4]

$$T^{\mu\nu}=(\epsilon+\Pi)u^\mu u^\nu-\Pi g^{\mu\nu}. \tag{4}$$

Here, ϵ and Π denote the proper energy density and pressure, respectively, and u^μ is the dimensionless 4-velocity of the matter in the volume element at which $T^{\mu\nu}$ is calculated. In Σ, the capacitor and the gas are motionless, and $u^\mu=(1,0,0,0)$.

The charge is located on the inner surfaces of the plates. Thus the plates themselves are free of pressure. Therefore, their energy–momentum tensor takes the following simple form:

$$T_P^{\mu\nu}=\begin{pmatrix} \epsilon_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Using (4), one finds that the energy–momentum tensor of the gas is

$$T_G^{\mu\nu}=\begin{pmatrix} \epsilon_G & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & \Pi & 0 \\ 0 & 0 & 0 & \Pi \end{pmatrix}. \tag{6}$$

The energy–momentum tensor of electromagnetic fields is[5]

$$T_F^{\mu\nu}=\frac{1}{4\pi}\left(F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta}+\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\right), \tag{7}$$

where the field tensor is

$$F^{\mu\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{8}$$

In particular,

$$T_F^{00}=\frac{1}{8\pi}(E^2+B^2) \tag{9}$$

is the energy density and

$$T_F^{i0}=\frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})_i \tag{10}$$

is c times the ith component of the momentum density. In the present case we have no magnetic field and the electric field is parallel to the x axis. Hence, the field's energy–momentum tensor is

$$T_F^{\mu\nu}=\frac{E^2}{8\pi}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{11}$$

As expected, an examination of (5), (6), and (11) shows that the momentum density T^{i0}/c vanishes everywhere. The energy included in the volume considered here is obtained from the energy density T^{00} of each of these tensors. This quantity is uniform in each of the corresponding rectangular parallelepipeds R_0, R_1, and R_2. Thus, the energy of each constituent of the system is obtained as a simple multiplication of the energy density by the corresponding volume of the rectangular parallelepiped. It follows that the energy–momentum 4-vector of each physical element of the capacitor is

$$P_P^\mu=2V_1\epsilon_P(1/c,0,0,0), \tag{12}$$

$$P_G^\mu=V_0\epsilon_G(1/c,0,0,0), \tag{13}$$

and

$$P_F^\mu=\frac{V_0E^2}{8\pi}(1/c,0,0,0). \tag{14}$$

The energy-momentum 4-vector of the entire capacitor is

$$P^\mu=P_P^\mu+P_G^\mu+P_F^\mu=[2V_1\epsilon_P+V_0(\epsilon_G+\Pi)](1/c,0,0,0), \tag{15}$$

where relation (3) was used.

The dynamics of the system can be described as follows. The electric field and the gas do not interact with each other. Each of these elements exchanges momentum with the plates but the effects of these interactions cancel each other. Thus it will be shown that the plates' expression (12) is a genuine 4-vector. On the other hand, the gas' expression (13) as well as that of the field (14) are actually false 4-vectors. Obviously, the energy and momentum of the entire system (15) are components of a genuine 4-vector.

Let us now examine the capacitor from another inertial frame Σ'. In Σ, this frame is seen moving in velocity $\mathbf{u}=-u\mathbf{i}$. The Lorentz transformation from Σ to Σ' is

$$L_\nu^\mu=\begin{pmatrix}\gamma&\gamma u/c&0&0\\\gamma u/c&\gamma&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \tag{16}$$

where $\gamma=(1-u^2/c^2)^{-1/2}$. Assume a 4-vector P^μ is seen in Σ. Then (16) yields its form in Σ'

$$P'^\mu=L_\nu^\mu P^\nu. \tag{17}$$

Similarly, a tensor having two indices transforms as follows:

$$T'^{\mu\nu}=L_\alpha^\mu L_\beta^\nu T^{\alpha\beta}. \tag{18}$$

Applying this Lorentz transformation to the energy–momentum tensors (5), (6), and (11), one obtains the corresponding quantities at Σ'

$$T_P'^{\mu\nu}=\begin{pmatrix}\gamma^2\epsilon_P&\gamma^2u\epsilon_P/c&0&0\\\gamma^2u\epsilon_P/c&\gamma^2u^2\epsilon_P/c^2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \tag{19}$$

$$T_G'^{\mu\nu}=\begin{pmatrix}\gamma^2(\epsilon_G+u^2\Pi/c^2)&\gamma^2u(\epsilon_G+\Pi)/c&0&0\\\gamma^2u(\epsilon_G+\Pi)/c&\gamma^2(\Pi+u^2\epsilon_G/c^2)&0&0\\0&0&\Pi&0\\0&0&0&\Pi\end{pmatrix} \tag{20}$$

and

$$T_F'^{\mu\nu}=\frac{E^2}{8\pi}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{21}$$

Note that in the present case, the tensor of the electromagnetic fields does not change and $T_F'^{\mu\nu}=T_F^{\mu\nu}$. This is consistent with Lorentz transformation of electromagnetic fields which are parallel to the boost. As is well known, in this case the fields do not change.[6]

In the frame Σ', the volume of each of the rectangular parallelepipeds undergoes a Lorentz contraction by a factor γ. Hence, one multiplies $T'^{\mu0}$ of (19), (20), and (21) by $2V_1/c\gamma$ or by $V_0/c\gamma$, respectively, and finds that in Σ', the energy–momentum 4-vectors of the capacitor's constituents (which are either genuine or false ones) are as follows:

$$P_P'^\mu=\gamma 2V_1\epsilon_P(1/c,u/c^2,0,0), \tag{22}$$

$$P_G'^\mu=\gamma V_0(\epsilon_G/c+u^2\Pi/c^3,u(\epsilon_G+\Pi)/c^2,0,0), \tag{23}$$

and

$$P_F'^\mu=\frac{V_0E^2}{8\pi\gamma}(1/c,0,0,0). \tag{24}$$

Summing (22), (23), and (24) and using (3), one obtains the energy–momentum 4-vector of the entire capacitor

$$P'^\mu=\gamma[2V_1\epsilon_P+V_0(\epsilon_G+\Pi)](1/c,u/c^2,0,0). \tag{25}$$

This result agrees with the one obtained from an application of the Lorentz transformation (16) to the capacitor's energy–momentum 4-vector (15).

This outcome reflects the self-consistency of special relativity. On the other hand, it is interesting to note the different behavior of the energy and momentum of the capacitor's constituents. The plate is matter which is free of pressure, and in Σ' its energy–momentum 4-vector (22) fits the transformed version of (12). This property does not hold for the false 4-momenta of the gas and of the electric field. *Only the sum of these quantities transforms as a 4-vector.*

B. A microscopic analysis of pressure

Let us consider the microscopic constituents of the gas. For simplicity assume that we have a "gas" made of point particles which move parallel to the x axis. These particles are reflected elastically from the plates and do not interact with each other. Consider one particle M_k of this gas whose rest mass is m. The purpose of the calculation is to find the mean energy and mean momentum of M_k in Σ and in the Σ'

and the relations between these quantities. Ignoring negligible fluctuations, one finds that the sums of mean energy and mean momentum of the entire ensemble of particles correspond to the instantaneous values of these quantities.

In the inertial frame Σ, the velocity of M_k is

$$v_+ = v\mathbf{i} \tag{26}$$

and its 4-momentum is

$$P_+^\mu = m\gamma_v(c,v,0,0). \tag{27}$$

(In this section, a subscript appended to γ indicates the specific velocity pertaining to it.) The duration of this motion is

$$t_+ = d/v. \tag{28}$$

After hitting the right-hand plate, M_k reverses its velocity and we have

$$v_- = -v\mathbf{i}. \tag{29}$$

The 4-momentum takes the form

$$P_-^\mu = m\gamma_v(c,-v,0,0). \tag{30}$$

The duration of this motion is the same as (28)

$$t_- = t_+ = d/v. \tag{31}$$

Using these expressions, one finds that in the frame Σ, the mean energy–momentum of M_k is

$$\bar{P}^\mu = m\gamma_v(c,0,0,0), \tag{32}$$

where the bar denotes a mean value. This quantity is calculated for a time interval consisting of a complete cycle, $T = t_+ + t_-$. It is shown below that (32) is a false 4-vector.

Let us now calculate the mean energy and momentum of this particle in the frame Σ'. The transformation of v_+ yields[7]

$$v'_+ = \frac{v+u}{1+vu/c^2}. \tag{33}$$

The 4-momentum is obtained by application of the Lorentz transformation (16) to (27)

$$P'^\mu_+ = m\gamma_u\gamma_v(c+uv/c,u+v,0,0). \tag{34}$$

In Σ', M_k is seen moving toward the right plate with velocity (33), whereas this plate recedes with velocity u (see Fig. 2). The distance between the plates undergoes a Lorentz contraction and we have

$$d' = d/\gamma_u. \tag{35}$$

The time elapsed while the particle moves toward the right plate is obtained from the relation $v'_+ t'_+ = d' + ut'_+$. [An alternative method is to apply the Lorentz transformation (16) to the 4-vector $(t_+,d,0,0)$]. Hence, using (33) and (35), one finds

$$t'_+ = \frac{d}{\gamma_u(v'_+ - u)} = \frac{\gamma_u d(1+uv/c^2)}{v}. \tag{36}$$

Repeating the calculation for the motion toward the left-hand plate, one finds

$$v'_- = \frac{u-v}{1-vu/c^2}, \tag{37}$$

$$P'^\mu_- = m\gamma_u\gamma_v(c-uv/c,u-v,0,0), \tag{38}$$

and

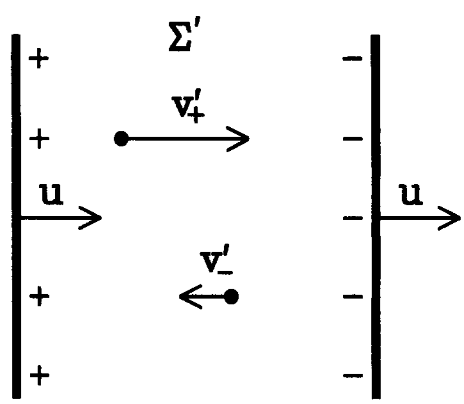

Fig. 2. The capacitor's plates and a particle M_k as seen from Σ'. u denotes the uniform velocity of the plates. v'_+ is the velocity of M_k at an instant when it moves to the right. v'_- is its velocity at another instant when it moves to the left. The direction of v'_- refers to the case where $|u|<|v|$ (see the text).

$$t'_- = \frac{\gamma_u d(1-uv/c^2)}{v}. \tag{39}$$

The quantities (33), (34), and (36)–(39) enable the calculation of the mean energy and the mean momentum of the particle, as seen in Σ'. Using the relativistic expression for the energy of a particle $W = cP^0 = mc^2\gamma$, one finds the mean value of P'^0

$$\bar{P}'^0 = mc\,\frac{\gamma_{v'_+}t'_+ + \gamma_{v'_-}t'_-}{t'_+ + t'_-} = mc\gamma_u\gamma_v(1+u^2v^2/c^4), \tag{40}$$

where

$$\gamma_{v'_\pm} = \gamma_u\gamma_v(1\pm uv/c^2). \tag{41}$$

The calculation can also be done for the x component of the relativistic momentum $P'^1 = m\gamma v_x$. Thus the mean of the x component of the momentum is

$$\bar{P}'^1 = m\,\frac{v'_+\gamma_{v'_+}t'_+ + v'_-\gamma_{v'_-}t'_-}{t'_+ + t'_-} = m\gamma_u\gamma_v u(1+v^2/c^2). \tag{42}$$

At this point it is possible to compare the results of the corpuscular analysis carried out here with those of the continuous one presented in Sec. II A. To this end, let us define the mean energy density of the particle M_k in Σ on the basis of (32)

$$\bar{\epsilon}_G = mc^2\gamma_v/V_0. \tag{43}$$

Similarly, the mean pressure of M_k is

$$\bar{\Pi} = m\gamma_v v^2/V_0 = v^2\bar{\epsilon}_G/c^2. \tag{44}$$

Substituting these expressions into (40) and (42), one finds

$$\bar{P}'^0 = \gamma_u V_0(\bar{\epsilon}_G/c + u^2\bar{\Pi}/c^3), \tag{45}$$

$$\bar{P}'^1 = \gamma_u u V_0(\bar{\epsilon}_G + \bar{\Pi})/c^2. \tag{46}$$

Results (45) and (46) agree completely with the corresponding components of (23) (note that γ of Sec. II A is denoted here as γ_u). This outcome proves that the *apparent* violation of covariance in the transformation of the false

4-momentum of the gas (13) and (23) is just a wrong impression and that these expressions have a solid covariant foundation. Indeed, as shown here, these results stem from the mechanical properties of the particles whose motion creates the gas' mean energy and momentum.

The corroboration of the covariance of the transformation laws of the gas carries implications for those of the electromagnetic part of the system. Indeed, at this point the plates and the gas transformations, (22) and (23), are accepted as correct relativistic expressions. The transformation (25) of the entire capacitor is obviously a correct covariant expression. It follows that the false 4-vector of the electromagnetic part (14) and its transformation (24) should not be altered. Thus the analysis of the capacitor provides an example of the correctness of the general expression of the electromagnetic energy–momentum tensor (7) and of that of macroscopic matter (4) as well.

Summarizing: It is shown that in Σ' the correct expression of the gas' energy and momentum is (23). This expression, together with (44), shows that the relative "correction" to the transformed momentum is $\Pi/\epsilon_G = v^2/c^2$. For example, if one considers the pressure of air at room temperature, the correction is on the order of 10^{-12}. In (23), the correction to the energy density is much smaller because of the additional factor u^2/c^2. It should be noted that a part of the results found above have been published earlier.[8,9]

III. STANDING WAVES

It is proved here that the same results are obtained in a system where standing electromagnetic waves replace the (one-dimensional) gas and act as a source of pressure. To this end, assume that the inner faces of the capacitor's plates are perfect reflectors. The standing wave is polarized in the y direction and its fields are[10]

$$\mathbf{E}_w = A \sin(\omega x/c)\sin \omega t\mathbf{j}, \tag{47}$$

$$\mathbf{B}_w = A \cos(\omega x/c)\cos \omega t\mathbf{k} \tag{48}$$

and the angular frequency ω satisfies

$$\omega d/c = n\pi, \tag{49}$$

where n is a (large) integer. The amplitude A will be determined later. Note that the subscript w distinguishes between the fields of the standing wave and the electrostatic field (1).

The energy-momentum tensor (7) of the fields of the standing wave is

$$T_w^{\mu\nu} = \frac{1}{8\pi}\begin{pmatrix} \lambda & 2E_wB_w & 0 & 0 \\ 2E_wB_w & \lambda & 0 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix}. \tag{50}$$

In order to reduce the size of some arrays, the following notation is introduced:

$$\lambda = E_w^2 + B_w^2, \tag{51}$$

$$\eta = E_w^2 - B_w^2. \tag{52}$$

Note that there is also a time-dependent Poynting vector associated with the constant electric field (1) and the magnetic field (48) of the standing wave: $c\mathbf{E}\times\mathbf{B}_w/4\pi$. However, the mean value of this quantity vanishes. The same is true for the E_xE_{wy} terms of the total energy–momentum tensor of the fields. Therefore, cross terms between the electrostatic field (1) and the standing waves (47) and (48) can be ignored in the energy–momentum tensors discussed here.

Let us find a condition on A which renders the plates free of external force. The x component of the electromagnetic force exerted on the plates is associated with the T^{11} component of the electromagnetic energy–momentum tensors[11] (11) and (50). Using (47)–(49), one finds that on the plates $\mathbf{E}_w = 0$ and

$$\mathbf{B}_w = A \cos \omega t\mathbf{k}. \tag{53}$$

Thus the mean value of B_w^2 at the plates is $\frac{1}{2}A^2$. Examining T^{11} of the energy–momentum tensor (11) of the electrostatic field and (50), one finds that the mean force exerted on the plates vanishes, provided the amplitude A of (47) and (48) satisfies the following relation to the electrostatic field (1):

$$A = \sqrt{2}|\mathbf{E}|. \tag{54}$$

The energy of the standing wave located within the rectangular parallelepiped R_0 is obtained from the integration of T^{00} of (50) on the volume V_0. Using (54), one finds

$$P_w^0 = \frac{1}{8\pi c}\cdot\int (E_w^2 + B_w^2)d^3x = E^2V_0/8\pi c. \tag{55}$$

Notice that the final quantity is time independent. Note also that the component $T^{10} = E_wB_w$ integrates to zero. Hence, the overall momentum of these fields vanishes. Thus the energy and momentum of the standing wave (which are components of a false 4-momentum) at the rectangular parallelepiped R_0 are

$$P_w^\mu = \frac{E^2V_0}{8\pi}(1/c,0,0,0). \tag{56}$$

Adding (14) and (56), one obtains the entire electromagnetic part of the energy–momentum 4-vector

$$P_{\mathrm{EM}}^\mu = \frac{E^2V_0}{4\pi}(1/c,0,0,0). \tag{57}$$

Let us examine the energy and momentum of the standing wave in Σ'. Here one can use the Lorentz transformation (16) to transform the fields (47) and (48). Using these fields, one constructs the energy–momentum tensor (7). An alternative and equivalent method is to transform the energy–momentum tensor (50) which is written in terms of the fields in Σ. The latter method is adopted here because it is shorter. (The reader is invited to try the first method.) For notational simplicity the components T^{01} and T^{10} of (50), which integrate to zero, are omitted. Applying the Lorentz transformation (16) to the tensor (50), one finds

$$T_w'^{\mu\nu} = L_\alpha^\mu L_\beta^\nu T_w^{\alpha\beta} = \frac{1}{8\pi}\begin{pmatrix} \gamma^2(1+u^2/c^2)\lambda & 2\gamma^2u\lambda/c & 0 & 0 \\ 2\gamma^2u\lambda/c & \gamma^2(1+u^2/c^2)\lambda & 0 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix}. \tag{58}$$

Integrating $T_w'^{\mu 0}$ over the volume V_0' of R_0 (where $V_0' = V_0/\gamma$) and using (47), (48), (51), and (54), one obtains the false 4-momentum

$$P_w'^\mu = \frac{\gamma E^2V_0}{8\pi}(1/c + u^2/c^3, 2u/c^2, 0, 0). \tag{59}$$

Adding the false 4-momentum of the static field (24) and that of the standing wave (59), one finds

$$P'^{\mu}_{\rm EM}=\frac{\gamma E^2 V_0}{4\pi}\,(1/c,u/c^2,0,0). \qquad (60)$$

This result is precisely the 4-vector obtained by applying the Lorentz transformation (16) to the 4-vector (57). This outcome shows that the discussion of the present section is consistent with that of Sec. II.

IV. AN APPLICATION

The main result of the present work is that matter in motion which is under pressure carries within itself an additional amount of momentum (and of energy as well), as given in Eqs. (46) and (45). The additional part of the momentum $\gamma_u u \Pi V_0/c^2$, is called hidden momentum.

At this point one can analyze the system of static electric and magnetic fields[2] whose overall electromagnetic field momentum $(1/4\pi c)\int \mathbf{E}\times\mathbf{B}d^3r$ is finite. This problem has already been discussed in the literature[12] and will be mentioned here only briefly. Let us examine an incompressible charged fluid flowing inside a circular tube. The tube is a charged insulator that screens the electric field of the charged fluid. This system is a model for a classical magnetic dipole. The linear momentum associated with the magnetic field of such a tube and the electrostatic field of a system of charges is[13]

$$\mathbf{P}_{\rm EM}=\frac{1}{c^2}\int \Phi \mathbf{J} d^3x, \qquad (61)$$

where Φ is the electrostatic potential and $\mathbf{J}$ is the electric current that produces the magnetic field.

In order to balance the Lorentz force exerted by the electrostatic field of the electric charges, a pressure gradient builds up in the charged fluid. By now the reader will be familiar with this situation: there will be a mechanical hidden momentum in the moving fluid, a hidden momentum that precisely cancels (61). Thus the *total* linear momentum of a stationary system vanishes.

V. CONCLUDING REMARKS

The analysis carried out in this work confirms that the *total* energy–momentum 4-vector of the closed systems examined transforms as a 4-vector. Thus they can be viewed as examples illustrating a theorem of special relativity.[14] The example discussed in Sec. II is of particular interest. Here one has two distinct forces which cancel one another, a mechanical one and an electromagnetic one. There can be no doubt concerning the transformation laws of the mechanical motion of the gas' particles, and yet, the mechanical part of the 4-momentum, *by itself*, does *not* transform as a 4-vector. A glance at (13) and (23) confirms this conclusion. The same is true for the electromagnetic part of the energy–momentum 4-vector, as can be seen in (14) and (24). The analogous result is obtained in Sec. III, where the pressure is itself electromagnetic (due to standing waves).

The fact that the energy and momentum of each constituent of the system do not transform as a 4-vector should not be considered a surprise. Indeed, as pointed out in Sec. I, it can be shown[3] that the energy and momentum obtained from an energy–momentum tensor transform as 4-vector *if the energy–momentum tensor is divergenceless* $T^{\mu\nu}_{,\nu}=0$. This condition holds for any closed system. In the present examples, it holds also for the tensor of the capacitor's plates (5). On the other hand, it does not hold separately for the gas' and fields' tensors (6) and (11). Indeed, the gas' pressure vanishes outside the inner capacitor's volume, proving that this pressure behaves like a step function at the vicinity of the inner part of the plates. It follows that $T^{\mu\nu}_{G,\nu}\neq 0$ on the boundaries of the inner capacitor's volume. An analogous relation holds for $T^{\mu\nu}_F$ of the electrostatic field. These examples show that the condition $T^{\mu\nu}_{,\nu}=0$ may be violated for the energy–momentum tensors of *separate components* of the system, and in the cases discussed here, the corresponding energy and momentum do *not* transform as a 4-vector. It is also shown that besides the hidden momentum, a false 4-momentum carries hidden energy.

The device discussed in Sec. II contains a gas under pressure. In Sec. III electromagnetic radiation takes the gas' role. Analyzing *separately* the energy–momentum tensors of bound fields and of radiation, one arrives at the same conclusions as in Sec. II. This outcome proves that a property of the energy–momentum tensor, namely, $T^{\mu\nu}_{,\nu}\neq 0$, yields false 4-momenta that conceal hidden momentum. It is explained here that hidden momentum emerges as a result of applying an unjustified intuition which is inconsistent with relativity. This hidden momentum is associated with the mechanical subsystem. However, the analogy between the one-dimensional gas of Sec. II B and the radiation of Sec. III indicates that, *in principle*, one might arrive at analogous unjustified conclusions pertaining to energy–momentum of the electromagnetic subsystem.

Romer's Question[2] has stimulated this work. Three Answers[15–17] to this Question have been published before the completion of the revised version. It appears that all authors agree on the key role of the energy–momentum tensor of the entire system, which satisfies $T^{\mu\nu}_{,\nu}=0$. One of the main results obtained here is that there exists a mechanical subsystem (like the gas discussed in Sec. II) whose energy and momentum do not transform as components of a genuine 4-vector, thereby emphasizing the usefulness of the notion of a false 4-vector. This outcome pertains to the standard definitions of the electromagnetic energy and momentum densities of a system whose charge density can be treated as a continuous object. These definitions can be relied upon as valid expressions which hold in all frames. The results obtained here are compatible with the relevant arguments of one Answer[17] but differ in some respects from the other two.[15,16]

An application of the outcome of this work is discussed in Sec. IV. It is explained there how this outcome settles the problem of the linear momentum of a stationary classical system of a magnet and charges.

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[1]S. Coleman and J. H. Van Vleck, "Origin of 'hidden momentum forces' on Magnets," Phys. Rev. **171** (5), 1370–1375 (1968), and references therein.

[2]R. H. Romer, "Electromagnetic field momentum," Am. J. Phys. **63**, (9), 777–779 (1995).

[3]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), pp. 77–80.
[4]See Ref. 3, p. 86.
[5]See Ref. 3, p. 81.
[6]E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985), p. 239; see Ref. 3, p. 62.
[7]See Ref. 3, p. 12.
[8]W. Rindler and J. Denur, "A simple relativistic paradox about electrostatic momentum," Am. J. Phys. **56**, 795 (1988).
[9]E. Comay, "Lorentz transformation of electromagnetic system and the 4/3 problem," Z. Naturforsch. **46a**, 377–383 (1991).
[10]E. W. Jordan and K. G. Balmain, *Electromagnetic Waves and Radiating Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1968), pp. 137, 138.
[11]See Ref. 3, p. 80.
[12]L. Vaidman, "Torque and force on a magnetic dipole," Am. J. Phys. **58**, 978–983 (1990), case III.
[13]See the discussion in 8 of Ref. 2 above.
[14]See Sec. II of Ref. 1 above.
[15]K. McDonald, "Answer to Question #26," Am. J. Phys. **64** (1), 15–16 (1996).
[16]F. Rohrlich, "Answer to Question #26," Am. J. Phys. **64**, (1), 16 (1996).
[17]B. R. Holstein, "Answer to Question #26," Am. J. Phys. **64** (1), 17 (1996).

Uniqueness of the Airy packet in quantum mechanics

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The general form of a nonspreading wave packet in one-dimensional free space is derived from first principles, employing a decomposition of the quantum mechanical evolution operator of the free particle. In agreement with the classical analysis of Berry and Balazs, the corresponding probability density is proportional to the square of an Airy function, and the packet propagates with constant acceleration.

I. INTRODUCTION

Though the Airy wave packet, introduced by Berry and Balazs[1] as a solution of the free particle Schrödinger equation which propagates with uniform acceleration and without spreading, has been the subject of some subsequent discussion,[2–4] a formal derivation of this expression and a quantum mechanical proof of its uniqueness are yet to be supplied. Berry and Balazs[1] drew attention to the fact that the one-dimensional free particle Schrödinger equation (for convenience, we use atomic units such that $m=\hbar=1$)

$$i\frac{\partial}{\partial t}\psi(x,t)=\frac{1}{2}p^2\psi(x,t) \tag{1}$$

admits a nonstationary solution

$$\psi(x,t)=e^{(iB^3t/2)(x-B^3t^2/6)}\mathrm{Ai}\left\{B\left(x-\frac{B^3t^2}{4}\right)\right\}, \tag{2}$$

where B is an arbitrary positive constant, whose probability density accelerates toward positive x without change of shape. They further pointed out that a classical analysis of trajectories corresponding to this Airy packet leads to the conclusion that this is the only solution having this property. In this paper, we present a fully quantum mechanical proof of this assertion, which also constitutes a derivation from first principles of Eq. (2).

II. GENERAL FORM OF A NONSPREADING WAVE PACKET IN FREE SPACE

The time evolution of the wave function $\psi(x,t)$, Eq. (1), is formally given by

$$\psi(x,t)=e^{-itp^2/2}\psi(x,0). \tag{3}$$

If $\psi(x,t)$ is to be a nonspreading wave packet, we should have $|\psi(x,t)|=|\psi(x+\eta(t),0)|$, where $\eta(t)$ is a function of t. The simplest decomposition of the evolution operator which accomplishes this, if it exists, would be of the form

$$e^{-itp^2/2}=e^{i\delta(t)}e^{i\gamma(t)g(x)}e^{i\eta(t)p}, \tag{4}$$

where δ, γ, and η are scalar (c number) functions of t, and g is some function of x. This, however, is impossible, as can be seen on operating with $\exp(-i\eta(t)p)$ from the right on both sides of Eq. (4). We, therefore, look for a general decomposition of the form

$$e^{-itp^2/2}=e^{i\delta(t)}e^{i\gamma(t)g(x)}e^{i\eta(t)p}e^{if(t)H(x,p)}, \tag{5}$$

where H is an operator depending on x and p, but not on t, whose form can be determined as described below. If we now choose $\psi(x,0)$ to be an eigenfunction of H satisfying $H\psi(x,0)=E\psi(x,0)$, Eq. (3) becomes

$$\psi(x,t)=e^{i\delta(t)}e^{i\gamma(t)g(x)}e^{iEf(t)}\psi(x+\eta(t),0); \tag{6}$$

that is, we have a nonspreading wave packet.

