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Numbers, Magnitudes, Ratios, and Proportions in Euclid's *Elements*: How Did He Handle Them?

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For a century or so much Greek mathematics has been interpreted as algebra in geometric and arithmetical disguise. But especially over the last 25 years some historians of mathematics have raised objections to this interpretation, finding it to be misleading and anachronistic, and even wrong. Accepting these criticisms, I consider Euclid's *Elements* in this context: if it cannot be read in this algebraic manner, how did he conceive and handle his various types of quantity? The question is not merely of historical interest, for it raises issues about basic relationships between algebra, arithmetic, and geometry.

In de afgelopen eeuwen is de Griekse wiskunde vooral gezien als algebra in een meetkundige of rekenkundige gedaante. Maar vooral in de laatste 25 jaar hebben enkele historici van de wiskunde betoogd dat deze interpretatie misleidend, anachronistisch of zelfs onjuist is. Op basis van deze kritiek beschouw ik de Elementen van Euclides met de volgende vraag: als dit werk niet op deze algebraische manier gelezen kan worden, hoe vatte Euclides dan de diverse soorten 'grootheid' op en hoe ging hij ermee om? Deze vraag is niet alleen van historisch belang, maar heeft ook te maken met de fundamentele relaties tussen algebra, rekenkunde en meetkunde.

Εδώ και έναν αιώνα περίπου, ένα μεγάλο μέρος των αρχαίων Ελληνικών μαθηματικών έχει ερμηνευτεί σαν άλγεβρα κάτω από γεωμετρική ή αριθμητική αμφίεση. Ομωś τα τελευταία 25 χρόνια, κάποιοι ιστορικοί των μαθηματικών εναντιώθηκαν σε τούτη την ερμηνεία, θεωρώντας τήν αναχρονιστική και αποπροσανατολιστική, έως και λανθασμένη. Συμφωνώντας με αυτή την κριτική, θεωρώ τα Στοιχεία του Ευκλείδη κάτω από το εξής πρίσμα: Αν το έργο αυτό δεν έχει την δυνατότητα να διαβαστεί με αλγεβρικό τρόπο, τότε πώς ο Ευκλείδης συνέλαβε και χειρίστηκε τις διάφορες μορφές ποσότητας στο έργο του; το ερώτημα αυτό έχει κάτι παραπάνω από απλό ιστορικό ενδιαφέρον, ακριβώς επειδή θέτει προβλήματα γύρω από τις βασικές σχέσεις μεταξύ άλγεβρας, αριθμητικής και γεωμετρίας.

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1. INTRODUCTION: EUCLID AS AN ALGEBRAIST

Although Euclid's *Elements* was not the first work of its kind when it was written around 300 B.C., it seems to have soon become a focal point for later writers, with innumerable commentaries, editions, and translations over the centuries. For a long time it was an active source for new results; then it became a classic, upheld by some as a fine tool for mathematical education.

In the course of this change of status, a change in interpretation gradually emerged. The *Elements* was usually taken at face value to be an account of geometry and arithmetic; but when algebra began to develop among some Arabic mathemati-

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cians from the 8th century and in Europe from the late Middle Ages, parts of the work were held to be algebraic in character and were even rewritten in algebraic terms and notations. An interesting example from the early 19th century is provided by François Peyrard, librarian of the *École Polytechnique,* who produced editions and translations of various Greek mathematicians. Those of Euclid (partial in 1804, full in 1814–1818) did not contain much commentary;[1] but in the notes added to his translation [26] of Archimedes in 1807 he rewrote several results as algebra, and was widely praised for so doing by various contemporaries, especially J. L. Lagrange, who advocated the conversion of mathematical theories into algebraic forms as a general principle.

Such praise, and the status of France as by far the leading mathematical country of that time, must have given the algebraic reading of Greek mathematics much greater status. In any event, from the mid 19th century a line of thought evolved in which much Greek mathematics was seen as *algebraic in disguise,* as it were. Expounded by Georg Nesselmann in a volume of 1847 on "the algebra of the Greeks" [24], this thesis took a still more specific form in the 1880s, especially from Paul Tannery in 1882 [34] and Hieronymus Zeuthen four years later [43]. They interpreted much of the *Elements* and some other Greek mathematics as "geometric(al) algebra" (their phrase), that is, common algebra with variables, roughly after the manner of Descartes though without necessarily anticipating his exact concerns, and limited to three geometrical dimensions. The earlier Books of the *Elements* were held to deal with simple algebraic identities and forms, while many of the later constructions correspond to the extraction of roots from equations, normally quadratic but sometimes quartic. For some commentators the pertaining operations constituted a "geometric arithmetic"; for convenience I shall subsume this aspect under the term "geometric algebra."

This interpretation of much Greek mathematics, especially the *Elements,* soon became popular. An important example is the commentaries given by Sir Thomas Heath in his English translation of 1908, based upon a text established by J. L. Heiberg in the 1880s. I shall use it here, in the second edition [13] of 1926, citing an item by its Book and proposition (or definition or postulate) number; thus, "(2.prop.1)" is Book 2, proposition 1, while "(*2.prop.1)" refers to that proposition as an *example* of a point also evident elsewhere in the work.

Around 1930 the interpretation was adopted by figures such as Otto Neugebauer, who then sought the algebraic origins in Babylonian mathematics. It was picked up by his follower Bartel L. van der Waerden, whose articles, and especially the book *Science Awakening* [40] of 1954, first published in Dutch in 1950, have been influential sources. Heath's follower, Ivor Thomas, was a similar influence, especially with his edition [36] (1939–1941) of Greek texts. It came to be normal historiography

[1] Peyrard's full edition (3 vols., 1814–1818, including Books 14 and 15 and the *Data*) was based upon a Vatican manuscript; a quotation from his introduction appears in Section 11. Only one of the *Data* problems was algebraicized in his commentary. A distinguished French example just before Peyrard is J. E. Montucla in 1799 [20, 278]; he multiplied magnitudes, unfortunately, but at least he used "::" in proportions (see Section 8).

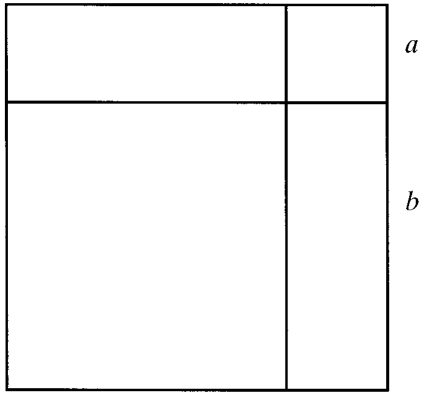


FIGURE 1

of mathematics during this century and remains influential; for example, most general histories of mathematics adopt some form of it in their account of the *Elements,* usually without much discussion.

A typical simple example of the interpretation is (2.prop.4; see Fig. 1): "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments." This is held to be, at base,

$$(a + b)^2 = a^2 + 2ab + b^2, \tag{1.1}$$

when the sides are the lines a and b as shown. A more sophisticated example is (6.prop.28), which uses an important construction called "application of areas," where the task is to set a parallelogram P of given size, and similar to but smaller than another given parallelogram, upon a given line as base. In geometric algebra, this comes down to solving a quadratic equation in x, a length important in the construction of P. We return to this example in Section 11.

In this article, I examine the credentials and verisimilitude of geometric algebra as an interpretation of the *Elements*. Especially during the last quarter century, critical voices against geometric algebra have been heard, answered by defenders. Here, I find in favor of the critics. First, I shall summarize the main objections, and also introduce three categories of algebraicization which refine considerably the clarity of the positions on each side of the dispute. Then I state three features of the *Elements* which seem to deserve a central place in the discussion.

The main purpose of this article is to present, in a manner convenient for fellow nonspecialists in Greek mathematics, a *general* answer to the questions posed by my title: the quantities and propositions with which Euclid works, and the manner in which he handles them. I develop my interpretation in Sections 4–9, where drawing especially on the three features just mentioned, I shall suggest *why,* and not just *that,* the *Elements* cannot be read as geometric algebra. Finally, the general methodological questions are reviewed, using, in Section 11, four examples from the disputed literature. Throughout I use quotation marks not only for quoted texts and names but also to indicate the mention rather than the use of words or symbols.

Some limitations of the paper need explanation. First, I am not a Greek scholar, but I have checked various technical terms with experts. Heath's reputation for producing a (welcomely) literal translation from the Greek appears to be quite justified; indeed, if points of my interpretation falter because of his translation, then we are all in trouble! Similarly, I shall not discuss any philological or etymological questions, or scribal practices in manuscript editions.

Second, while there have been editions of Euclid based on sources other than that established by Heiberg and translated by Heath, my examination of some others suggests that the differences between them are not significant for my interpretation, although, of course, differences of detail arise. Hubertus Busard and Menso Folkerts have recently published an important comparative edition [3] of an influential medieval Latin version where such minor variants are evident, but no more.

Third, I take the *Elements* as it stands (in its 13 Books, the two numbered 14 and 15 being later interpolations). I am not concerned with the pre- or post-history of the work, the consequences of its interpretation to other figures, or possible sources in Babylonian mathematics. David Fowler has recently questioned most standard wisdom on these issues [7]. Further, I shall mention only one other of Euclid's own works.

Fourth, there are well known epistemological difficulties in the *Elements*: definitions which are not really definitions, constructions and theorems muddled together ((6.prop.28) above is an example), and so on. I shall pass over these matters as much as possible.

Fifth, being only concerned with the general principles of Euclid's handling of his quantities, I shall ignore most of the theories developed from them: perfect numbers, the "binomial" and "apotome" magnitudes of Book 10 (surd expressions to the algebraists), and so on. I shall also not discuss his various methods of proof.

Finally, while I use some algebraic letters to represent Euclid's procedures and results, I try to minimize the differences from his arithmetic and geometric concerns.

These limitations do *not* reflect any dismissal on my part of the merit of the issues raised; they require the skills of specialists. Several of them are addressed or noted in the literature cited below.

2. THE CASE AGAINST GEOMETRIC ALGEBRA

Heath himself saw limitations to reading parts of Euclid as geometric algebra: "The algebraical method has been preferred to Euclid's by some English editors; but it should not find favour with those who wish to preserve the essential features of Greek geometry as presented by its greatest exponents, or to appreciate their point of view" [13, 1:373–374]. At the time of his comments in the mid 1920s, some nonadherents to geometric algebra began to express themselves. For example, E. J. Dijksterhuis avoided it in his presentation of the *Elements* in Dutch in 1929–

1930 [4]. During the mid 1930s, in an important article on "Greek Logistic and the Origins of Algebra," Jacob Klein was guarded in its use, precisely because he compared and contrasted Greek mathematics with the genuine symbolic algebra of the European Renaissance [15]. His essay was Englished in 1968, appearing at a time when more modern critics began to appear. Árpád Szabó attacked geometric algebra in 1969 in an appendix to a largely philological examination of the origins of Greek mathematics [32].

The most forthright critic was Sabetai Unguru, who wrote a polemical paper [37] in 1975 against it. He received replies of similar tone from three mathematicians with historical interests: van der Waerden in 1976 [41], Hans Freudenthal in 1977 [8], and André Weil in 1978 [42]. The culprit replied in 1979 in another journal [38], and, with David Rowe, he developed his position in more detail soon afterward [39]. Other recent critics of geometric algebra include Wilbur Knorr in 1975 on the evolution of the *Elements,* with a special emphasis on incommensurability [16]; Knorr again in 1986, where much Greek mathematics is viewed as concerned with geometric problem-solving [18]; and Ian Mueller, who proposed in 1981 a different kind of algebraic interpretation [21]. The year 1987 was a rich one for further discussions in various contexts: by Fowler, who stressed analogies between Euclid's theory of ratios and continued fractions (without imposing the latter *theory* on the Greeks) [5]; by R. H. Schmidt, on the issue of analytic and synthetic proof-methods [31]; and by Roger Herz-Fischler, in a detailed study of the method of "division in extreme and mean ratio" [14].

The quality of the dispute over geometric algebra has been somewhat dimmed by the frequent failure on each side to make clear the epistemological place which algebra is held to occupy (or not) in the *Elements*. A very useful distinction was made in 1837 by William Rowan Hamilton in his paper [12] on irrational numbers. He distinguished these three categories of algebraic mathematics:

"Practical," like an instrument; algebra just produces a convenient set of abbreviations by letters or simple signs for quantities and operations, and rules for the subject at hand (for example, arithmetic);

"Philological," like a formula; algebra furnishes in some essential way the language of the pertinent theory; and

"Theoretical," like a theorem; algebra provides the epistemological basis for the theory.

Weak forms of the claim of geometric algebra (and arithmetic) invoke the practical category; that we can use algebra simply as a means of representing or abbreviating (some of) Euclid's theorems and definitions. Strong claims, which have been the center of concern, state that the *Elements* uses algebra theoretically (although not philologically), that is, assert the advocates and deny the critics, the work is at root algebraic in its conception, even though arithmetical and geometric in its content.

The principal criticisms of the claim may be summarized as follows.

2.1. The algebra is simply the *wrong* style: there are no equations, or letters used in an algebraic way, in the *Elements*. In other words, the absence of the philological category is quite crucial.

2.2. Had Euclid been thinking algebraically, he would have presented constructions corresponding to easy manipulations of (1.1) (for example) which, in fact, are absent from the *Elements*.

2.3. Information is lost when the algebra is introduced, in particular concerning shapes of regions. Thus, using "$p + q$" to denote adding, say, two rectangles does not distinguish between their being adjoined at the top, bottom, left, or right (Fig. 1 gives examples). Again, theorems about parallelograms are often (mis-)written in terms of corresponding theorems about rectangles (*6.prop.28).

2.4. Common algebra is associated with analysis in the sense of reasoning from a given result to principles already accepted. Euclidean geometry goes in the reverse, synthetic, direction. Hence, proofs may well be warped.

2.5. Euclid never *measures* a geometrical magnitude of any kind. For example, there is nothing in the *Elements* directly pertaining to π, in any of its four roles for circles and spheres; apparently such mathematics was not Element-ary for him. Hence the association with algebra leads to an emphasis on arithmetic which cannot be justified.

2.6. If the Greeks really possessed this algebraic root, why did they not bring it to light in the later phases of their civilization? Why, one might add, did that philosophically sophisticated culture not introduce a word to denote, even if informally, this important notion? This point is strengthened by Klein's real history of algebra [15] from later Greek figures (especially Diophantos) through the Arabs to the Renaissance and early modern Europeans, for a gradual process in three stages is revealed: (1) using and maybe abbreviating words to denote operations and known and unknown quantities, (2) replacing these words by symbols or single letters, and (3) allowing letters also to denote variables as well as unknowns and extending notational systems for powers. The interpretation of Euclid as a geometrical algebraist requires him to have passed all three stages; and while he might have skated through them with greater ease than did his successors, the total silence over his achievement among his compatriots is indeed surprising.

3. THREE FEATURES OF THE *ELEMENTS*

To me these criticisms appear quite correct; to borrow a word and notation from Lagrange's algebraic version of the calculus, the geometric algebraist constructed from the *Elements* is not Euclid but Euclid′, a fictional figure derived from Euclid's text by means which he helped to inspire in successors but did *not* possess himself. I return in section 11 to the general methodological point involved here.

The following three features of the geometry in the *Elements* seem to provide clear evidence of central differences between Euclid and Euclid′.

3.1. In his geometry, Euclid *never* multiplies a magnitude by a magnitude; for example, the line of length b is never multiplied by itself to produce the square b^2. This is particularly clear in (1.prop.46), where he constructs a square (already defined in (1.def.22)): not even there, and nor anywhere else in the *Elements,* is it stated, assumed, or proved that the area of the square is the square of a side. Thus, for example, Pythagoras's theorem, which follows at once with its converse (1.props.47–48), states that two squares are equal to a third one, and the well-known proof works by shuffling around regions of various shapes according to principles of congruence and composition; nowhere are area *formulae* involved. To make an analogy (and no more) with arithmetic, this theorem deals with, say, $9 + 16 = 25$, but not with $3^2 + 4^2 = 5^2$. The same point applies to his extension (6.prop.31) of the theorem, concerning similar rectangles laid out on each side of the triangle.

In other words, in Euclid's geometry *the square* ***on*** *the side is not the square* ***of*** *the side*, *or the side squared*; it *is* a planar *region* which has this size. In the same way, he never construes the side of a square as the square root of the square; a square can *have* an associated side (*10.prop.54). Heath always translates Euclid's description of such figures by phrases such as "square on" [τετράγωνον ἀπό] AB (*1.prop.47), or the rectangle "contained" [περιεχόμενον] by its sides AB, CB (*2.prop.4 above); but unfortunately, in his own commentaries, he writes algebraic equivalents of AB^2 and $AB \times CB$. The standard German translation [35] of the *Elements* by Clemens Thaer between 1933 and 1937, is even more disappointing in this respect; for there terms such as "AB^2" are used *in* Euclid's text. Of course, the Greeks *knew* the area property as well as anyone (especially in the taxation office); so its avoidance by Euclid is deliberate. Indeed, he avoided multiplying *all* kinds of geometric magnitude, for reasons explained in Section 5.

To take another important example, the method of "application of areas" to a line means that a rectangle (or maybe parallelogram) is constructed with that line as a side and equal to some given region, perhaps under further conditions also ((*6.prop.28) in Section 1). But no theorems about areas *as areas*—that is, quantities with arithmetically expressible properties—are involved.

This eschewal of geometrical products seems to refute the reading as geometric algebra on its own, in all three of Hamilton's categories. For example, (1.1) is an algebraic travesty of Euclid's geometry, since none of its four magnitudes involved appears in his theory.

By contrast, in Euclid's arithmetic numbers *can* be multiplied: for example, he even calculates triples of numbers, involving squares, which satisfy Pythagoras's theorem (10.prop.28, Lemmas 1–2). Thus, the algebraic version of his arithmetic is free of this objection (though not of others, as we see in Section 10).

3.2. While he speaks of the equality of numbers and of magnitudes, Euclid never says that ratios are "equal" to each other, only that they are "in the same ratio [ἐν τῷ αὐτῷ λόγῳ]," or that one ratio "is as" the other in a proportion proposition. Thus, the use of "=" for ratios, normal in geometric algebra, is again a travesty (see Section 7).

3.3. Euclid's method of compounding ratios is not at all the same as multiplication, although the two theories exhibit structural similarity (Section 7). Similarly, his theory of a lesser integer l being "parts" of a greater one g is *not* one of rational numbers l/g smaller than unity.

Following the critics of geometric algebra, I claim that Euclid studies numbers, geometrical magnitudes, and ratios in the *Elements*. I shall use "quantity" as a neutral umbrella term to cover these three "types," and when clear I shall just say "magnitudes" for the second one. They are different from each other as objects, and they have a different ensemble of means of combination and of comparison, although some structural similarity between them is evident. In the next six sections, I shall amplify this claim, treating the quantities in the above order. Like most other commentators on Euclid, I find mysterious the order in which the Books lie: numbers are formally treated in Books 7–9, but they appear in many of the other ten, which are basically concerned with magnitudes; ratios first appear in Book 5 and regularly thereafter, usually within proportions (which are described in section 7).

4. EUCLID'S NUMBERS

Euclid presents a theory of positive integers [ἀριθμός] starting with 2; 1 is a unit, and there is no zero (7.defs.1–2). The basic combinations and comparisons are given in Book 7.

Numbers can be combined under the operations of addition, subtraction "of the lesser from the greater" to ensure that the resultant number is positive, and multiplication (7.defs.5, 7, and 15 and props.18 and 3–4). Numbers may be "equal to" [ἴσος], "greater" [μείζων], or "less" [ἐλάσσων] than each other; the basic properties of equality are covered by three basic "common notions" in Book 1.

Euclid does not divide integers to produce rational numbers (contrary, once again, to the geometric algebraists' reading discussed in Section 2.5). Instead, a lesser number l is "part" [μέρος] or "parts" [μέρη] of a greater one g according as l "measures" g or not (7.defs.3–5)—to us, whether l is a factor of g or not. For example (my own), 3 is part of 9, 3 is parts of 7, and 6 is the "same parts" (*7.prop.6) of 14; but the rational numbers $\frac{3}{9}$, $\frac{3}{7}$, and $\frac{6}{14}$ are not constructed thereby. Similarly, Euclid's rule for finding the least common multiple of numbers, ratios of them, and properties of part (7.props.36–39) cannot be so read.

The only exception is the use of part numbers, mostly "a half" (*13.prop.13), and occasionally cases such as "a third part" (12.prop.10) on the volume of a cylinder) and a "fifth" (lemma to (13.prop.18), on measuring angles). They correspond to 1 as part of 2 (or 3, or 5, or ...) rather than to unit fractions. Presumably their Egyptian parentage gave them a special status [17]; an explanation from Euclid would have been helpful! In addition, he uses numbers *in connection with* geometrical magnitudes in a way described in the next section.

In (7.defs.16–19) and thereafter in the Books on arithmetic, Euclid presents numbers on occasion as lines, their squares as geometrical squares, and cubes as solids; and he says that a square number has a "side" [πλευρά] (8.prop.11). This overloads the link, however, for he is treating arithmetic within geometry, which is the realm of his theory of magnitudes—and a quite different theory, as we shall now see.

5. EUCLID'S GEOMETRICAL MAGNITUDES: GLOSSARY

Euclid's theory of numbers deals with *discrete* quantities; the *continuous* ones are handled in geometry. Here he works with ten kinds (another general term of mine) of magnitude [μέγεθος] in five pairs, again with no zero;[2] many of them are defined, or so he thought, in (1.defs.1–23). The pairs divide on the property of being straight or curved for each magnitude. In fact, the latter are confined to circles and spheres, although the ideas seem applicable somewhat more broadly, at least to simple concave or convex curves and surfaces.

The pairs of magnitudes are listed below, with an example of each given in brackets. As usual, I distinguish a magnitude from any possibly arithmetical value that it may take: a line has length, a region or surface has area, a solid has volume, and an angle has measure.

Kind	Straight	Curved
lines	straight (line)	planar curved (arc of circle)
regions	planar rectilinear (rectangle)	planar curvilinear (segment of a circle)
surfaces	spatial rectilinear (pyramid)	spatial curvilinear (sphere)
solids	rectilinear (cube)	curvilinear (hemisphere)
angles	planar	solid planar[3]

In these examples I distinguish between, say, a circle as a closed curved line and as a convex region, or a cube as a closed surface and as a convex solid; Euclid does the same, without confusion. He does not consider curves in space or surfaces or solids set upon some kind of rectilinear base; probably he had no examples in mind of the latter.

The distinction between straight and curved magnitudes is a natural one; for example, it plays a key role in the theory of limits. Its place in Euclid (and in the work of many other Greek mathematicians) is evident in Book 10, where proofs of theorems relating rectilinear and curvilinear solids work by double contradiction ($A = B$ because both $A < B$ and $A > B$ lead to absurdities) rather than a direct process of limit-taking, which would cross this conceptual boundary.

[2] In Book 3, Euclid deals with tangents to circles and touching circles; while he skirmishes with curvilinear angles (*3.prop.16), he does not seem to use a zero as such.

[3] "Solid angles" (Euclid's name) appear only in (11.def.11 and props.20–26), and in the proof that there are only five regular solids (addendum to (13.prop.18)); but they are handled in the same way as other magnitudes. Similarly, the theory of solids is far less developed than that of lines and regions—to Plato's distraction in *Republic* 528a–d [5, 117–121].

In addition, Euclid has on occasion "multitude" [πλῆθος] of a quantity, an informal idea referring to an unspecified number of them (*7.prop.14) for numbers, ((*10.def.3) for magnitudes). He also speaks of "equal in multitude," which we would treat as a 1–1 correspondence between members of two such collections (*5.prop.1) for magnitudes).

This leaves the status of points. The famous (1.def.1), defining a point "as that which has no part," is well recognised as a failed definition. I take it to be a principle of *atomicity,* asserting the existence of a point as a "primitive" part of a magnitude, such as something inside a sphere, for example, or the place where two lines intersect or where a line has an extremity. It is striking to note that Euclid actually wrote of a "sign" [σημεῖον], and, in particular cases, said "this A," not "the point A." Perhaps inspired by Plato, the change from "point" [στιγμή] was maintained by most of Euclid's contemporaries and successors, although Aristotle used both words [23, 376–379]. But the old tradition was to prevail again from the 5th century, when in the spirit of Roman empiricism translators and commentators Martianus Capella and Boethius rendered "σημεῖον" as "punctum" (see Heath in [13, 1:155–156]). However, the reformers had a good case; "sign" makes clear that the objects denoted do not admit the means of combination or comparison to which magnitudes are subject, and which we now examine.

6. EUCLID'S GEOMETRICAL MAGNITUDES: HANDLING WITHIN EACH KIND

The key feature of Euclid's treatment of magnitudes is that, with an important exception to be noted in Section 9, the means of combinations and comparisons are treated *between magnitudes of the same kind*—in my view, *consciously and intentionally.* Since his magnitudes vary considerably in character (from a line to a solid), he does not always use the same word for the same sort of combination, and in some cases no word at all; but they may be fairly characterised as follows.

Magnitudes of the same kind may be added or subtracted, the latter combination restricted to positive resultant magnitudes.[4] They may also be multiplied by numbers; when the number is an integer, a "multiple" of the magnitude is produced, but there may also arise, for example, the one-and-a-half of a square in (13.prop.13) of Section 4. The converse never happens; that is, numbers are never multiplied by magnitudes. This difference shows on its own than they are different types of quantity.

Their comparisons are "equal to," "greater than," or "less than" (1.common notions.4–5, post.5). Euclid uses the same words as for numbers, but only structurally

[4] Euclid would not have regarded *every* possible combination as meaningful: for example, an arc to a circle (considered as a curvilinear line), or at the end of an infinitely long line (as permitted to exist (1.def.23) by the parallel postulate (1.post.5)).

similar notions are involved, not identical ones (see Section 10 below). He does not equate a rectilinear region with a curvilinear one; indeed, in connection with the famous problem of squaring the circle, his commentator, Proclus (5th century A.D.), explicitly mentioned this possibility as a worthwhile *research* topic [28, 334–335]. Euclid may well have deemed this problem, and similar ones such as squaring lunes, as not Element-ary—and with good justice!

Once it is recognized that Euclid handles magnitudes of the same kind, the reasons for his avoidance of their multiplication become clear. First, *the kind would be changed;* for example, the product of a straight line and a straight line is a rectilinear region. Second, many of the multiplications cannot be defined anyway (angle with angle, line with angle, solid with line, and so on); so for uniformity he omits *all* of them.

This point must not be confused with the fact that many theorems *involve* magnitudes of different kinds at the same time; for example, there are many in Book 10 in which lines and rectilinear regions appear together. However, no means of combination or comparison *between* kinds occurs.

7. EUCLID'S RATIOS AND PROPORTIONS

The ratio [λόγος] is Euclid's third type of quantity. It is specified only between two numbers or between two magnitudes of the same kind. Following the 17th-century English astronomer, Vincent Wing, I shall write "$a:b$" for the ratio of two such quantities a and b.

Euclid's presentation is not clear. The ratio of numbers is not formally defined at all, but it creeps in first in (7.def.20) in a proportion about the sameness of two pairs of ratios, and is used in a theorem about subtraction (7.prop.11). Further, he does not stress that the ratio, say, $3:7$, is *different in type* from 3 being parts of 7, which is a property *within* arithmetic (recall Section 4 above).

Something like a definition of the ratio of magnitudes appears in (5.def.3): "A ratio is a sort of relation in respect of size between two magnitudes of the same kind." The word "size" [πηλικότης] denotes the measure of the ratio. This definition is best interpreted as creative, although, of course, he had no *theory* of the sort that has been developed in this century—that is, a definition is creative relative to a theory if there exist theorems in the theory which cannot be proved without using the definition.

Comparisons between a pair of ratios are expressed in the companion theory, that of proportions [ἀναλογία] between a pair of ratios. In addition to bringing some clarity to the role of these mysterious ratios, the main advance of proportion theory over magnitude theory is that ratios of magnitudes *of two different kinds* can be compared (for example, between lines and regions in (6.prop.1)), and/or compared with ratios of numbers, by comparison of their sizes. For magnitudes a ratio may be "in the same ratio as," "greater than" (5.defs.5–7), or "less than" (not formally defined) the other one. The comparison in question is established in Eudoxus's manner by examining (in)equalities between multiples of the magnitudes involved.[5] The two theories of sameness are quite different, showing again that numbers and magnitudes are distinct types of quantity.

Euclid's total avoidance of the word "equals" [ἴσος] for ratios shows in the clearest manner possible that he saw them as a third type of quantity distinct from numbers and magnitudes (or perhaps as a relation between its components). Hence, I follow the practice initiated by Wing's wise contemporary, William Oughtred, in the 17th century in symbolizing sameness by "::". Writing a, b, c, d for magnitudes, we never find equations such as, say,

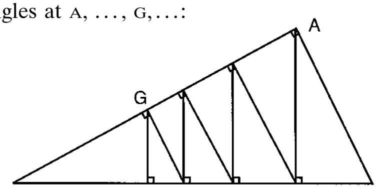
$$a : 2b = c : d, \text{ but the proportion } a : 2b :: c : d. \tag{7.1}$$

The habit of the geometric algebraist of writing "=" between ratios is inadmissible, as also is the habit of drawing consequences involving the multiplication of magnitudes, such as

$$a \times d = 2b \times c. \tag{7.2}$$

One of the most extensive uses of magnitudes and ratios in the *Elements* is the theory of (in)commensurable and (ir)rational[6] magnitudes (mostly straight lines and rectilinear regions) in the virtuoso Book 10. These properties are based upon "anthyphairesis," the successive subtraction of lesser from greater magnitudes of the same kind so as (not) to produce a ratio the same as that of two integers, respectively (10.props.2 and 5–6). Note that it is not a theory of irrational *numbers*. Indeed, he proves that two magnitudes are incommensurable if and only if the ratio "have not to one another the ratio which a number has to a number" (10.props.7–8). Therefore, an error in type of quantity is committed when it is compared with Dedekind's theory of irrationals, unless one invokes from Dedekind's side the so-called "Cantor–Dedekind axiom" to set each irrational number in isomorphism with a geometric length. Indeed, Euclid may have taken up some existing theory of such numbers and translated it into

[5] Euclid never presents proportions between trios or greater multitudes of numbers or of magnitudes of the same kind. Presumably, the reasons were that the case of nonsameness cannot be defined (sameness presents no difficulties), and that anthyphairesis becomes very hard to execute. Similarly, his theorems in Book 8 on numbers in "continued proportion" (geometric progressions to us, and also to Weil in his rather marginal objection [42] to Unguru) work by taking only neighboring *pairs* of numbers together. A geometric version of such sequences would be easy to produce, as pairs of bases of these triangles, which have successive right angles at A, . . . , G, . . . :



[6] With little enthusiasm, I follow the tradition of translating "(ἀ)λόγος" as "(ir)rational"; something like "(in)expressible" is much better (as Heath himself noted in [13, 3:525]). Euclid defines irrationality from incommensurability by assigning a line as a basis relative to which such ratios can be defined his way (the rather messy (10.defs.3–4), which cover also (in)commensurability in square).

geometrical terms, since only in geometry can the *continuum* of magnitudes upon which it depended be guaranteed.

However, numbers do have a role in anthyphairesis, as counters of repetitions. For example, the magnitude 7 subtracts from 46 six times, with remainder 4; 4 from 7 once with remainder 3; and so on, generating the sequence 6, 1, ...[7] These ellipsis dots can encompass an unending sequence of remainders, sometimes periodic, with remarkable properties [6]. They are still not well studied, although the (arithmetical) theory of continued fractions bears some structural similarity to them [5, Chaps. 5 and 9].

8. EUCLID'S RATIOS: A MUSICAL BACKGROUND?

Why did Euclid always avoid speaking of equal ratios? It is plausible that, like his contemporaries and predecessors, he understood ratio theory as *generalized music,* at least culturally [2] and perhaps even philosophically. Whatever the Pythagoreans did or did not do (and doubtless more sources of information were available then than now), properties of strings and comparisons between tones and lengths were a major part of the mathematics of their time and had remained so until Euclid's.[8] In such contexts, the notion of equality is not as natural as with numbers or magnitudes. *We* usually say that the intervals F#–A and B–D are the same interval (here a minor third) rather than that they are equal—for the first is placed a major fifth above the second one. Further, the trio of Euclidean quantities, number–magnitude–ratio, surely bears an intentional cultural correlation with three subjects of the Aristotelian quadrivium, arithmetica–geometria–harmonia.

This association with music is evident also in the method of combination to which ratios are subject: they are to be "compounded" [$\sigma\upsilon\gamma\kappa\varepsilon\acute{\iota}\mu\varepsilon\nu o\nu$], tardily in (6.prop.23). In the simplest case, the ratio $a:b$ may be put together with $b:c$ to produce the ratio $a:c$ (5.def.9), the "duplicate ratio" of the original pair. This procedure is clearly similar in structure to taking the musical intervals (say) D#–F# and F#–C to produce D#–C. Further, as with music, the process may be repeated, to produce the "triplicate ratio" $a:d$ after three stages (5.def.10), and the "ex equali" proportion which arises from the ratio $a:n$ achieved after N stages (5.def.17).

Clearly, one cannot follow (some) geometric algebraists and identify this procedure with multiplication, although there is obvious structural similarity between the two means of combination (in the practical category but not the theoretical, Hamilton might say). But note that even if the quantities are all numbers, a compounded ratio results, not an arithmetical product. For example, with the symbol "·" denoting compounding, note the difference between the propositions

[7] Maybe anthyphairesis inspired Euclid's use of the curious term "parts" for integers not in factorial form (recall Section 5): 7 is parts of 46 because 46 : 7 leads to repetition numbers 4, 1, . . . , whereas 7 is part of 42 because 42 : 7 yields 6 "straight off." The term may also have come from talk of the time on unit fractions and calculations.

[8] In an undeservedly neglected essay [19], Ernest McClain argues that ratios were a fundamental theory in many cultures, and he suggests novel interpretations for several aspects of Plato. Before Euclid, other applications had been made of ratios, for example, to astronomical motions and calendars.

$$(3:7)\cdot(7:11) :: 3:11 \quad \text{and} \quad (3/7)(7/11) = 3/11; \tag{8.1}$$

not only do they differ by type of quantity, but the latter proposition cannot even be stated in the *Elements* (recall Section 4).

Further, unlike a general definition of multiplication, Euclid does not offer the *general* definition of compounding *any* two ratios of magnitudes, for it would read

$$(a:b)\cdot(c:d) :=? ([a \times c]:[b \times d]), \tag{8.2}$$

which involves the forbidden product of magnitudes. The product could be replaced by regions in certain cases such as the rectangles contained by a and c and by b and d if all four magnitudes were lines—but *not* in full generality, for the reasons explained in Section 6. Nevertheless, it appears in (6.prop.23), the interesting but isolated proposition which substitutes for the multiplication (1.1) of sides: it states that the ratio of two equiangular parallelograms is the compound of the ratios of the respective pairs of sides.[9] In the proof (see Fig. 2), the given parallelograms ADCB and CGFE are aligned to admit the intermediate parallelogram DHGC in the (gnomon-like) figure, so that

$$\square \text{ADCB} : \square \text{CGFE} :: (\square \text{ADCB} : \square \text{DHGC})\cdot(\square \text{DHGC} : \square \text{CGFE}) \tag{8.3}$$

$$:: (\text{BC}:\text{CG})\cdot(\text{CD}:\text{CE}). \tag{8.4}$$

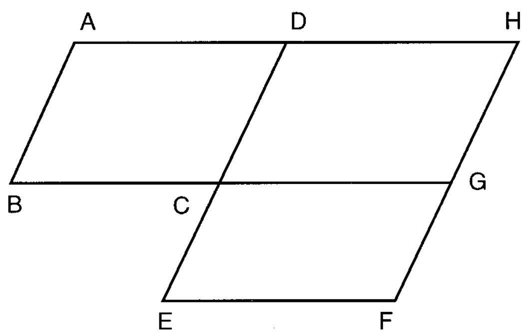


FIGURE 2

[9] This is the proposition in which Euclid first uses the word "compounded"; (6.def.5) is an interpolation, not always included in editions. While the theory as such seems clear, his presentation is not good, either at (6.prop.23) or elsewhere. Compounding occurs in (6.props.19–20) on taking ratios of areas of triangles and polygons; in (8.prop.5) on plane numbers and (8.props.11–12) on determining mean proportions between numbers; in (12.props.12–18 *passim*) on various propositions mentioned in Section 9 concerning the volumes of cones, cylinders, and spheres; and in (13.prop.11) in making a square 25 times bigger than another one. Contrary to Heath's editorial remarks, I do not see compounding necessarily present in (5.props.20–23), where various theorems on ratios of the same value are proved. Saito provides a good discussion of compounding (unfortunately notated "+") in Euclid and also in Apollonius; however, he considers possible theorems involving (8.2) [30, 32, 38, 59], and uses "=" in proportions. Compounding ratios is not to be confused with their composition (5.def.14): given $a:b$, form $(a + b):b$.

9. MIXED KINDS: THE ALTERNATION THEOREM

One feature in Euclid's theory of ratios needs to be resolved: the alternation theorem (5.prop.16), according to which, in the notation of (7.1) for magnitudes,

$$\text{if } a:b::c:d, \quad \text{then} \quad a:c::b:d. \tag{9.1}$$

There is a surprise: Euclid asks only that the pairs a and b, and c and d, be each of the same kind, not that the whole quartet be so. Thus, as it stands, the theorem asserts in its second proposition the sameness of two "mixed ratios" (my name), where a pair of ratios of the same different magnitudes are related. This feature has often been regarded as a slip (by Heath, for example, in his commentary on the theorem [13, 2:165]). In the influential earlier edition by Robert Simson, the seemingly missing clause was even interpolated into the text.

But in my view Euclid has *not* made a mistake here (although, regrettably, he supplies no explanation either), for in one context he uses mixed ratios. This is in Book 12, where he presents his version of Eudoxus's theory of exhaustion of curvilinear by rectilinear regions. The first occasion arises when he proves that "circles are to one another as the squares on the diameters" (12.prop.2). There, he constructs regular polygons inside each circle and then uses (9.1) to show the sameness of the ratios of each polygon to its parent circle.[10] The same procedure occurs in the companion theorem about spheres (12.prop.18) and in two theorems about cones and cylinders on circular bases (12.props.11–12). The proofs work by double contradiction, the type mentioned in Section 5.

Euclid's extension of proportion theory to mixed ratios is minimal, in that the magnitudes involved are at least of the same dimension each time. But it goes beyond the practice elsewhere in the *Elements*—and as an extension it escaped Heath's attention.

10. EUCLID'S ONTOLOGY

Euclid treats his quantities in a very direct way: he *has* numbers, angles, lines, regions, ratios, and so on. No constructed object depends upon the means of construction in a reductionist sense, such as a square *as* the product of its sides. This directness is also evident in his *Data,* where he presents a string of geometric exercises; for each one is expressed in the form that "if so-and-so properties of [say] a triangle are given, then such-and-such properties are given." The phrase "are (is) given" [δεδομένον] is the *Leitmotiv* of the entire work.[11]

[10] There may be a slip in the diagram attached to this theorem. Two "areas" S and T play roles in the proof, and mainly in the comparison of equality to circles. Therefore, even granted that both rectilinear and curvilinear regions are involved in the proof, surely the latter shapes are more appropriate than the rectangles that have been drawn. This feature seems to be of long standing; for example, it is evident in the recent medieval Latin version of the *Elements* edited by Busard and Folkerts [3, 1:294]. Little seems to be known about the history of the diagrams in the *Elements*.

[11] Taisbak argues that "are given" refers to static states of affairs rather than dynamic ones of potential movement [33]. This may also be a motive, though I prefer to give priority to a simpler explanation—that Euclid means just what he says. Maybe "be the case that" would be a better translation. See also Schmidt [31] on this matter.

In his work, Euclid uses three different types of quantity, often together but with their theories distinguished. Much similarity of structure obtains between them; but there are also essential differences, so that identity of content *cannot* be asserted. This point is an important one in the philosophy of real mathematics when analogy plays a prominent role [10]. Unfortunately, the geometric algebraists miss it, for they assert that the *same* algebra obtains for each type of quantity. As Nesselmann put it in 1847 [24, 154]: "Allow us, however, to consider and to treat as arithmetical under our kind of thinking this theory which Euclid proposed geometrically.... thus it would be hard to lay down a strong boundary between form and content as a ground for division" [24, 154]. While a perfectly legitimate reading of Euclid′, it is a distortion of Euclid, and can lead to confusions—for instance, in identifying his theory of irrational magnitudes with modern ones of irrational numbers, or regarding numbers as special kinds of magnitude (see, for example, Heath in [13, 2:124, 113], respectively).

Each type of quantity in Euclid has to be considered *separately* for its possible algebra. Converse to the phrase "geometric algebra," I shall use "algebraic" as an *adjective* in each case, to show that it qualifies the succeeding quantity noun.

First, there might be a case for algebraic arithmetic, though at most only for the practical category and *only* for integers and unit fractions. However, the fact that (unfortunately) Euclid gave numbers a geometric interpretation reduces the quality of the case even there—which seems to be the only place in the *Elements* where the phrase "geometric arithmetic" could be justified. Among his successors, a philological category for algebraic arithmetic begins to emerge only with Diophantos in the 3rd century A.D., and then not fully.

Second, any theory of algebraic magnitudes must be radically different from geometric algebra. The avoidance of multiplication of magnitudes rules it out on its own, and the other criticisms rehearsed in Section 2 are very powerful. Mueller has given a cogent account of several of them [21]; however, his alternative, a sort of logico-functorial algebra, seems also to take us far from Euclid's numbers and magnitudes, although in a different direction from that of the geometric algebraists—to Euclid″, say.[12] Abstract algebra, such as groups and fields, would set us off down yet another irrelevant track.

Finally, a case for algebraic ratios might be argued, as long as compounding is not identified with arithmetical multiplication. However, the background in music provides a far more faithful orientation.

If one wishes to pursue algebraisation at all, symbols corresponding to the shapes might be introduced, such as "□ a" for the square on side a and "O a" for a circle with that diameter; Heath used such notations occasionally (*12.prop.7). However, even modern computers do not always readily supply the required symbols. Another

[12] Mueller [21] has, for example, SIMSOLID (k, l) for solid (that is, cubed) numbers k and l, followed by a definition in terms of mathematical logic. But a significant anachronism is evident here; for his use of quantification deploys set-theoretic interpretations of the quantities involved, whereas Euclid's own treatment of collections follows the very different part–whole tradition of handling collections.

good strategy is that of Dijksterhuis [4], who used functorial letters such as "T(a)" for the square on side a, and "O(a,b)" for the rectangle with sides a and b.

Should algebra-like symbols be desired, different symbols *must* be used for the "same" means of combination and comparison when applied to different quantities. A possible choice could be these:

Quantity	Addition	Subtraction	Division[13]	Multiplication/ compounding	Equality/ sameness	Greater than	Less than
Numbers	+	−	:	×	=	>	<
Magnitudes	‡	∼	:		≈	⩾	⩽
Ratios				·	::	**:**:	:**:**

Operations on magnitudes apply within each kind, of course, with the extension to cover exhaustion theory as described in Section 9. The same symbol ":" is proposed for the ratios of numbers and of magnitudes precisely because a ratio is created on each occasion. In the case of magnitudes and ratios, the means of combination are to be understood in a rather abstract, operational sense. While still far away from Euclid, a closer sense to him than geometric algebra is furnished by the connotation of the interior and exterior products which Hermann Grassmann proposed in his *Ausdehnungslehre* of 1844. It is a great irony that his work was gaining publicity in the 1880s and 1890s, *exactly* when Heiberg and Tannery were trying to convert Euclid into Descartes!

11. THE TENACITY OF ALGEBRAIC THINKING

As the case of Euclid ≠ Euclid′ exemplifies, algebra is not among the ancient roots of mathematics, and to impose it on Euclid distorts and even falsifies his intentions. Moreover, the point is not restricted to the Greeks, for claims of similarity between their mathematics and that of the Babylonians and the Chinese have been grounded in the alleged common factor of geometric algebra. The principle seems to be a (mistaken) application to history of Euclid's own (1.ax.1), that "things which are equal to the same thing are equal to each other"!

Although not an ancient source for mathematics, algebra and algebraic thinking and styles have long assumed a central role in much mathematics and mathematical education. (Recall how often children identify mathematics with equations and formulae, often in disgust.) The case, and even the *name* of "geometric algebra," shows the tenacity of the algebraic style [11]; for it characterizes an algebra which is geometric, not even a geometry (or arithmetic) which might be algebraic. Note, by contrast, that nobody in the late 19th century suggested that Aristotle's syllogistic logic was a logical algebra, although it influenced substantially (though partly by reaction) the development of algebraic logic at that time [9].

The underlying philosophy behind the interpretation of the *Elements* as geometric algebra, common to mathematicians' and sometimes even historians' understanding

[13] I use the word "division" in its modern sense, not in a way alluding to anthyphairesis.

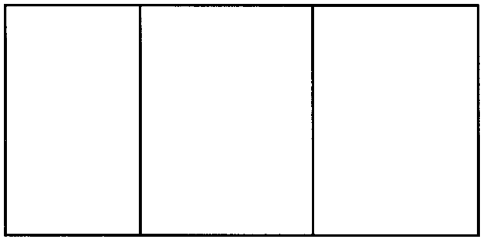


FIGURE 3

of the history of mathematics in general, is *historical confirmation theory:* Suppose that a theory T_1, created during epoch E_1, is followed by a later theory T_2 in epoch E_2; then at epoch E_3 the (non-)historical reading is proposed that T_1 was conceived by its creators as an *intended* draft of T_2, and thereby confirms it. Roll on history, deterministically [27].

While this reading might be correct or at least arguable in some cases, it should be treated with caution, for it may well propose as actual developments what were only potential. For example, Euclid's *Elements* undoubtedly influenced the real development of algebra among the Arabs and then (in a rather different form) during the Western Renaissance; but it does not at all follow that Euclid himself had been trying to be a geometric algebraist [38]. The history of the various readings that Euclid has received over the centuries regarding his types of quantity and their handling before the era of geometric algebra would be a valuable contribution to historiography. He may have been better understood by (some!) later Greeks and in the Middle Ages than later, when algebraicization began to develop in the stages outlined in Section 2. The final quartet of specific examples illustrates the point.

First, Heath's edition provides a frequent oscillation between Euclid in English and Euclid′ in algebra, sometimes to such an extent that he actually attributes a different procedure to Euclid. The example (6.prop.28) of Section 1 on constructing a certain parallelogram is a very good case. "To exhibit the exact correspondence between geometrical and the ordinary algebraical method of solving the equation," Heath sets up a Euclid′-style quadratic equation (in which Euclid's parallelograms are—consciously—replaced by rectangles . . .), and after some working he calculates an expression for a certain line GO in the original proof. However, two pages earlier Euclid mentions GO only once [13, 2:262–265].

Second, in his reply to Unguru, van der Waerden clearly explains that he uses the word "algebra" "for expressions like $(a + b)^2$, and how to solve linear and quadratic equations" [40, 200]. He then takes (2.prop.1), in Heath's translation (see Fig. 3): "If there be two straight lines, and one of them be cut into any number of segments, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments." Geometrically, this theorem just means that every rectangle can be cut into rectangles by lines parallel to one of the sides. This is evident: everyone sees it by just looking at the diagram. Within the framework of geometry there is no need for such a theorem: *Euclid* never makes use of it in his first four books.

But Euclid's text and van der Waerden's version say different things: Euclid builds up the full rectangle R from its components to *form* rectangles and state a property of them [30, 54–60]; van der Waerden breaks it up into those components from R. Of course, the theorem is very simple; hence its location at the head of Book 2. But the construction embodied here (recall from Section 2.4 that Euclid often conflated them with the theorems themselves) underlies or at least relates to many theorems from Book 2 onwards, starting with (2.prop.2), long before Book 5 is reached.

Third, Freudenthal's reply [8] to Unguru defending geometric algebra is unintentionally amusing; for he explicitly praises Dijksterhuis's edition as a source to read Euclid, and then makes all the mistakes that Dijksterhuis avoids—multiplication of magnitudes, equality between ratios, compounding as multiplication, and so on!

The final pair of examples are small but interesting ones, since they come respectively from a critic and an agnostic of geometric algebra. Both involve the unfortunate practise of setting "=" between ratios in a proportion. Szabó makes it in the very book which later on contains his attack on geometric algebra mentioned in Section 2 [32, 131]. So does Benno Artmann, in a nice recent short survey of the *Elements,* in which he regards the issue of geometric algebra as "moot" [1, 7, 45–47].

Many other examples of this and other dubious procedures can be found, even in several other items in the bibliography here. Algebraic thinking holds on tight; as Peyrard declared already in 1814, "Mr. Lagrange . . . often repeated to me that Geometry was a dead language" [26, ix].

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