# Hermann Minkowski and the Postulate of Relativity

by Leo Corry



## 1. Introduction

In the history of two of Einstein's chief scientific contributions—both the special and the general theories of relativity—two of the leading Göttingen mathematicians of the beginning of this century each plays a significant role: Hermann Minkowski (1864-1909) and David Hilbert (1862-1943). Einstein published his famous paper on the electrodynamics of moving bodies in 1905. Beginning in 1907, Hermann Minkowski erected the new theory of relativity on what was to become its standard mathematical formulation and devised the language in which it was further investigated. In particular, Einstein's adoption of Minkowski's formulation—which he had initially rejected—proved essential to his own attempts to generalize his theory to cover gravitation and arbitrarily accelerated systems of reference. After a long and winding process that spanned at least three years of intense work and included the publication of several versions he later deemed incorrect, Einstein presented to the Prussian Academy of Sciences in Berlin his generally-covariant field equations of gravitation on November 25, 1915. But, as it happened, David Hilbert—

the undisputed, foremost living mathematician in the world and the lifelong close friend and collaborator of the by then deceased Minkowski—had already presented to the Göttingen Academy his own equivalent version of the same equations a few days earlier, on November 20.

Although Minkowski and Hilbert accomplished their most important achievements in pure mathematical fields, their respective contributions to relativity should in no sense be seen as merely occasional excursions into the field of theoretical physics. Minkowski and Hilbert were motivated by much more than a desire to apply their exceptional mathematical abilities opportunistically, jumping onto the bandwagon of ongoing physical research by solving mathematical problems that physicists were unable to. On the contrary, Minkowski's and Hilbert's contributions to relativity are best understood as an organic part of their overall scientific careers. It is remarkable that although the close professional and personal relationship between Minkowski and Hilbert is well-known, no direct connection between their respective contributions in these fields has hitherto been established or even suggested.[1] The history of the special and the general theories of relativity has more often than not been told from the perspective of Einstein's work and achievements, and the roots and true motivations of Minkowski's and Hilbert's contributions to this field have therefore remained only partially and incorrectly analyzed.

A detailed examination of their careers makes it evident that a keen interest in physics was hardly ever distant from either Hilbert's or Minkowski's main focus of activity in pure mathematics. Minkowski's interest in physics dates back at least to his Bonn years (1885-1894), during which he was in close contact with Heinrich Hertz.[2] In 1888 he published an article on hydrodynamics in the proceedings of the Berlin Academy (Minkowski 1888). From his correspondence with Hilbert,[3] we know that during his Z?rich years Minkowski kept alive his interest in mathematical physics, and in particular in thermodynamics. In 1902 he moved to Göttingen, following Hilbert's strong pressure

1. For example, no such connection is considered in oft-cited accounts of Minkowski's work: Galison 1979, Pyenson 1977, Miller 1981. 238-244. Neither is it discussed in accounts of Hilbert's contribution to general relativity: Earman and Glymour 1978; Mehra 1974; Pais 1982, 257-261; Vizgin 1994, 54-69.

2. See R?denberg and Zassenhaus (eds.) 1973, 39-42, and Hilbert 1909, 355.

3. See R?denberg and Zassenhaus (eds.) 1973, 110-114.

on Felix Klein (1849-1925) to create a professorship for his friend. It is well known that during his last years there, Minkowski's efforts were intensively dedicated to electrodynamics. But this was not the only field of physics to which his attention was attracted. Minkowski was commissioned to write an article on capillarity for the physics volume of the *Encyclopädie der mathematischen Wissenschaften,* edited by Arnold Sommerfeld (Minkowski 1906). At several meetings of the Göttingen Mathematical Society he lectured on this, as well as on other physical issues such as Euler's equations of hydrodynamics and Nernst's work on thermodynamics, and the evolution of the theory of radiation through the works of Loretnz, Rayleigh, W. Wien, and Planck.[4] He also taught advanced seminars on physical topics and more basic courses on continuum mechanics, and gave exercises in mechanics and heat radiation.[5]

Perhaps under Minkowski's influence, Hilbert also developed a strong attraction to physics from very early on. He followed the latest developments in physics closely and taught courses and seminars on almost every current physical topic. Hilbert elaborated the principles of his axiomatic method between 1894 and 1899 as part of his current interest in problems related to the foundations of geometry; but to a considerable extent, he also reflected throughout these years on the relevance of the method for improving the current state of physical theories. Influenced by his reading of Hertz's *Principles of Mechanics*, Hilbert believed that physicists often tended to solve disagreements between existing theories and newly found facts of experience by adding new hypotheses, often without thoroughly examining whether such hypotheses accorded with the logical structure of the existing theories they were meant to improve. In many cases, he thought, this had led to problematic situations in science which could be corrected with the help of an axiomatic analysis of the kind he had masterfully performed for geometry. In a course in Göttingen in 1905 on the logical principles of mathematics, Hilbert gave a quite detailed overview of how such an axiomatic analysis would proceed in the case of several specific theories, including mechanics, thermodynamics, the kinetic theory of gases, electrodynamics, probabilities, insurance mathematics and psychophysics.[6]

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4. As registered in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (*JDMV*). See Vol. 12 (1903), 445 & 447; Vol. 15 (1906), 407; Vol. 16 (1907), 78.

5. See the announcement of his courses in *JDMV* Vol. 13 (1904), 492; Vol. 16 (1907), 171; Vol. 17 (1908), 116.

After his arrival in Göttingen, Minkowski was deeply involved in all the scientific activities of Hilbert, including his current interests in the axiomatization of physics. An ongoing interchange of ideas between them—if not an actual collaboration—should be taken into account by the historian as important in the evolution of the conceptions of each throughout their careers, and especially during their shared years at Göttingen. More specifically for our present concerns, in 1905 Hilbert and Minkowski, together with other Göttingen professors, organized an advanced seminar that studied recent progress in the theories of the electron. In 1907, the two conducted a joint seminar on the equations of electrodynamics. Beginning at least in 1907 and until his death in 1909, Minkowski devoted all his efforts to the study of the equations of electrodynamics and the postulate of relativity. Hilbert certainly followed Minkowski's work in this field with great interest. In his study of electrodynamics, Minkowski also addressed the question of gravitation, and formulated some preliminary ideas concerning the possibility of a Lorentz covariant theory to account for it. An account of Hilbert's way to his later work on general relativity obviously calls for an exploration of Minkowski's work between 1907 and 1909.

To what extent Hilbert actively contributed to the consolidation of Minkowski's specific ideas on electrodynamics and the principle of relativity, and to what extent Minkowski influenced Hilbert's conceptions on physical issues, is hard to determine with exactitude, but it seems safe to assume that the two shared many basic conceptions concerning these matters. In the present article I claim that a proper understanding of Minkowski's incursion into the field of electrodynamics and relativity must take into account its proximity to the kind of ideas put forward in Hilbert's program for the axiomatization of physics. Minkowski undertook a systematic examination—like those found in Hilbert's 1905 lectures on the axiomatic method—of the logical, mathematical and physical implications of adding to the existing building of physics the newly formulated hypothesis known as the principle of relativity. Given Minkowski's own physical background and mathematical interests—which differed in several respects from Hilbert's—and given the latest developments in physics, Minkowski's analysis implied a direction of

6. I have presented a detailed account of the origins and early stages of Hilbert's program for the axiomatization of physics from 1894 to 1905, including his 1905 course, in Corry 1997. The present article should ideally be read as a follow-up of that earlier one. For an overview of Hilbert's work on physical issues until 1915, see Corry 1997a. For Hilbert's work on General Relativity see Corry 1997b.

thinking that Hilbert did not cover—and perhaps could not even imagine possible—when teaching his 1905 course. Yet the very motivations for such an analysis, as well as many of the questions addressed in it, are clearly reminiscent of Hilbert's own and are clarified by association with the latter. In fact, one of the important insights afforded by this reading of Minkowski is that it also stresses the kind of questions that Minkowski was *not* pursuing in his work. In particular, the point of view adopted here suggests a reinterpretation of the rôle of Minkowski's work in the debates of the first decade of the century—much discussed in the secondary literature—concerning the ultimate nature of natural phenomena.

Between 1907 and 1910, the years in which Minkowski was vigorously pursuing his ideas on electrodynamics and relativity, Hilbert himself did not publish or lecture on physical issues at all. In fact, after his 1905 course on axiomatization and the joint seminar of 1907 with Minkowski, Hilbert taught a course on physics again only in 1910, when he lectured on mechanics.[7] In a section of that course dealing with the "new mechanics", we find the first evidence of Hilbert's referring to Minkowski's contributions. Hilbert stated that those contributions were the starting point for his own presentation in that course. Therefore, in the absence of direct evidence to the contrary, my default assumption will be that Minkowski's published work can be taken as a faithful expression of Hilbert's own views between 1907 and 1913, and as the starting point for his own study of physical topics after Minkowski's death. This will be important in tracing Hilbert's own way to general relativity, a task which I intend to undertake in the near future.

## 2. The Principle of Relativity

Minkowski's ideas concerning the postulate of relativity have been preserved in the manuscript and published versions of three public talks, as well as through an article posthumously published by Max Born, based on Minkowski's papers and on conversations between the two. Minkowski presented his ideas on electrodynamics and relativity in public for the first time in November 5, 1907, in a talk delivered to the Göttingen Mathematical Society under the name of "The Principle of Relativity."[8] One month before the talk, Minkowski had written to Einstein asking for a reprint of his 1905 paper, in order to study it in his joint seminar with Hilbert.[9]

7. See the appendix to Corry 1997a.

Recent developments in the electromagnetic theory of light—Minkowski said in opening his talk—had given rise to a completely new conception of space and time as a four-dimensional, non-Euclidean manifold. Whereas physicists are still struggling with the new concepts of the theory, painfully trying to find their way through the "primeval forest of obscurities," Minkowski added, mathematicians have long possessed the concepts with which to clarify this new picture. The physicists Minkowski associated with this trend were Lorentz, FitzGerald, Poincaré, Planck and Einstein. Minkowski thought that a proper elaboration of their ideas could become one of the most significant triumphs in applying mathematics to understanding the world, provided—he immediately qualified his assertion—"they actually describe the observable phenomena."[10] This latter, brief remark characterizes very aptly the nature of Minkowski's incursion into the study of the electrodynamics of moving bodies: along the lines of Hilbert's analysis of the axioms of other physical disciplines, he would attempt to understand and simplify the conceptual structures of electrodynamics and mechanics—presently in a state of great confusion, in view of the latest discoveries of physics. He would sort out the fundamental statements that lie at the basis of those structures, statements that must be confronted by experiment in order to validate or refute the relevant theories. The principle of relativity would then be shown to play a fundamental role in these new developments of physics.

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8. Published as Minkowski 1915. For details on the printed and manuscript versions of Minkowski's work see Galison 1979, 119-121. The original typescript of this lecture was edited for publication by Arnold Sommerfeld. After comparing the published version with the original typescript, Lewis Pyenson (1977, 82) has remarked that Sommerfeld introduced a few changes, among them a significant one concerning the role of Einstein: "Sommerfeld was unable to resist rewriting Minkowski's judgment of Einstein's formulation of the principle of relativity. He introduced a clause inappropriately praising Einstein for having used the Michelson experiment to demonstrate that the concept of absolute space did not express a property of phenomena. Sommerfeld also suppressed Minkowski's conclusion, where Einstein was portrayed as the clarifier, but by no means as the principal expositor, of the principle of relativity." The added clause is quoted in Galison 1979, 93.

9. See Stachel et al (eds.) 1989, 267.

10. Minkowski 1915, 927: "... falls sie tatsächlich die Erscheinungen richtig wiedergeben,..."

Minkowski's 1907 talk comprised four sections: electricity, matter, dynamics, and gravitation. In the first two sections, Minkowski elaborated on ideas that had been discussed recently in his joint seminar with Hilbert. In this seminar, geometrical space had been described as filled with three different kinds of continua—ether, electricity and matter—whose properties must be characterized by suitable differential equations.[11] This particular conception was not in itself new. In fact, the study of the connection between ether and matter in motion had sharply intensified after the 1898 meeting of the Society of German Scientists and Physicians in D?sseldorf, in which the subject was discussed. On that occasion Lorentz described the problem in the following terms:

> Ether, ponderable matter, and, we may add, electricity are the building stones from which we compose the material world, and if we could know whether matter, when it moves, carries the ether with it or not, then the way would be opened before us by which we could further penetrate into the nature of these building stones and their mutual relations.[12]

This development comprised two different perspectives: the microscopic theories of the electron and the macroscopic theories of optical and electromagnetic phenomena in moving media.[13] Whereas Einstein's 1905 relativistic kinematics concerned only Lorentz's microscopic electron theory, it was Minkowski who first addressed the formulation of a relativistic electrodynamics of moving media. Thus his three public lectures on the postulate of relativity deal mainly with the macroscopic perspective, while the posthumous article published by Born focused on the microscopic one.

The Göttingen seminars of 1905 and 1907 on electrodynamics were ostensibly conducted in the context of the intense activity developed by German-speaking physicists on these questions, following the D?sseldorf meeting.[14] But the differential equations briefly discussed in the 1907 seminar were reformulated in Minkowski's talk in an innovative way: Minkowski introduced here four-vectors of four and of six components (he

11. Notes of this seminar were taken by Hermann Mierendorff, and they are kept in Hilbert's *Nachlass* (Cod Ms 570/5). For more details on the seminar see Pyenson 1977, 83.

12. Lorentz 1898, 101. Translation quoted from Hirosige 1976, 35.

13. On the development of these two perspectives before Einstein and Minkowski, see Stachel et al (eds.) 1989, 503-504.

14. On these activities, see Hirosige 1976, 36-41.

called the latter "*Traktoren*") as the mathematical tool needed to bring to light all the symmetries underlying the physical questions involved.[15] Minkowski explicitly claimed that it is precisely the four-vector formulation that makes evident the kind of invariance characteristic of Lorentz's equations for the electron (which also describe the behavior of an electromagnetic field in pure ether and of an electric field filling infinite space, i.e., the first and second of the three continua mentioned above). Moreover—he remarked—the way in which this purely formal property of the equations is presented here had not been noticed before even by authors like Poincaré.[16] Although Minkowski in his talk did not actually write the Maxwell equations in Lorentz-covariant form, he showed sketchily that if these equations are formulated in terms of in four-vectors, their invariance under any transformation of the four coordinates that leaves invariant the expression $x_1^2 + x_2^2 + x_3^2 + x_4^2$ (where $x_4^2 = it$) follows as a simple mathematical result. In Minkowski's formulation, the Lorentz transformations represent rotations in this four-dimensional space.

In the second part of the talk Minkowski investigated how the equations are affected when matter is added to pure ether. Minkowski, very much like Hilbert in his 1905 lectures, stressed that his theory does not assume any particular world view: it treats first electrodynamics and only later mechanics, and its starting point is the assumption that the correct equations of physics are still not entirely known to us.[17] Perhaps one day a reduction of the theory of matter to the theory of electricity might be possible, he said, but at this stage only this much is clear: that experimental results, and especially the Michel-

15. For the place of Minkowski's contribution in the development of the theory of tensors, see Reich 1994, 168-184.

16. Minkowski 1915, 929: "Ich will hier, was ?brigens bei keinem gennaten Autoren, selbst nicht bei Poincaré, geschehen ist, jene Symmetrie von vornherein zur Darstellung bringen, wodurch in der Tat die Form der Gleichungen, wie ich meine, äu?erst durchsichtig wird."

17. Passages like this one have often been quoted in the secondary literature as evidence to support the claim that Minkowski completely adhered to the electromagnetic world-view. For instance, Galison 1979, 92, translates the original "Hier stellen wir uns auf den Standpunkt ..." as "Here we find ourselves at a standpoint where the true physical laws are not yet completely known to us." I read this differently as "We place ourselves here at the standpoint ...", namely, this is not a standpoint imposed upon us, as it were, but rather one we deliberately adopt in order to avoid debate on this particular question.

son experiment, have shown that the concept of absolute rest corresponds to no property of the observed phenomena. This situation, Minkowski asserted, can easily be clarified if one assumes that the equations of electrodynamics still remain invariant under the Lorentz group *after* matter has been added to the field. It is precisely here that the principle of relativity enters the picture of physics. Minkowski declared the principle of relativity—i.e., invariance under Lorentz transformations—to be a truly new kind of physical law: it is not one that has been deduced from observation, but rather *it is a demand we impose on yet to be found equations describing observable phenomena*.[18] Applying this postulate to the situation in question, Minkowski showed that assuming Lorentz covariance and using the four-vector formulation, the assumption of the "Galilean principle of inertia" implies that the speed of light must be infinite. Similarly, he derived the electrodynamic equations of a moving medium, making evident and stressing their invariance under the Lorentz group. From the kind of reasoning applied here—he remarked in the third part of the lecture—it follows, that if the principle of relativity is actually valid also for matter in motion, then the basic laws of classical mechanics should be understood as only approximately true. But then, the above-mentioned impossibility of detecting the motion of the earth relative to the ether confirms that this is indeed the case.[19] Moreover, he quoted some elaborate technical reasoning taken from Max Planck's recent contribution to a relativistic thermodynamics (Planck 1907), as additional arguments for rejecting the classical principle of inertia.[20]

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18. Minkowski 1915, 931: "Hier tritt nun das Relativitätsprinzip als ein wirkliches neues physikalisches Gesetz ein, indem es ?ber noch gesuchte Gleichungen f?r Erscheinungen eine Forderung stellt."

19. Minkowski 1915, 934-935: "Nachdem, was ich bereits ?ber das Verhältnis der Relativitätsprinzipes zum Trägheitsgesetze gesagt habe, ist von vornherein klar, da? die bisherigen Grundgesetze der Mechanik nur als eine Approximation an die Wirklichkeit gelten können, falls auch in der Mechanik das Relativitätsprinzip gelten soll. Das m??te aber wieder der Fall sein, weil sonst doch wieder eine Möglichkeit vorliegen w?rde, eine Bewegung der Erde relativ zum Äther konstatieren."

20. Minkowski 1915, 935-937. For an account of Planck's paper, see Miller 1981, 360-362.

The fourth part of Minkowski's lecture contained a brief discussion on gravitation. Naturally, if the principle of relativity is to be truly universal it should account also for phenomena of this kind. Minkowski mentioned a similar discussion that had appeared in Poincaré's relativity article, and endorsed Poincaré's conclusion that gravitation must propagate with the velocity of light. The purely mathematical task thus remained open, to formulate a law that complies with the relativity principle, and at the same time has the Newtonian law as its limiting case. Poincaré had indeed introduced one such law, Minkowski said, but his law is only one among many possible ones, and Poincaré's results had hitherto been far from conclusive. Minkowski left a more elaborate treatment of this point, for a later occasion.

## 3. The Basic Equations of Electromagnetic Processes in Moving Bodies

Minkowski's second talk on electrodynamics and relativity was given less than two months after his first one, this time at the meeting of the Göttingen Scientific Society on December 21, 1907. Two weeks earlier, on December 10, Felix Klein had lectured at the regular meeting of the Göttingen Mathematical Society on the possible applications of the quaternion calculus to the theory of the electron and its relation to the principle of relativity. Following Klein's lecture, Minkowski showed how the equations of electrodynamics can be simplified if the electric and magentic magnitudes are jointly represented by means of bi-quaternions, namely, quaternions with complex components, and how this is related to the study of the significance of the principle of relativity.[21]

The printed version of Minkowksi's second talk, entitled "The Basic Equations of Electromagnetic Processes in Moving Bodies", was Minkowski's only publication on this topic to appear before his death in 1909. It contained his most detailed mathematical treatment of the differential equations of electrodynamics. It also presented an illuminating conceptual analysis—once again, very similar in spirit to Hilbert's axiomatic treatment of physical theories—of the main ideas involved in the current developments of the theories of the electron and of the role played by the principle of relativity in those theories. Minkowski

21. See the announcement in the *JDMV* Vol. 17 (1908), pp. 5-6.

distinguished three possible different meanings of this principle. First, the plain mathematical fact that the Maxwell equations, as formulated in Lorentz's theory of electrodynamics, are invariant under the Lorentz transformations. Minkowski called this fact the "*theorem* of relativity." It seems natural to expect, Minkowski said, that the domain of validity of the theorem—a mathematically evident theorem, in his opinion—might be extended to cover *all* laws governing ponderable bodies, including laws that are still unknown. This is the "*postulate* of relativity;" it expresses a confidence (*Zuversicht*) rather than an objective assessment concerning the actual state of affairs. One can embrace this confidence, Minkowski explicitly stressed, *without thereby committing oneself to any particular view of the ultimate relationship between electricity and matter.*[22] He compared this postulate to the postulation of the validity of the principle of conservation of energy, which we assume even for forms of energy that are not yet known. Lastly, if we can assert that the expected Lorentz covariance actually holds as a relation between directly observable magnitudes relating to a moving body, then this particular relation is called the "*principle* of relativity."

It is interesting to compare this analysis of Minkowski's with a similar one advanced by Hilbert in a course on the kinetic theory of gases in the winter semester of 1912-13. Facing the enormous mathematical difficulties raised by the theory, Hilbert stressed the need to approach it using a "physical" perspective, namely, through a thorough application of the axiomatic method, in order to point out clearly those parts of the theory in which physics enters into mathematical deduction. In this way, Hilbert proposed to separate three different components of a physical theory: first, what is arbitrarily adopted as definition or assumed as the basis of all experience; second, what we *a priori* expect should follow from these assumptions, but which the current state of mathematics does not yet allow us to conclude with certainty; and third, what is truly proven from a mathematical point of view.[23] Thus, both Minkowski and Hilbert stressed the need to separate in a clear way the various assumptions, physical and mathematical, involved in a theory, and this is precisely what Minkowski attempted to do here.

22. Minkowski 1908, 353: "Nun kann man, ohne noch zu bestimmten Hypothesen ?ber den Zusammenhang von Elektrizität und Materie sich zu bekennen, erwarten, jenes mathematisch evidente Theorem werde seine Konsequnezen so weit erstrecken, da? dadurch auch die noch nicht erkannten Gesetze in bezug auf ponderable körper irgendwie eine Kovarianz bei den Lorentz-Transformationen ?bernehmen werden."

Minkowski's analysis allows one to understand more clearly his own views about the specific contributions of various physicists to the theory of the electrodynamics of moving bodies. Lorentz, Minkowski thought, had discovered the theorem and had also set up the postulate in the form of the contraction hypothesis. Einstein's contribution was, according to Minkowski, that of having very clearly claimed that the postulate is not an artificial hypothesis, but rather, that the observable phenomena force it upon us as part of a new conception of time. Minkowski did not mention Poincaré this time, but given the latter's conception of the general validity of the theorem, Minkowski would presumably have classified Poincaré's contribution as having also formulated the "relativity postulate." In fact, it was Poincaré who had first suggested extending the domain of validity of Lorentz invariance to all laws of physics. In 1904, for instance, Poincaré formulated the principle as an empirical truth, still to be confirmed or refuted by experiment, according to which the laws of physics should be the same for any two observers moving with rectilinear, uniform motion relative to each other.[24]

Minkowski claimed that the principle had never been formulated for the electrodynamics of moving bodies in the way in which he was doing it. The aim of his presentation was to deduce an exact formulation of the equations of moving bodies from the principle of relativity. This deduction, he claimed, should make it clear that none of the formulations hitherto given to the equations is fully compatible with the principle. In other words, Minkowski believed that his axiomatic analysis of the principle of relativity and of the electrodynamic theories of moving bodies was the best approach for unequivocally obtaining the correct equations.

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23. Hilbert 1912-13, 1: "Dabei werden wir aber streng axiomatisch die Stellen, in denen die Physik in die mathematische Deduction eingreift, deutlich hervorheben, und das voneinander trennen, was erstens als logisch willk?rliche Definition oder Annahme der Erfahrung entnomen wird, zweitens das, was a priori sich aus diesen Annahmen folgern liesse, aber wegen mathematischer Schwierigkeiten zur Zeit noch nicht sicher gefolgert werden kann, und drittens, das, was bewiesene mathematische Folgerung ist."

24. See Poincaré 1905, 176-77; 1906, 495. And again in 1908 Poincaré wrote: "It is impossible to escape the impression that the Principle of Relativity is a general law of nature... It is well in any case to see what are the consequences to which this point of view would lead, and then submit these consequences to the test of experiment." See Poincaré 1908, 221.

As in his former lecture, in the first part of the present one Minkowski discussed the equations of a pure electromagnetic field, i.e., ether without matter. As part of his discussion of the invariance of these equations under the Lorentz group of transformations, Minkowski introduced the new mathematical tool that allowed him to put forward his own version of the principle of relativity and that turned into the standard language of all future developments of electrodynamics and relativity: the four-vectors of four and six components (which he called "space-time vectors of type I and II", respectively).[25] He stressed throughout the invariance of the metric element $x_1^2 + x_2^2 + x_3^2 + x_4^2$, where $x_4 = it$, and showed that the invariance of the equations expressed in the four-vector language follows from simple symmetry considerations.

Minkowski dedicated a separate section of the first part to a discussion of the changes in the concept of time brought about by introducing the Lorentz transformations into kinematics, and in particular the impossibility of speaking about the simultaneity of two events. His explanation was based on formal properties of the transformations, discussed in an earlier section: if in a certain reference system we are given a space point $A$ at time $t_0 = 0$, and a second point $P$ at a different time $t$, and if $t - t_0 < PA$ ($PA$ being the time required for light to traverse the distance between the two points), then it is always possible to choose a Lorentz transformation that takes both $t_0$ and $t$, to the value $t' = 0$. The same is true if we are given two space points at $t_0 = 0$ and a third one at $t$, or three non-collinear points in space at $t_0 = 0$ and a fourth one at $t$ (again, $t - t_0$ satisfying a similar condition like that just mentioned). However, if we are given four non-coplanar events it is no longer possible to find the desired transformation. Minkowski's arguments can essentially be construed, in hindsight, as locating points outside or inside the light-cone—as the case may be—of a given space-time event. Such a formulation would seem indeed to suggest itself in this context, yet Minkowski did not introduce those concepts and arguments at this stage. In the closing sections of this lecture he came much closer to those ideas, and they finally appeared fully-fledged only in his best-known article on this issue, the famous lecture on "Space and Time." One should also notice that, since Minkowski's discussion was intended as an axiomatic investigation of the specific implications of the various assumptions involved, it is significant that he raised the question of simultaneity at the end

25. Vectors of type II correspond to modern second-rank, antisymmetric tensors.

of the section dealing with the equations in empty ether. We learn from this that, for Minkowski, *the relativity of simultaneity is a consequence of the Lorentz theorem for the equations in empty ether, and it is therefore independent of whatever conception of the nature of matter one may adopt*. Minkowski concluded this section with a remark that clarifies his understanding of the basic motivations behind Einstein's contribution to the latest developments in electrodynamics: mathematicians—Minkowski said—accustomed as they are to discuss many-dimensional manifolds and non-Euclidean geometries, will have no serious difficulties in adapting their concept of time to the new one, implied by the application of the Lorentz transformation; on the other hand, the task of making physical sense out of the essence of these transformations had been addressed by Einstein in the introduction to his 1905 relativity article.[26]

As in his earlier 1907 talk, the second part of the December 1907 paper considered how the equations change when matter is added to the ether. For the case of a body at rest in the ether, Minkowski simply relied on Lorentz's version of Maxwell's equations, and analyzed the symmetry properties of the latter. He formulated the equations as follows:

$$\text{(I)} \qquad \operatorname{curl}\boldsymbol{m} - \frac{\partial \boldsymbol{e}}{\partial t} = \boldsymbol{s}$$

$$\text{(II)} \qquad \operatorname{div}\boldsymbol{e} = \rho$$

$$\text{(III)} \qquad \operatorname{curl}\boldsymbol{E} + \frac{\partial \boldsymbol{M}}{\partial t} = 0$$

$$\text{(IV)} \qquad \operatorname{div}\boldsymbol{M} = 0$$

Here ***M*** and ***e*** are called the magnetic and electric intensities (*Erregung*) respectively, ***E*** and ***m*** are called the electric and magnetic forces, ρ is the electric density, ***s*** is the electric current vector (*elektrischer Strom*).[27] The properties of matter, in the case of isotropic bodies, are characterized by the following equations:

$$\text{(V)} \qquad \boldsymbol{e} = \varepsilon\boldsymbol{E}, \quad \boldsymbol{M} = \mu\boldsymbol{m}, \quad \boldsymbol{s} = \sigma\boldsymbol{E},$$

26. Minkowski 1908, 362: "Dem Bed?rfnisse, sich das Wesen dieser Transformationen physikalisch näher zu bringen, kommt der in der Einleitung zitierte Aufsatz von A. Einstein entgegen."

27. In Einstein & Laub 1908, 1908a, in which Minkowski's article is referred to, the vector ***M*** in these equations is called the dielectric displacement, whereas ***e*** is the magnetic induction.

where $\varepsilon$ is the dielectric constant, $\mu$ is the magnetic permeability, and $\sigma$ is the conductivity of matter.

From the basic properties of the equations for bodies at rest, Minkowski deduced the fundamental equations for the case of a body in motion. This deduction is where the detailed axiomatic derivation is realized: Minkowski assumed the validity of the previously discussed equations for matter at rest to which he added three axioms. He then sought to derive the equations for matter in motion exclusively from the axioms together with the equations for rest. Minkowski's axioms are:

1. Whenever the velocity $\boldsymbol{v}$ of a particle of matter equals 0 at $x, y, z, it$ in some reference system, then equations (I)-(V) also represent, in that system, the relations among all the magnitudes: $\rho$, the vectors $\boldsymbol{s}$, $\boldsymbol{m}$, $\boldsymbol{e}$, $\boldsymbol{M}$, $\boldsymbol{E}$, and their derivatives with respect to $x, y, z, it$.

2. Matter always moves with a velocity which is less than the velocity of light in empty space (i.e., $|\boldsymbol{v}| = v < 1$).

3. If a Lorentz transformation acting on the variables $x, y, z, it$, transforms both $\boldsymbol{m},-i\boldsymbol{e}$ and $\boldsymbol{M},-i\boldsymbol{E}$ as space-time vectors of type II, and $\boldsymbol{s},i\rho$ as a space-time vector of type I, then it transforms the original equations exactly into the same equations written for the transformed magnitudes.[28]

Minkowski called this last axiom, which expresses in a precise way the requirement of Lorentz covariance for the basic equations of the electrodynamics of moving matter, the principle of relativity. That is to say: it is only after establishing the equations for empty ether, and proving the Lorentz theorem of invariance, that we can speak of the principle of relativity, which, together with two additional assumptions, yields the electrodynamics of moving matter. It is relevant to see in some detail how Minkowski in this section applies the axioms to derive the equations.

Since $v < 1$ (axiom 2), Minkowski could apply a result obtained in the first part, according to which the vector $\boldsymbol{v}$ can be put in a one-to-one relation with the quadruple

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28. See Minkowski 1908, 369. For the sake of simplicity, my formulation here is slightly different but essentially equivalent to the original one.

$$w_1 = \frac{v_x}{\sqrt{1-v^2}}, \ \ w_2 = \frac{v_y}{\sqrt{1-v^2}}, \ \ w_3 = \frac{v_z}{\sqrt{1-v^2}}, \ \ w_4 = \frac{i}{\sqrt{1-v^2}}$$

which satisfies the following relation:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = -1.$$

Again from the results of the first part, it follows that this quadruple transforms as a space-time vector of type I. Minkowski called it the "velocity space-time-vector." Now, if $\boldsymbol{v} = 0$, by axiom 1, equations (I)-(V) are also valid for this case. If $\boldsymbol{v} \neq 0$, since $|\boldsymbol{v}| < 1$, again the results of earlier sections allow the introduction of a transformation for which

$$w_1' = 0,\ w_2' = 0,\ w_3' = 0,\ w_4' = i\ .$$

In this case, we also obtain a transformed velocity $\boldsymbol{v}' = 0$. According to axiom 3, whatever the basic equations may be that hold for this case must remain invariant when written for the transformed variables $x',y',z',t'$ and the transformed magnitudes $\boldsymbol{M}'$, $\boldsymbol{e}'$, $\boldsymbol{E}'$, $\boldsymbol{m}'$, $\rho'$, $\boldsymbol{s}'$, and the derivatives of the latter with respect to $x',y',z',t'$. But, since $\boldsymbol{v}' = 0$, the transformed equations are (by axiom 1) just (I′)-(IV′), obtained from (I)-(IV) by tagging all variables. The same is true for equation (V) (although there is no need to apply axiom 3), but with $\varepsilon$, $\mu$, and $\sigma$ remaining unchanged. Finally, one applies the inverse of the Lorentz transformation originally applied and, by axiom 3, it follows that the form of the basic equations for the original variables is in fact precisely (I)-(IV). Minkowski thus concluded that the basic equations of electrodynamics for moving bodies are the same as the equations for stationary bodies, and the effects of the velocity of matter are manifest only through those conditions in which its characteristic constants $\varepsilon$, $\mu$, and $\sigma$ appear. Also, Minkowski concluded, the transformed equation (V′) can be transformed back into the original equation (V).

The particular kinds of argument advanced in this section by Minkowski seem somewhat out of place amidst the elaborate mathematical and physical arguments displayed throughout the talk. They find a natural place, however, in the light of the kind of axiomatic conceptual clarification promoted by Hilbert in his lectures on physics, for, like Hilbert, Minkowski was stressing here precisely that kind of task. Minkowski, in addition, went on to check to what extent different existing versions of the equations satisfied the principle as stated in his axioms. Since nothing similar to his analysis had been attempted before, Minkowski's implicit assumption was that only equations which comply with his

own version of the principle can be accepted as correct. Without going any further into details here, I will only point out that Minkowski found the macroscopic equations for moving media which were formulated in Lorentz's *Encyclopädie* article (Lorentz 1904) to be in certain cases incompatible with his principle.[29] Minkowski also discussed the equations formulated in 1902 by Emil Cohn, pointing out that they agree with his own up to terms of first order in the velocity.[30] After having formulated the equations and discussed their invariance properties, Minkowski dealt in detail, in three additional sections, with the properties of electromagnetic processes in the presence of matter.

Minkowski's paper has an appendix discussing the relations between mechanics and the postulate (not the principle!) of relativity. It is in this appendix that the similarity of Minkowski's and Hilbert's treatment of physical theories is most clearly manifest: the appendix is an exploration of the consequences of adding the postulate of relativity to the existing edifice of mechanics, and of the compatibility of the postulate with the already established principles of this discipline. The extent to which this addition can be successfully realized provides a standard for assesing the status of Lorentz covariance as a truly universal postulate of all physical science.

29. Minkowski 1908, 372 (Italics in the original): "Danach entsprechen die allgemeinen Differentialgleichungen von Lorentz f?r beliebig magnetisierte Körper *nicht* dem Relativitätsprinzipe."

30. Minkowski cited here Cohn 1902. For Cohn's electrodynamics see Darrigol 1993, 271-276; Hirosige 1966, 31-37; Miller 1981, 181-182. Miller gives a long list of works that critically discussed Cohn's theory, but Minkowski's article is not mentioned in this context. On the other hand, Miller describes Cohn's theory in the following terms: "Cohn speculated on neither the nature of the ether, nor the nature of electricity (his theory was not based upon an atomistic conception of electricity), nor did he attempt to reduce the laws of electromagnetism to those of mechanics." Moreover, adds Miller, Cohn suggested that the ether should be utilized as a "heuristic concept," that should not acquire an importance relative to the theory in question." Given the views of Minkowski as presented here, these remarks suggest a possible, direct or indirect, influence of Cohn's work on Minkowski (Although according to Pyenson 1979, Cohn's articles were not among the texts studied in the 1905 seminar on electron theory.) A more detailed discussion of this point must be left for a future occasion.

Minkowski showed—using the formalism developed in the earlier sections—that in order for the equations of motion of classical mechanics to remain invariant under the Lorentz group it is necessary to assume that $c = \infty$. It would be embarrassing or perplexing (*verwirrend*), he said, if the laws of transformation of the basic expression

$$-x^2 - y^2 - z^2 + c^2t^2$$

into itself were to necessitate a certain finite value of $c$ in a certain domain of physics and a different, infinite one, in a second domain. Accordingly, the postulate of relativity (i.e., our confidence in the universal validity of the theorem) compels us to see Newtonian mechanics only as a tentative approximation initially suggested by experience, which must be corrected to make it invariant for a finite value of $c$. Minkowski not only thought that reformulating mechanics in this direction was possible; in terms very like those that can be found in Hilbert's lecture notes, he asserted that such a reformulation seems considerably to perfect the axiomatic structure of mechanics.[31]

Naturally, all the discussion in this section is couched in the language of space-time coordinates $x, y, z, t$. But Minkowski referred throughout to the properties of *matter* at a certain point of *space* at a given *time*, clearly separating the three elements, and focusing on the path traversed by a particle of matter along all times $t$. The space-time line of that piece of matter is the collection of all the space-time points $x$, $y$, $z$, $t$ associated with that particle, and the task of studying the motion of matter is defined as follows: "For every space-time point to determine the direction of the space-time line traversed by it." Likewise, the collection of all space-time lines associated with the material points of an extended body is called its space-time thread (*Raum-Zeitfaden*). One can also define the "proper time" of a given matter particle in these terms, generalizing Lorentz's concept of local time. One can also associate a positive magnitude (called *mass*) to any well-delimited portion of (three-dimensional!) space at a given time. These last two concepts lead to the definition of a rest-mass density, which Minkowski used to formulate the principle of conservation of mass involving all these concepts. Thus, Minkowski relied here on the

31. Minkowski 1908, 393 (Italics in the original.): "Ich möchte ausf?hren, da? durch eine *Reformierung der Mechanik, wobei an Stelle des Newtonschen Relativitätspostulates mit c = ∞ ein solches f?r ein endliches c tritt*, sogar der axiomatische Aufbau der Mechanik erheblich an Vollendung zu gewinnen scheint."

four dimensional language as an effective mathematical tool providing a very concise and symmetric means of expression, but his appeal to the four-dimensional geometry does not seem to convey at this stage any direct evidence of a new, articulated conception of the essence of the relation between space and time, like the one that characterizes Minkowski's famous 1908 Köln lecture (discussed below).

Using this language, then, Minkowski analyzed the compatibility of the postulate of relativity with two accepted, basic principles of mechanics: Hamilton's principle and the principle of conservation of energy. Compatibility with the former he discussed in a way anlaogousto his discussion on electrodynamics in earlier sections. As for the conservation of energy, Minkowski stressed with particular emphasis the full symmetry of the equations obtained for all four variables *x, y, z, t*. Integrating the terms of the equations of motion derived using the Hamilton principle, he deduced four new differential equations

$$m\frac{d}{d\tau}\frac{dx}{d\tau} = R_x ,$$

$$m\frac{d}{d\tau}\frac{dy}{d\tau} = R_y ,$$

$$m\frac{d}{d\tau}\frac{dz}{d\tau} = R_z ,$$

$$m\frac{d}{d\tau}\frac{dt}{d\tau} = R_t .$$

Here $m$ is the constant mass of a thread, $\tau$ is the proper time, and $R$ is a vector of type I: the *moving force* of the material points involved. The full symmetry obtained here by the adoption of the postulate of relativity struck Minkowski as very telling, especially in relation to the status of the fourth equation. As in the previously considered, analogous case of electrodynamics, he claimed, here too there is a high degree of physical evidence in its favor.[32] Moreover, he concluded—again in terms strikingly similar to those found in Hilbert's lectures on physics—the derivation presented here justifies the assertion that if the postulate of relativity is placed on top of the building of mechanics, the equations of motion can be fully derived from the principle of conservation of energy alone.[33]

32. Minkowski 1908, 401 (Italics in the original): "*...gleichsam eine höhere physikalische Evidenz zuzuschreiben ist.*"

So much for the basic principles of mechanics and the laws of motion. But clearly, the truly universal validity of the postulate of relativity could only be expected if one could show that its assumption does not contradict the observable phenomena related to gravitation. To that end, in the closing passages of the talk, he sketched his proposal for a Lorentz covariant theory of gravitation, much more elaborate than his earlier one. As in his former talk, Minkowski again mentioned Poincaré's similar attempt, but declared that his own followed a different direction.

Minkowski elaborated his four-dimensional formulation even further here, introducing ideas quite close to the notion of a light cone and the kind of reasoning associated with it. It is pertinent to present briefly the basic terms of his derivation of the law of gravitation, since they convey a distinct geometric flavor (in the basic, intuitive sense of the term "geometric", though in four dimensions instead of the usual three)—a flavor that is often adduced in connection with Minkowski's approach to relativity, but which appears only in this section, and not in his previous ones on electrodynamics or even on mechanics.

In order to adapt Newton's theory of gravitation to the demand of Lorentz covariance Minkowski described in four-dimensional geometrical terms the force vector acting on a mass particle $m$ at a certain point $B$. This vector has to be orthogonal to the world-line of the particle at $B$, since four-force vectors are orthogonal to four-velocity vectors. To remain close to Newton's theory, Minkowski also assumed that the magnitude of this vector is inversely proportional to the square of the distance (in ordinary space) between any two mass particles. Finally, he also assumed that the actual direction of the orthogonal vector to the world-line of $m$ is in fact determined by the line connecting the two attracting particles. These requirements must all be satisfied by any adaptation of Newton's laws to Lorentz covariance, but of course, Minkowski still had to be more specific in his choice of such a law. He did so in the following way: Take a fixed space-time point $B^*(x^*,y^*,z^*,t^*)$, and consider all the points $B(x,y,z,t)$ satisfying the equation

$$(x-x^*)^2+(y-y^*)^2+(z-z^*)^2=(t-t^*)^2, \quad (t-t^*\geq 0).$$

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33. Minkowski 1908, 401 (Italics in the original): "*Wird das Relativitätspostulat an die Spitze der Mechanik gestellt, so folgen die volständigen Bewegungsgesetze allein aus dem Satze von der Energie*."

This is called the "light-structure" of $B^*$, and $B^*$ is a light-point in the set of all the points located towards the concave side of the 3-surface defined by the light-structure. Using the language introduced later by Minkowski himself, one can say that $B^*$ can communicate by light signals with all points of which it is a light-point. If in the above relation $B^*$ is taken as variable and *B* as fixed, then Minkowski claimed that for an arbitrarily given space-time line there exists only one point $B^*$ which is a light-point of *B*. This latter conclusion is valid only if the space-time line is (using the terminology introduced later) time-like, which is impicit in Minkowski's definition of space-time lines as world-lines of matter.[34] Given two matter points $F, F^*$ with masses $m, m^*$, respectively, assume *F* is at space-time point *B*, and let *BC* be the infinitesimal element of the space-time line through *F*. This space-time line is nothing but the (modern language) word-lines of the particles at those events, with masses $m, m^*$. Minkowski claimed that the moving force of the mass point *F* at *B* should (*möge*) be given by a space-time vector of type I, which is normal to *BC*, and which equals the sum of the vector described by the formula

$$mm^*\left(\frac{OA'}{B^*D^*}\right)^3 BD^*, \qquad \text{(N)}$$

and a second, suitable vector, parallel to $B^*C^*$. The following figure may help in understanding Minkowski's train of thought:

34. Minkowski 1908, 393.

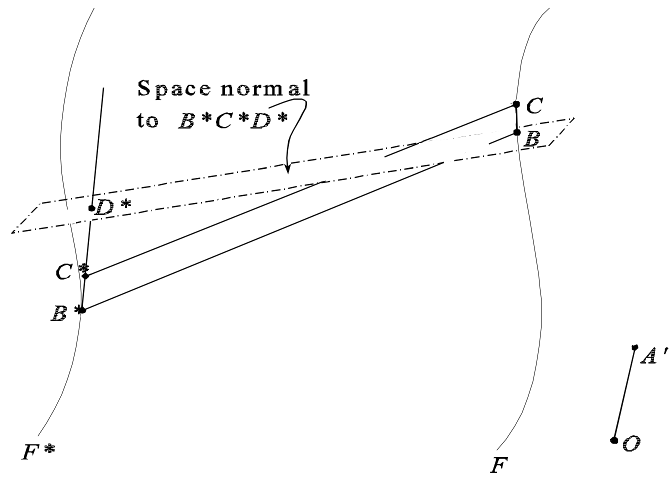


Figure 1

The additional space-time points that appear here are defined by Minkowski (without himself using any figure) as follows: $B^*$ is the light-point of $B$ along the space-time line of $F^*$; $O$ is the origin of the coordinate system and $OA'$ is a segment parallel to $B^*C^*$ ($C^*$ being the light-point along the world-line of $F^*$, of space-time point $C$) whose endpoint $A'$ lies on the four-dimensional hyperbolic surface

$$-x^2 - y^2 - z^2 + t^2 = 1.$$

Finally, $D^*$ is the intersection point of the line through $B^*C^*$ and the normal to $OA'$ passing through $B$.

Using Fig. 1, some further explanations may help to clarify Minkowski's somewhat obscure treatment of gravitation. In developing this topic, Minkowski adds the assumption that the material point $F^*$ moves uniformly, i.e., that $F^*$ describes a straight line. Thus, at the outset Minkowski has presumably assumed that $F^*$ moves arbitrarily (as described in Fig. 1 above). In this more general case, $BC$ and $B^*C^*$ represent the tangent vectors to the curves $F$ and $F^*$, and they can be physically interpreted as the four-velocities of the masses with world-lines $F$ and $F^*$, respectively. Now, Minkowski's gravitational

force must be orthogonal to the four-velocity of $F$ at $B$, and therefore orthogonal to $BC$. $B^*C^*$, on the other hand, helps to determine the distance between $F$ and $F^*$ in the rest-frame of the attracting body $F^*$, a magnitude necessary to make the gravitational law inversely proportional to it. In effect the velocity of $F^*$ at $B^*$ is parallel to $B^*C^*$, and by extending the latter into $B^*D^*$, Minkowski is determining the plane on which the desired distance should be measured, i.e., a plane which is normal to $B^*D^*$ and passes through $B$. The space distance (not space-time) between the two points is thus given by $BD^*$.

Now the quantity $BD^*$ also appears in formula (N) and in fact it gives the direction of the vector represented by the latter. But, as said above, the gravitational force should be orthogonal to $BC$, which is not necessarily the case for $BD^*$. Minkowski corrected this situation by adding to the first vector a second "suitable" one, parallel to $B^*C^*$. Thus the "suitable" vector that Minkowski was referring to here is one that, when added to ($N$) yields a third vector which is orthogonal to $BC$.

Equation (N) involves the product of the masses $m$ and $m^*$, and to that extent it directly corresponds to the Newtonian law. But does this equation really embodies an inverse square law in the present situation? It seems that Minkowski's additional assumption, i.e., that $F^*$ moves uniformly, could serve to answer this question (although Minkowski does not explicitly elaborate on this point). In fact, after this assumption is added, the situation in Fig. 1 can be represented as follows:

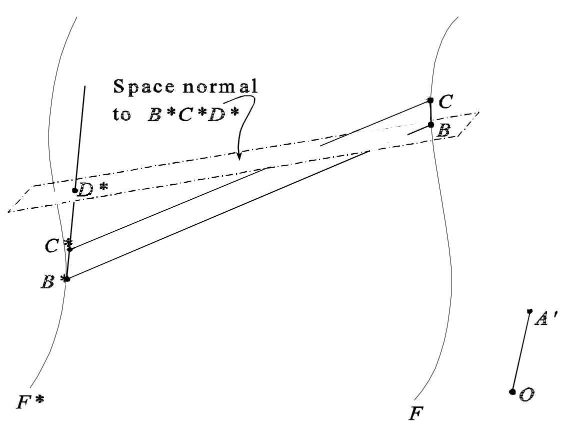


Figure 2

If one sets the coordinates of $B^*$ to be $(0,0,0, \tau^*)$, then the origin $O$ lies on $F^*$. Moreover, the following values of the magnitudes involved in the equation can be deduced directly from their definitions:

$$OA' = 1; \quad B^*D^* = t - \tau^*; \quad (BD^*)^2 = x^2 + y^2 + z^2.$$

But $B^*$ is a light point of $B$, and therefore

$$(B^*D^*)^2 = (t - \tau^*)^2 = x^2 + y^2 + z^2$$

Equation ($N$) is thus reduced to the following:

$$mm^*\left(\frac{OA'}{B^*D^*}\right)^3 BD^* = \frac{mm^*}{(x^2 + y^2 + z^2)},$$

which is the desired inverse square law of gravitation. Moreover, the assumption that $F^*$ moves uniformly also prepares the way for Minkowski's discussion of the solar system at the end of his article (see below), by letting $F^*$ represent the inertial motion of the sun and $F$ the non-inertial motion of an orbiting planet.

Although many details of Minkowski's argument (such as those presented here) do not appear in the printed version of his article, all the discussion was fully conducted in the framework of space-time geometry, using only four-vectors defined on world-points and word-lines. Minkowski could thus conclude, without further comment, that the above determination of the value of the moving force is covariant with respect to the Lorentz group.

Minkowski went on to determine how the space-time thread of $F$ behaves when the point $F^*$ undergoes a uniform translatory motion. He asserted that starting from equation (N) as the value of the attracting force, the following four equations could be obtained:

$$\text{(A)} \qquad \frac{d^2x}{d\tau^2} = -\frac{m^*x}{(t-\tau^*)^3}, \quad \frac{d^2y}{d\tau^2} = -\frac{m^*y}{(t-\tau^*)^3}, \quad \frac{d^2z}{d\tau^2} = -\frac{m^*z}{(t-\tau^*)^3},$$

and

$$\text{(B)} \qquad \frac{d^2t}{d\tau^2} = -\frac{m^*x}{(t-\tau^*)^2}\frac{d(t-\tau^*)}{dt}.$$

Since the relation $x^2 + y^2 + z^2 = (t - \tau^*)^2$ holds true, equations (A) are similar to the motion equations of a material point under the Newtonian attraction of a fixed center, as Minkowski stated, substituting instead of the time *t* the proper time τ of the particle. Equation (B), on the other hand, establishes the dependence between the proper time of the particle and the time *t*. Using these equations, Minkowski added some brief calculations concerning the orbits and expected revolution times of planets and inferred—using the known values of the mass of the Sun as *m** and of the axis of the Earth's orbit—that his formulas yielded values for the eccentric anomalies in the planetary orbits of the order of 10-8. He concluded with two remarks: first, that the kind of attraction law derived here and the assumption of the postulate of relativity together imply that gravitation propagates with the velocity of light. Second, that considering the small value obtained above for Kepler's equation for eccentric anomalies, the known astronomical data cannot be used to challenge the validity of the laws of motion and modified mechanics proposed here and to support Newtonian mechanics.[35]

Minkowski's treatment of gravitation was extremely sketchy and tentative. An attentive reading of it raises more questions that it seems to answer. Some of these questions have been raised in the foregoing paragraphs, but more can be added to them. For instance: Is Minkowski's gravitational force in any sense symmetric with respect to *F* and $F^*$? What kind of conservation laws arise within such a theory? Minkowski did not address any of these issues, either in the article or elsewhere. Rather than addressing the issue of gravitation in detail, when writing this article Minkowski's main concern was clearly to investigate the logical status of the principle of relativity as applied to all physical domains and the plausibility of assuming that it must also hold when dealing with gravitation.

Still, the theory sketched here was, together with Poincaré's, the starting point of the attempts to extend the validity of relativity to cover gravitation as well. Einstein himself addressed the same task in an article submitted for publication on December 4, 1907, in which he raised for the first time the question whether the principle of relativity could be extended to cover accelerated, rather than only inertial reference systems (Einstein 1907a, 476). Although Einstein formulated here for the first time what he later called the principle of equivalence—a fundamental principle of his general theory of relativity—his 1907 attempt did not directly lead to an extension of the validity of relativity. Einstein did

35. Minkowski 1908, 404 (Italics in the original): "In Anbetracht der Kleinheit des periodischen Termes ... d?rfte eine Entscheidung *gegen* ein solches Gesetz und die vorgeschlagene modifizierte Mechanik zugunsten des Newtonschen Mechanik aus den astronomischen Beobachtungen nicht abzuleiten sein."

not return to this topic until 1911, when his actual efforts to generalize relativity really began.[36] In his 1907 paper Einstein mentioned neither Minkowski nor Poincaré. Nor did Minkowski mention this article of Einstein, and one wonders if he ever read it. Minkowski's approach to electrodynamics and the principle of relativity came to provide the standard language for future investigations, but his specific argumentation on gravitation attracted little if any attention. Minkowski himself, for instance, mentioned the issue of gravitation once again in his next article, "Space and Time," but only in passing. Arnold Sommerfeld (1868-1951), in a 1910 article that contributed more than any other work to systematize and disseminate Minkowski's four- and six-vector formalism, claimed that Minkowski's approach to gravitation was no better than Poincaré's, and that if they differed in any respect—as Minkowski had claimed in his article—it was in their methods rather than in their results.[37] Unfortunately, we don't know how Minkowski would have reacted to Sommerfeld's interpretation on this point. But perhaps more interesting than Sommerfeld's is Hilbert's attitude. Neither in any of his physical lectures after 1905, nor in his published articles on physical issues (his works on general relativity included) did he discuss or comment on Minkowski's ideas on gravitation. In the winter semester of 1913-14, Hilbert's lectures dealt with electromagnetic oscillations. In these lectures he addressed the "high desirability" of solving the still unsettled question of how to explain gravitation exclusively from the assumption of an electromagnetic field, from the Maxwell equations, and from some additional, "simple" hypotheses, such as the existence of rigid bodies. By this time Hilbert had already studied with great interest Gustav Mie's electromagnetic theory of matter, which was to become the basis of his own relativistic theory of gravitation in 1915.[38] But the three works he mentioned in his 1913-14 course as meaningful attempts to solve this question were neither Minkowski's nor Mie's. Rather he went back as far as LeSage's corpuscular theory of gravitation, originally formulated in 1784 and reconsidered in the late nineteenth century by J.J. Thomson;[39] to Lorentz's 1900

36. See Norton 1984, 105-107; Norton 1992, 20-35; Stachel et al (eds.) 1989, 274.

37. Sommerfeld 1910, 687. On pp. 684-689 one finds a somewhat detailed account of the physical meaning of Minkowski's sketch for a theory of gravitation, and a comparison of it to Poincaré's. On Sommerfeld's place in disseminating Minkowski's ideas see Walter 1997, § 4.

38. From a reply letter of Mie to Hilbert, dated October 22, 1913 (Hilbert *Nachlass*, NSUB Göttingen - Cod Ms David Hilbert 254 - 1), we know that by this time the latter had already begun studying the articles of the former.

article on gravitation,[40] and to the more recent work by Erwin Madelung—an assistant at the Göttingen physics department—which itself elaborated the ideas of Lorentz.[41] Of course, Hilbert was here seeking a physical explanation of the phenomenon of gravitation; thus Minkowski's theory, which was no more than an attempt to adjust Newton's laws to the demand of Lorentz covariance, could be of little help to him.

We can summarize the foregoing account by assessing Minkowski's brief incursion into gravitation in its proper, rather limited, context. Arguing again existing opinions, Minkowski sought to investigate, in axiomatic terms, the conceptual consequences of applying the postulate of relativity in domains other than electrodynamics.[42] In this framework he addressed the phenomena related to gravitation and showed how an argument could be worked out for the claim that there was no *prima facie* reason to assume that the postulate of relativity contradicts the observable effects of such phenomena. Therefore, one could envisage the possibility of a truly articulate Lorentz-covariant theory of gravitation which would approximate the Newtonian theory as a limiting case.[43] However, neither Minkowski himself in his next writings, nor Hilbert in his own, returned to this theory. It seems then, that neither of them considered it as anything more than a very preliminary attempt. On the other hand, it clarifies very well the kind of motivations underlying Minkowski's investigation of the place of the principle of relativity in physics. Moreover, this particular article of 1908 shows very clearly how the geometric element ("geometric" taken here in its intuitive-synthetic, rather than in its formal-analytical, sense) entered Minkowski's treatment only gradually, and that an immediate visualization, in geometruc terms, of the consequences of the adoption of the principle of relativity in mechanics was not an initial, major motivation behind his attempt.

39. On the Le Sage-Thomson theory see North 1965, 38-40.

40. Lorentz 1900. On this theory, see McCormmach 1970, 476-477.

41. Hilbert 1913-14, 107. Hilbert referred to Madelung 1912.

42. Minkowski opened the appendix on mechanics (p. 392), with the words: "Nun sagen viele Autoren, die klassische Mechanik stehe im Gegensatz zu dem Relativitätspostulate, das hier f?r die Electrodynamik zugrunde gelegt ist."

43. A similar assessment of Minkowski's theory appears in Norton 1992, 21. Norton's article refers to this theory only in passing and offers such an assessment only as a conjecture. The argument presented here should provide a more compelling basis for that conjecture.

## 4. Space and Time

Minkowski presented his views on relativity for the first time outside Göttingen nine months later, on September 21, 1908, when he delivered a lecture to the meeting of the German Association of Natural Scientists and Physicians in Köln. The text of his lecture would become the article “Raum und Zeit”, Minkowski’s best known contribution to the special theory of relativity and to the new conception of space and time associated with it. Both the opening and the closing passages of the text have repeatedly been quoted as encapsulating the essence of Minkowski’s views on these issues. In the opening passage Minkowski declared: “Henceforth, space by itself, and time by itself, are doomed to fade away in the shadows, and only a kind of union of the two will preserve an independent reality.” In the closing passage he concluded: “The validity without exception of the world-postulate, I would like to think, is the true nucleus of an electromagnetic image of the world, which, discovered by Lorentz, and further revealed by Einstein, now lies open in the full light of day.” These two passages have helped to consolidate the image of Minkowski’s geometrically motivated approach to relativity and of his alleged commitment to the electromagnetic view of nature. I will proceed next to examine Minkowski’s “Space and Time,” from the perspective provided by the foregoing analysis of his earlier works and against the background of Hilbert’s program for the axiomatization of physical theories and its concomitant views.

Two kinds of invariance arise in connection with the equations of Newtonian mechanics, Minkowski opened. First, the invariance associated with an arbitrary change of position, and second, the one associated with uniform translation. Moreover, he added, our choice of a particular point to stand as $t = 0$ does not affect the form of the equations. Although these two kinds of invariance can be equally expressed in terms of the groups of invariance they define with respect to the differential equations of mechanics, traditional attitudes towards the corresponding groups had been utterly different: whereas the existence of the group corresponding to the first invariance is usually seen as expressing a fundamental property of space, the existence of the second one (i.e., the group of Galilean transformations) has never attracted any special interest as such. At best, it has been accepted with disdain (*Verachtung*) in order to be able to make physical sense of the fact

that observable phenomena do not allow us to decide whether space, which is assumed to be at rest, is not after all in a state of uniform translation. It is for this reason, Minkowski concluded, that the two groups carry on separate lives with no one thinking of combining them. Now this separation, Minkowski thought, had a counterpart in the way the axiomatic analysis of these two scientific domains is usually undertaken: in the axiomatization of mechanics, the axioms of geometry are usually taken for granted, and therefore the latter and the former are never analyzed simultaneously, as part of one single undertaking.[44] We know precisely what Minkowski meant by this latter assertion: in Hilbert's 1905 lectures on the axiomatization of physics, he had discussed the axiomatization of the laws of motion by adding to the already accepted axioms of geometry separate axioms meant to define time through its two basic properties, namely, its uniform passage and its unidimensionality (*ihr gleichmä?iger Verlauf und ihre Eindimensionalität*). In order to study motion, Hilbert had said, one starts by assuming space and adds to it time[45]—and this is indeed what he did. Minkowski's brilliant idea was, then, that the traditional separation of mechanics and geometry—more clearly accepted in relation to the respective invariance groups, but also implied in the way their axiomatic definitions have been introduced—should be ended, and that combining the two invariance groups together, would lead to a better understanding of the reality of space and time, and of the laws of physics. Explaining the implications of this integration was the aim of his talk.

Minkowski's audience was composed of natural scientists rather than mathematicians. This certainly conditioned the kinds of arguments and emphases he chose to adopt. In particular, he stressed from the outset that the ideas presented in the lecture were independent of any particular conception of the ultimate nature of physical phenomena. As in the two earlier lectures, Minkowski intended his arguments to be an exploration of the logical consequences of adopting the postulate of relativity in the various domains of physics, without necessarily committing himself to any particular view. Therefore, he put forward his arguments in a way intended to prevent any physicist, whatever his basic conception of

44. Minkowski 1909, 431: "Man ist gewohnt, die Axiome der Geometrie als erledigt anzusehen, wenn man sich reif f?r die Axiome der Mechanik f?hlt, und deshalb werden jene zwei Invarianzen wohl selten in einem Atmenzuge gennant." The standard English translation of Minkowski's lecture (Minkowski 1952) is somewhat misleading here, as in many other passages.

45. Hilbert 1905, 129.

physical phenomena, from reacting to these ideas with *a propri* suspicion or hostility. They were meant to be compatible with any possible belief concerning the ultimate nature of mass, electromagnetic processes and the ether, and the relationships among these: “In order not to leave a yawning void anywhere,” Minkowski said, “we want to imagine, that at any place in space at any time something perceptible exists. In order not to say matter or electricity, I will use the word ‘substance’ to denote this something.”[46] *Substance* in general, then, rather than a particular choice between mass, ether, electricity or any other candidate. In a later passage in which he referred to the velocity of light in empty space, he exercised again the same kind of caution: “To avoid speaking either of space or of emptiness, we may define this magnitude in another way, as the ratio of the electromagnetic to the electrostatic unit of electricity.”[47]

Assuming that we are able to recognize a substantial point as it moves from a first four-coordinate “world-point,” to a second one, Minkowski declared in the introduction that the world can be resolved into world-lines, namely, collections of all the world-points associated with a substantial point when $t$ acquires all values between $-\infty$ and $\infty$. He added that the laws of physics attain their most perfect expression when formulated as relations between such world-lines.

Minkowski began the development of his argument by describing the relationship between the groups defined by the Lorentz transformations and by the Galilean transformations. In his first talk on the principle of relativity in 1907, Minkowski had already shown that the assumption of the principle of inertia implies that the velocity of propagation of light in empty space is infinite. This time he discussed this impliction, while focusing on certain formal properties of these groups. Referring back to the two groups mentioned in the introduction, Minkowski explained that the first of them expresses the fact that if the $x,y,z$ axes for $t = 0$ are rotated around the origin of coordinates, then the expression

$$x^2 + y^2 + z^2$$

46. Minkowski 1909, 432: “Um nirgends eine gähnende Leere zu lassen, wollen wir uns vorstellen, da? allerorten und zu jeder Zeit etwas Wahrnembares vorhanden ist. Um nicht Materie oder Elektrizität zu sagen, will ich f?r dieses Etwas das Wort Substantz brauchen.”

47. Minkowski 1909, 434 (1952, 79).

remains invariant. The second group expresses the fact that the laws of mechanics remain unchanged under the transformations that send $x,y,z,t$ to $x - \alpha t$, $y - \beta t$, $z - \gamma t$, $t$, with any constant coefficients $\alpha,\beta,\gamma$. Under these transformations, the $t$-axis can be given whatever upward direction we choose. But how is the demand for orthogonality in space, asked Minkowski, related to this complete freedom of the $t$-axis? Minkowski answered this question by looking at four-dimensional space-time and considering a more general kind of transformation, namely, those transformations that leave invariant the expression $ct^2 - x^2 - y^2 - z^2 = 1.$ These properties turn out to depend on the value of the parameter $c$ and thus classical mechanics appears as a special case of a more general class of theories. He stressed the geometrically intuitive elements of his arguments, by focusing on the case $ct^2 - x^2 = 1,$ which is graphically represented as a hyperbola on the plane $x,t$:

Fig. 3

Here $OB$ is the asymptote ($ct - x = 0$), and the orthogonal segments $OC$ and $OA$ have the values $OC = 1$ and $OA = 1/c$. Choose now any point $A'$ on the hyperboloid, draw the tangent $A'B'$ to the hyperbola at $A'$, and complete the parallelogram $OA'B'C'$. If $OA'$ and $OC'$ are taken as new axes, $x',t'$ respectively, and we set $OC' = 1$, $OA' = 1/c$, then the expression for the hyperbola in the new coordinates retains its original form $ct'^2 - x'^2 = 1.$ Hence, $OA'$ and $OC'$ can now be defined as being themselves orthogonal and thus the hyperbola construction helps to conceive orthogonality in a way that departs from the usual Euclidean intuition. The parameter $c$ determines in this way a family of transformations that, together with the rotations of space-time around the origins of coordinates, form a group, the group $G_c$. But then—again from geometric considerations—one sees that when $c$ grows infinitely large, the hyperbola approximates the $x$-axis and, in the limit case, $t'$ can be given any upward direction whatever, while $x'$ approaches $x$ indefinitely. This geometrical argument thus shows that $G_\infty$ is nothing but the above described group of transformations $G_c$ associated with Newtonian mechanics.

Making explicit this illuminating connection between the main two groups of transformations that arise in physics allowed Minkowski to digress again and comment on the relation between mathematics and physics. He thus said:

> This being so, and since $G_c$ is mathematically more intelligible than $G_\infty$, it looks as though the thought might have struck some mathematician, fancy-free, that after all, as a matter of fact, natural phenomena do not possess an invariance with the group $G_\infty$, but rather with a group $G_c$, $c$ being finite and determinate, but in ordinary units of measure, *extremely great*. Such a premonition would have been an extraordinary triumph for pure mathematics. Well, mathematics, though it can now display only staircase-wit, has the satisfaction of being wise after the event, and is able, thanks to its happy antecedents, with its senses sharpened by an unhampered outlook to far horizons, to grasp forthwith the far-reaching consequences of such a metamorphosis of our concept of mature. (Minkowski 1909, 434 [1952, 79])

It is not evident, on first reading, what Minkowski meant here when he said that $G_c$ is "mathematically more intelligible" than $G_\infty$, but apparently he was pointing to the fact that the group of Galilean transformations, which in itself had failed to attract any interest from mathematicians, becomes much more mathematically interesting when it is seen in the more general context of which it appears as a limiting case. In retrospect, Minkowski concluded, this situation might seem to suggest that mathematical insight could have sufficed to realize what is involved here, but in fact this was not the case, and physical considerations were necessary.

The invariance under the group $G_c$ of the laws of physics in a four-dimensional space-time has for Minkowski an additional, important consequence that reinforces—from a different perspective and in a much more compelling fashion—a point of view earlier elaborated in Hilbert's writings, namely, the view of geometry (i.e., the science of sensorial space) as a natural science on which all other physical sciences are grounded. Yet, what Hilbert had initially expressed as an epistemologically grounded conception, and had later developed when discussing the axioms of mechanics on the basis of the axioms of geometry, appears here in the opposite direction: the latest developments of physical science have raised the need to reconsider our basic conception of space and time in such a way as to recognize geometry as essentially embedded in physics. Thus, to conclude this section of his lecture Minkowski said:

> In correspondence with the figure described above, we may also designate time $t'$, but then must of necessity, in connection therewith, define space by the manifold of the three parameters $x',y,z,$ in which case physical laws would be expressed in exactly the same way by means of $x',y,z,t'$, as by means of $x,y,z,t$. We should then have in the world no longer *space*, but an infinite number of spaces, analogously as there are in three-dimensional space an infinite number of planes. Three dimensional geometry becomes a chapter in four-dimensional physics. (*ibid.*)

So much for the formal, geometrical considerations. But of course the question arises: what empirical facts compel us to adopt this new conception of space? Moreover: Does this conception never contradict experience? Is it useful in describing natural phenomena? These questions were discussed by Minkowski in the following three sections of his talk. First, he observed that by means of a suitable transformation the substance associated with a particular world-point can always be conceived as being at rest. This he considered to be a fundamental axiom of his theory of space-time. A direct consequence of the axiom is that every possible velocity in nature is smaller than $c$. In his second 1907 lecture Minkowski had taken this consequence in itself as a central axiom of the electrodynamics of moving bodies. Formulated in those terms, he felt, it had a somewhat "unpleasant" appearance that raised mistrust, but that in the present four-dimensional formulation it could be grasped more easily.

Minkowski then explained, in terms of the groups $G_c$ and $G_\infty$, the problems raised by the Michelson experiment, given the different invariance groups characteristic of different physical disciplines. He stressed that the concept of a rigid body has only meaning only in a mechanics based on the group $G_\infty$, and that the contraction hypothesis had been introduced by Lorentz in order to account for the divergence detected between theory and experiment. Admitting that this hypothesis in its original form "sounds extremely fantastical," he proceeded to show that it is entirely coherent with the new conception of space and time, and that the latter clarified the former completely. Minkowski's explanation was fully geometrical and it relied on a straightforward verification of the properties of a rectangle and a parallelogram drawn on the two-dimensional figure introduced in the first section. At this point Minkowski also characterized Einstein's contribution in this context, as explaining the nature of local time: whereas Lorentz had introduced the concept as a tool for better understanding the contraction hypothesis, Einstein "clearly recognized that the time of the one electron is just as good as that of the other."[48] Thus, Minkowski said, Ein-

stein had essentially undermined the idea of *time* as a concept unequivocally determined by phenomena. But then, in spite of the importance of this achievement, neither Einstein himself nor Lorentz undertook a similar attack on the concept of *space*; Minkowski considered such an attack to be indispensable in uncovering the full implications of the postulate of relativity, and he saw his own ideas as having contributed to the full achievement of that aim. It was in this framework that he introduced the term "world-postulate" instead of relativity:

> When [the attack on the traditional concept of space] has been undertaken, the word *relativity-postulate* for the requirement of an invariance with the group $G_c$ seems to me very feeble. Since the postulate comes to mean that spatio-temporal phenomena manifest themselves only in terms of the four-dimensional world, but the projection in space and in time may still be performed with certain liberty, I prefer to call it the *postulate of the absolute world* (or briefly, the world-postulate). (Minkowski 1909, 437)[49]

In the third part of the lecture, Minkowski showed that the world-postulate, by allowing a symmetrical treatment of the four coordinates $x,y,z,t$, provides a much clearer understanding of the laws of physics. In this section he introduced the concept—only implicit in his earlier lectures—of a light-cone (in fact, he only spoke separately of the front- and back-cones of a point $O$) and explored its usefulness, especially in dealing with the concept of acceleration.

In the last two sections, Minkowski addressed again the main point discussed in his earlier lecture, namely, the compatibility of the principle of relativity with existing physical theories, or, as he put it here, that "the assumption of the group $G_c$ for the laws of physics never leads to a contradiction." In order to show this, Minkowski understood that it was "unavoidable to undertake a revision of the whole of physics on the basis of this assumption." Such a revision had in fact already begun. Minkowski cited again Planck's recent article on thermodynamics and heat radiation (Planck 1907), as well as his own ear-

48. Minkowski 1909, 437 (1952, 83). In his obituary of Minkowski, Hilbert (1909, 90) repeated this assessment. For a discussion of the differences in the conception of time in Einstein's and in Minkowski's theories, see Walter 1997, § 3.5.

49. Minkowski's original sentence—"...noch mit einer gewissen Freiheit vorgenommen werden kann,..."— appears in the English translation (1952, 83) as: "... may still be undertaken with a certain degree of freedom." This seems to me somewhat misleading in this context.

lier lecture, already published, where the compatibility of the postulate of relativity with the equations of electrodynamics and of mechanics (retaining, he stressed, the concept of mass) had been addressed. With reference to the latter domain, Minkowski elaborated this time on the question of how the expressions of force and energy change when the system of reference changes. He then showed how the effects produced by a moving point-charge, and in particular the expression of its ponderomotive force, can be best understood in terms of the world postulate. He stressed the simplicity of his own formulation as compared with what he considered the cumbersome appearance of previous ones.

Finally, in a brief passage, Minkowski addressed the question of gravitation. He stressed that the adoption of the world-postulate for mechanics as well as for electrodynamics eliminated the “disturbing lack of harmony” between these two domains,. Referring back to his published lecture of 1907, he asserted that, by introducing in the equations of movement under gravitation the proper time of one of the two attracting bodies (which is assumed to be moving, while the other is at rest), one would obtain a very good approximation to Kepler’s laws. From this he concluded, as in his earlier lecture, that it is possible to reformulate gravitation so as to comply with the world-postulate.

In his closing remarks, Minkowski addressed the question of the electromagnetic world-view and the postulate of relativity, which he had expressly by-passed throughout the lecture. For Minkowski, it was not simply the case that all these physical domains were compatible with the world-postulate, because their equations had been derived in a particular way; the postulate had a much more general validity than that. It is in this light that we must understand the often-quoted closing passage of the lecture. The equations that describe electromagnetic processes in ponderable bodies completely comply with the world-postulate, Minkowski remarked, and moreover, as he intended to show on a different occasion, in order to verify this fact it is not even necessary to abandon Lorentz’s erudite (*gelehrte*) derivation of these fundamental equations, based on the basic conceptions (*Vorstellungen*) of the theory of the electron.[50] In other words, whatever the ultimate

50. Minkowski 1909, 444. Also here the translation (1952, 90-91) fails to convey the meaning of the original passage.

nature of physical processes may be, the world-postulate, i.e., the universal demand for invariance under the group $G_c$ of the equations expressing the laws of physical processes, must hold valid. This is what we have learnt from the latest developments in physics and this is what Minkowski expressed in his well-known assertion:

> The validity without exception of the world-postulate, I like to think, is the true nucleus of an electromagnetic image of the world, which, discovered by Lorentz, and further revealed by Einstein, now lies open in the full light of day. In the development of its mathematical consequences there will be ample suggestions for experimental verification of the postulate, which will suffice to conciliate even those to whom the abandonment of the old-established views is unsympathetic or painful, by the idea of a pre-established harmony between mathematics and physics. (Minkowski 1909, 444 [1952, 91])

Clearly, then, in reading this passage we need not assume that Minkowski was trying to advance the view that all physical phenomena, and in particular the inertial properties of mass, can be reduced to electromagnetic phenomena, as Lorentz and his supporters did. Nor must we assume that Minkowski had not understood Einstein's innovative point of view in his paper on the electrodynamics of moving bodies. Rather, Minkowski only claimed here that the electromagnetic world-view is nothing but what the world-postulate asserts: the belief in the general validity of the world-postulate is all that there is, and can be, to the electromagnetic world-view. A similar attitude may be found in Hilbert's 1905 lectures on physics; Hilbert analyzed in axiomatic terms the basic assumptions of a theory that are necessary for the derivation of its main theorems, but avoided, as much as possible, any commitment to a particular world-view. We also find in Minkowski's conclusion echoes of a further central feature of Hilbert's physical work: the purported identification of the "true nucleus" (*der wahre Kern*), as Minkowski said, of physical theories. This idea appears in several instances in Hilbert's Göttingen lectures but it is best known from the 1924 corrected republication of his earlier works on general relativity. While acknowledging that some changes had had to be introduced in his original proofs, and that the same might again be the case in the future, Hilbert nevertheless remarked that the principles of his theory will forever remain an "enduring core" (*ein bleibende Kern*) of any eventual reconstruction of physics.[51] In other words, both Minkowski and Hilbert believed that in constructing the mathematical skeleton of all physical theories, certain universal principles must be postulated (the world-postulate and general covariance, but also the energy principle and the continuity principle); even in the face of new empirical discoveries that

will force changes in the details of individual theories, these general principles will continue to hold true. Moreover, the idea of a pre-established harmony of mathematics and physics, so central to the conceptions of the Göttingen scientific community, can be traced to the belief in the existence of such universal principles, rather than to the specific contents of particular, probably provisional, physical theories expressed in mathematical terms. No wonder then, that Minkowski chose the somewhat bombastic name "world-postulate" for the universal validity of the postulate of relativity.[52]

## 5. Max Born, Relativity, and the Theories of the Electron

In the closing passages of "Time and Space", Minkowski declared that on a future occasion he intended to show that the universal validity of the postulate of relativity can be verified not only at the macroscopic level, as he had done, but also at the microscopic one, namely, starting from Lorentz's equations for the movement of the electron. On July 28, 1908, he gave a talk at the meeting of the Göttingen Mathematical Society on the basic equations of electrodynamics. Although no complete manuscript of this lecture is known, a very short abstract of it, published in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (Vol. 17, p. 111) seems to indicate that Minkowski addressed precisely the microscopic derivation of the equations using the principle of relativity. Be that as it may, whatever ideas he developed in this direction he was not able to publish before his untimely death on January 12, 1909. We nevertheless have a fair idea of what they were, from an article published by Max Born in 1910, explicitly giving credit for its contents to Minkowski.

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51. See Hilbert 1924, 2: "Ich glaube sicher, da? die hier von mir entwickelte Theorie einen bleibenden Kern enthält und einen Rahmen schafft, innerhalb dessen f?r den k?nftigen Aufbau der Physik im Sinne eines feldtheoretischen Einheitsideals gen?gender Spielraum da ist."

52. The very term "World-postulate" had been originally introduced by Hilbert in his 1905 lectures in the framework of a similar discussion. See Corry 1997, 59.

Born's first contact with Minkowski and Hilbert dates back to 1904, when he arrived as a student in Göttingen. He obtained his doctorate there in 1907, working with Carl Runge. At that time Born attended many of Hilbert's courses, and was asked to write up the notes of his 1905 lectures on "The Logical Principles of Mathematics." Born many times joined the two masters in their mathematical walks, and was deeply impressed by the breadth of their knowledge and by the critical attitude towards accepted ideas and institutions displayed in their conversations. According to Born's own testimony, Hilbert's lectures on physics, and particularly those on the kinetic theory of gases, deeply influenced all his work, including his contributions to the establishment of quantum mechanics between 1920 and 1925.[53] After military service and six months spent in Cambridge, England, Born returned to his native city of Breslau. There he read for the first time, at the beginning of 1908, Einstein's 1905 relativity paper which, according to his own report, fascinated him at once. What attracted Born in this paper was, in the first place, simply that it dealt with optics and electrodynamics, the subjects that had so strongly captivated his interest in the Hilbert-Minkowski seminar he had attended in Göttingen. As he encountered some difficulties in reading Einstein, Born wrote to Minkowski for advice. To his surprise Minkowski replied, explaining that he was himself working on those topics and inviting him to Göttingen as his assistant. Born eagerly accepted this invitation. He arrived in Göttingen for the second time in December of 1908, and thus was able to work with Minkowski only for a very short time. In these few weeks they held intense conversations concerning electrodynamics.[54] At the time of his death, Minkowski left a considerable amount of unfinished material pertaining to this domain, including many pages full of formulae, but with no additional comments. Hilbert commisioned Born to edit them. On February 8, 1910, Born lectured at the Göttingen Mathematical Society, and exposed the contents of Minkowksi's unfinished papers on electrodynamics.[55] Later that year he published in the *Mathematische Annalen*, a reconstruction of Minkowski's ideas, based on their conversations and on the latter's unfinished papers.[56]

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53. Born 1978, 99.

54. See Born's own account in Born 1978, 130-133.

55. See the announcement in the *JDMV* Vol. 19 (1910), p. 50.

56. See Minkowski 1910, 58-59. Born also presented these ideas at the meeting of the Göttingen Mathematical Society, on

Born attributed the contents of the paper totally to Minkowski, and to himself he attributed only the preparation of the material, as well as the writing of the introduction. The paper contains a detailed discussion of the Lorentz equations for the dynamics of the electron and their relation to the postulate of relativity. For the purposes of the present article, I will comment only on Born's introduction. The starting point of Minkowski's "*Grundgleichungen*..." was the assumption of the validity of the Maxwell equations for stationary bodies, inductively inferred from experience. This point of view, explained Born, differed from Lorentz's, which accounted for processes in material bodies in terms of certain hypotheses about the behavior of the electrons that conform to those bodies. Lorentz's equations for pure ether, which Born took from the latter's 1904 *Encyclopädie* article and which he described as an idealization of the Maxwell equations, provided the starting point. The Lorentz equations referred to by Born were the following:

$$\text{(I)} \qquad \mathrm{curl}\boldsymbol{M} - \frac{1}{c}\frac{\partial \boldsymbol{e}}{\partial t} = \frac{1}{c}\tilde{\rho}\tilde{\boldsymbol{w}}$$

$$\text{(II)} \qquad \mathrm{div}\,\boldsymbol{E} = \tilde{\rho}$$

$$\text{(III)} \qquad \mathrm{curl}\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{M}}{\partial t} = 0$$

$$\text{(IV)} \qquad \mathrm{div}\,\boldsymbol{M} = 0$$

Here $\boldsymbol{E}$ is the electric field intensity, $\boldsymbol{M}$ is the magnetic intensity of pure ether, $\tilde{\rho}$ *is the electricity density*, and $\tilde{\boldsymbol{w}}$ is the velocity of the electricity (or of the electron). The right-hand side of equations I,II contains the information concerning the charge and movement of the electron. Lorentz, according to Born, considered three kinds of electron. First, conduction electrons (*Leitungselektron*), whose movement is independent of matter and whose charge constitutes "true electricity." Second, polarization electrons, which provide a state of equilibrium inside molecules of matter; these electrons, however, can be dislocated from this state through the action of the electromagnetic field. The variable electricity density produced in this way is known as the "free electricity." Third and last were magnetization electrons, orbiting around central points inside matter, thus giving rise to magnetic phenomena. Lorentz's equations for electromagnetic processes in material bodies, Born now explained, are based on taking mean values of the magnitudes of the convection current due to the three types of electron. But as Minkowski had shown in his "*Grundgleichungen*", in certain cases—specifically, in the case of magnetized matter —the equations thus obtained contradict the postulate of relativity.

The specific aim of Born's article was to extend the validity of the postulate to cover all cases, including the problematic one pointed out by Minkowski in his former article. But for all the assumptions concerning the complex structure of matter that the above discussion implies, Born understood the need to stress, as Minkowski had done before him, the *independence* of this study from a particular conception of the ultimate nature of matter, ether or electricity. He thus explained that "among the characteristic hypotheses of the electron theory, the atomic structure of electricity plays only a limited role in Lorentz's derivation of the equations," given the fact that mean values have been taken over "infinitely small physical domains", so that all this structure is completely blurred, and the mean values, in the final account, appear as continuous functions of time and location. In this way Born justified his adoption of Lorentz's approach to the derivation of the equations, without thereby committing himself to any of Lorentz's ontological assumptions. He declared very explicitly:

> We hence altogether forgo an understanding of the fine structure of electricity. From among Lorentz's conceptions, we adopt only the assumptions that *electricity is a continuum that pervades all matter, that the former partially moves freely inside the latter and partially is tied to it, being able to carry out only very reduced motions relative to it.*
>
> If we want to come as close as possible to Lorentz, then all the magnitudes introduced below should be considered as Lorentzian mean values. It is however not necessary to differentiate among them, using special symbols, as if they were related to the various kinds of electrons, since we never make use of the latter.[57]

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57. Minkowski 1910, 61 (Italics in the original): "Wir verzichten daher ?berhaupt darauf, auf die feinere Struktur der Elektrizität einzugehen. Von den Lorentzschen Vorstellungen benutzen wir nur soviel, da? wir annehmen, die *Elektrizität sei ein Kontinuum, das die Materie ?berall durchdringt, zum Teil sich frei innerhalb derselben bewegen kann, zum Teil aber an sie gefesselt ist und nur sehr kleine Bewegungen relativ zu ihr ausf?hren kann.*

Will man näheren Anschlu? an Lorentz erreichen. so kann man alle im folgenden vorkommenden Grö?en als jene Lorentzschen Mittelwerte ansehen; es its dann aber hier nicht nötig, sie als solche durch besondere Zeichen von den auf die einzelnen Elektronen bezogenen Grö?en zu unterschieden, weil wir von den letzteren nirgends Gebrauch machen."

The ideas and points of view expressed in this article are Minkowski's rather than Born's own, as Born explicitly acknowledged. It would certainly be interesting to analyze in detail to what extent Born's other, contemporary, works on similar topics followed Minkowski's thinking, especially his axiomatic treatment of theories and concerning his unwillingness to take a clear stand in the debate about the ultimate nature of physical phenomena, electricity, matter and the ether. I will not undertake such an analysis here, but I think it relevant to comment briefly on some points connected to it.

Following Minkowski's death, Born went on developing his own ideas on relativity, which he had begun to consider following his reading of Einstein, and even before his return to Göttingen. A fundamental contribution of Born's in this context was the introduction of the Lorentz-invariant concept of a rigid body, a concept to which Born was led while working on the problem of the self-energy of the electron. As we saw above, Minkowski had already made it clear in "Space and Time" that the traditional concept of rigid body did not make sense outside Newtonian mechanics. Born's interest in this question implied an involvement in the Abraham-Lorentz debate concerning the independence or dependence of the mass of the (rigid or deformable) electron on its velocity, and, related to the former, in the question of the possible electromagnetic nature of the mass of the electron. In his autobiography, Born mentions that in their discussions of these issues, Minkowski "had not been enthusiastic about [Born's own ideas] but had raised no objections."[58] One wonders whether Minkowski's lack of enthusiasm was not perhaps connected to Born's particular interest in this latter topic, which Minkowski persistently tried to avoid in his own work. It is in any case pertinent to remark that Born lectured on his results before the Göttingen Mathematical Society and initially received a very unfavorable criticism from Klein. Several local mathematicians and physicists were then involved in an effort to offer Born a second opportunity to present his work in the same forum, and this was eventually successfull. Among those who interceded on behalf of Born was Hilbert (through the mediation of his student and Born's friend, Ernst Hellinger).[59] Hilbert evidently studied Born's work in detail, and from then onhe followed the latter's investigations with great interest.

58. Born 1978, 132.

59. See Born 1978, 133-138.

Both Abraham and Lorentz calculated the self-energy of a charged rigid body moving uniformly and used this energy as the Hamiltonian function for deriving the equations of motion. Born doubted the validity of an additional assumption implicit in their calculations, namely, that the energy calculated for uniform motion is the same for accelerated motion, since in an accelerated body different points have different velocities and therefore, according to the principle of relativity, different contractions. The classical concept of a rigid body is thus no longer applicable. The technical details of Born's derivation are beyond the scope of the present article. I will mention only that Born's definition is based on finding a Lorentz-covariant expression of the distance between any two space-time points; the classical distance between two points in a body is given by

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 ,$$

which is clearly not Lorentz-covariant.[60]

Born discussed the Lorentz-covariant definition of rigidity in two articles published in 1909. In the first, submitted on January 9 (three days before Minkowski's death), he discussed the relation between the concept of mass and the principle of relativity. This article is seldom, if ever, referred to in the secondary literature, perhaps because its main ideas appear in a more interesting fashion in the second 1909 article. For the purposes of the present article, however, the first article is more informative than the second, since it still reflects to some extent the direct influence of Minkowski's point of view. Born referred in the introduction of this article to the "Abraham-Sommerfeld theory of the rigid electron", whose main task he described as that of reducing the inertial mass of the electron to purely electrodynamic processes. The theory, however, does not satisfy the "Lorentz-Einstein principle of relativity." But on the other hand, said Born, the latter principle has in itself not led to a satisfactory explanation of inertial mass. The equations of motion formulated by Lorentz, Einstein and Minkowski are suggestive approximations of the Newtonian ones, which at the same time satisfy the relativity principle of electrodynamics, and the concept of mass is thus modified in the works of the three so as to fit that principle, without however explaining the concept in electrodynamical terms.

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60. For a more detailed discussion of Born's concept of rigid body and its impact, see Miller 1981, 243-257.

Born's treatment of mass was intended as an analogy to Minkowski's ideas, but applied in the framework of the Abraham-Sommerfeld theory. In his work, Minkowski had modified the Hamiltonian principle of classical mechanics so as to make the ensuing equations of motion fit the relativity principle. The variational equation to which this principle gives rise yields two integrals, one of which expresses the effect of the mass. Born intended to introduce a similar generalized Hamiltonian involving only electromagnetic magnitudes, and to derive the mass in a way similar to Minkowski's. However, it is noteworthy that for all of his interest in the Abraham-Sommerfeld theory and the point of view put forward in them, Born took pains to stress explicitly that his derivation is in no way dependent on any assumption concerning the ultimate nature of electricity—in particular, not those that underlie Abraham's and Lorentz's theories. Clearly alluding to the point of view adopted in the paper he had published under Minkowski's name, Born wrote:

> *It must be emphasized that no use will be made here of atomistic hypotheses*. In fact, the atom or the electron, imagined as rigid bodies, can in no way be incorporated into the system of electrodynamics built on the principle of relativity, in which no analog is known of a rigid body in arbitrary accelerated motion. However, given the fact that all the basic expressions of Lorentz's theory of the electron seem to be independent of the hypotheses concerning the atomistic electron, the inertia of a continuously flowing charge can be likewise electromagnetically established in the sense suggested above. Naturally, this conception in no way contradicts those physical facts that indicate an extraordinarily strong, variable (almost atomistic) spatial distribution of matter and electricity.[61]

61. Born 1909, 572-573 (Italics in the original): "*Dabei ist hervorzuheben, da? von atomistischen Vorstellungen kein Gebrauch gemacht wird*. In der Tat ordnet sich das Atom oder das Elektron, als starrer Körper vorgestellt, auf keine Weise in das System der auf das Relativitätsprinzip aufgebauten Elektrodynamik ein, in der ein Analogon zum starren Körper f?r beliebig beschleunigte Bewegungen nicht bekannt ist. Wie aber alle wesentlichen Aussagen der Lorentzschen Elektronentheorie von der Vorstellung des atomistischen Elektrons unabhängig zu sein scheinen, lä?t sich auch die Trägheit einer *kontinuierlich strömenden Ladung* in der angedeuteten Weise elektromagnetisch begr?nden. Nat?rlich widerspricht diese Auffassung in keiner Hinsicht denjenigen physikalischen Tatsachen, welche auf eine räumlich au?erordentlich stark wechselnde (nahezu atomistische) Verteilung der Elektrizität und der Materie hinweisen."

It is likely that Born himself was inclined to adopt wholly the electromagnetic view entailed by Abraham's theory (although I would suggest that this accepted view of Born should be at least carefully re-examined in view of the analysis presented here for Minkowski). In this article, however, he expressed complete allegiance to Minkowski's viewpoint, explaining that his results do not presuppose any particular conception of the ultimate nature of physical phenomena.

Born's second publication that year on the same topic is his better-known paper containing the definition of rigid bodies, submitted on June 13. Born asserted that his definition of rigidity would play a role in the Maxwellian electrodynamic similar to that of the classical rigid body in Newtonian mechanics. He was now more ready to express opinions on fundamental issues openly, yet he preserved much of Minkowski's characteristic caution. His theory, he thought, accounted for the atomistic structure of electricity in a way that Abraham's theory had not. It thus corresponded to the "atomistic instinct" of so many experimentalists, who found it very hard to support recent attempts to describe the movement of electricity as a fluid, unconstrained by any kinematic conditions, and affected only by the action of its own field.[62] But on the other hand, as his motivation for undertaking this analysis Born did invoke concerns like those repeatedly stressed by Minkowski in his own work: to allow for a further clarification of the conceptual relationship between electrodynamics and the principle of relativity. This view, which is manifest in various places in Born's paper, is best encapsulated in the following passage:

> *The practical value of the new definition of rigidity must manifest itself in the dynamics of the electron*. The greater or lesser transparency of the results obtained by means of it will also be used, to a certain extent, for or against making the assumption of the principle of relativity universally valid, since experiments have not yet provided a definite proof of it and perhaps never will.[63]

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62. Born 1909a, 5-6. Born cited here two recent papers by Tullio Levi-Civita.

63. Born 1909a, 4 (Italics in the original): "*Der praktische Wert der Neudefinition der Starrheit mu? sich also an der Dynamik des Elektrons erweisen*; die grö?ere oder geringere Durchsichtigkeit der dabei erzielten Resultate wird dann bis zu gewissem Grade auch f?r oder gegen die Annahme des Relativitätsprinzip ?berhaupt geltend zu machen, da die Experimente wohl noch keine eindeutige Weisung gegeben haben und vielleciht auch nicht geben werden."

## 6. Concluding Remarks

In this article I have argued that to understand the proper historical context of Minkowski's work on relativity one must consider it against the background of the ideas that animated Hilbert's program for the axiomatization of physics. The latter was initially conceived together with the consolidation of Hilbert's axiomatic treatment of geometry, and partly under the influence of Minkowski's current enthusiastic engagement with Hertz's ideas. Hilbert's program was announced in 1900 to a larger audience as the sixth of his list of twenty-three problems. Then in 1905, Hilbert lectured in Göttingen on the logical principles of mathematics and devoted a considerable part of his course to the details of the proposed implementation of the program for various physical disciplines. Just at this time, and over the following three years, Minkowski and Hilbert were collaborating intensively in teaching and lecturing on electrodynamics and related issues. Clearly, the ideas manifest in Hilbert's treatment of physical theories are also part of the scientific and mathematical background that informed Minkowski's work on electrodynamics and the principle of relativity.

Minkowski was interested in exploring the logical consequences of assuming the universal validity of Lorentz covariance for all physical disciplines. This assumption had been strongly suggested by experimental results obtained during the late nineteenth century, and its theoretical implications had been investigated from different perspectives in recent works, noticeably those of Lorentz, Poincaré and Einstein. Yet, in a spirit like that underlying Hilbert's program, Minkowski believed that the logical structure of the physical theories built on the principle of relativity had not been satisfactorily elucidated and he set out to do so. The postulate of relativity should be taken as a further axiom appearing at the base of each and every physical theory, together with the particular axioms of that theory. In his work Minkowski was able to prove for certain domains of physics that the ensuing theory indeed produced a consistent logical structure. For some other theories, such as gravitation, he was less successful, but he always declared his conviction that he had shown how a consistent Lorentz covariant theory of gravitation could eventually be worked out in detail.

But the postulate of relativity was for Minkowski not simply an additional axiom, with perhaps a wider domain of validity in physics than others. It was an axiom of a different nature: a principle that should be valid for every conceivable physical theory, even those theories that were yet to be discovered or formulated. This kind of universal physical principle had also been introduced by Hilbert in his treatment of physical theories. In particular Hilbert attributed an enormous importance to what he called the principle of continuity, and to the requirement that all physical processes be represented by continuous functions having at least one derivative. For several theories he elaborated in some detail the consequences of including these general principles among the axioms. Minkowski compared the status of the postulate of relativity with that of the principle of conservation of energy, whose validity we assume even for yet unknown forms of energy. Of course, this very comparison had been drawn earlier by Einstein (Einstein 1907). Minkowski was very likely aware of this specific article of Einstein, if only because it appeared in the *Annalen der Physik* as a reply to an earlier article of Paul Ehrenfest, who at that time was in Göttingen. But Einstein and Minkowski compared relativity and conservation of energy in different ways. Einstein spoke in his article of two "open" principles of physics, with a strong heuristic character. Unlike Minkowski and Hilbert, Einstein did not see the principle of relativity and the principle of energy conservation as parts of strictly deductive systems from which the particular laws of a given domain could be derived.[64] More generally, although Einstein introduced the principle of relativity together with the constancy of light at the beginning of his 1905 article as "postulates" of the theory (in some sense of the word), it is necessary to draw a clear difference between what he did and what I have stressed here as Minkowski's axiomatic analysis of the postulate of relativity.[65] In fact, one of the main explicit aims of Hilbert's program was to address situations like that created here by Einstein, which he saw as potentially problematic. As Hertz had pointed

64. Cf. Einstein 1907, 411: "Es handelt sich hier also keineswegs um ein 'System', in welchem implizite die einzelnen Gesetze enthalten wären, und nur durch Deduktion daraus gefunden werden könnten, sondern nur um ein Prinzip, das (ähnlich wie der zweite Hauptsatz der Wärmetheorie) gewisse Gesetze auf andere zur?ckzuf?hren gestattet."

65. On the other hand, Minkowski's axiomatic approach, and in particular his stress on universally valid principles in physics, strongly brings to mind Einstein's oft-quoted remarks on the differences between theories of principle and constructive theories. See, e..g, Stachel et al (eds.) 1989, xxi-xxii. It would be beyond of the scope of this article to compare these views in detail.

out in the introduction to his *Principles of Mechanics*—to which Hilbert used to refer when explaining the need for axiomatizing physical theories—it has often been the case in the history of physics that, faced with conflict between an existing theory and new empirical findings, physicists have added new hypotheses that apparently resolve the disagreement but perhaps contradict some other consequences of the existing theory. Hilbert thought that an adequate axiomatic analysis of the principles of a given theory would help to clear away possible contradictions and superfluities created by the gradual introduction of new hypotheses into existing theories. This was also what Minkowski was pursuing: he sought to verify that the introduction of the principle of relativity need not create such a problematic situation.

At this point I hasten to add that my refernces here to Einstein's treatment by no means imply that Minkowski's work in electrodynamics and on the postulate of relativity should be understood mainly in comparison with Einstein's. This comparison is likely to prove misleading not only with regard to the role of axioms in the theory, but also in a much broader sense. In fact, as stated in the introduction, one of the problems of historically interpreting Minkowski's and Hilbert's work in this domain has been precisely the excessive stress on Einstein's work as the adequate and inevitable main frame of reference. Whether Minkowski understood the significance of Einstein's contribution, and the extent to which their two theories correspond, are certainly two important questions that deserve attention; but by excessively focusing on them one risks losing sight of the more direct historical context of Minkowski's own endeavor. For the latter, Einstein's work was only one among a larger collection of significant contributions that—against the background of his own conception of the relationship between mathematics and physics—attracted Minkowski's attention and drew his interests to this domain.

One of the central points that emerges from studying Minkowski's work within its proper context, and one which is strongly suggested by the proximity of Hilbert's program, is the idea that the place of the postulate of relativity in physics could be fully analyzed without assuming, and certainly without committing oneself to, any particular conception of the ultimate nature of physical phenomena. I do not, however, mean to suggest that Minkowski had no clear position of his own on these issues; he most certainly had. We know, for example, that Hilbert was sympathetic to the mechanistic world-view until 1913, when he changed his position diametrically and adopted the electromagnetic world-view, based on Gustav Mie's theory of matter.[66] What then was Minkowski's posi-

tion in this respect? I have found no direct evidence to answer this question. However, Minkowski's admiration for Hertz and the fact that Hilbert sided with the mechanistic world-view in 1910 when lecturing on mechanics under the declared influence of Minkowski's ideas[67] may tend to suggest that this was also the latter's view. This question must remain open until further evidence is found.

The axiomatizing motivation behind Minkowski's work, which I have stressed throughout this article, provides the main perspective from which to understand the roots and the goals of his overall involvement with electrodynamics and relativity. At the same time, however, it must be repeated that this was only one among the elements that informed his much more complex mathematical and physical background. The geometric element of this background, for instance, is one that has received much attention in the secondary literature, and must certainly be taken into account. Still, there are several reasons why one should be cautious in assessing its actual significance. For one, the very terms "geometry" and "geometrical" are much too comprehensive and sometimes imprecise. They need to be sharpened and placed in proper historical context if they are to explain in some sense Minkowski's motivations or the thrust of his articles on electrodynamics.[68] One should be able to describe, for instance, what were Minkowski's views on some of the basic, foundational questions of geometry—questions to which many mathematicians dedicated their efforts during the last part of the nineteenth century and which led Hilbert to pursue this field actively.[69] Further research must be done on Minkowski's mathematics before questions like these can be answered properly. One particular, closely related, point can neverthless be briefly mentioned here.

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66. Hilbert's change of position on this issue is discussed in detail in Corry 1997b, § 6.

67. As manifest, i.e., in Hilbert's lecture notes: see Hilbert 1910-1, 295.

68. Galison 1979, for instance, interprets Minkowski's work mainly with reference to his "visual-geometric thinking." Without a more precise explanation of what "geometry" means in this context, however, I find it hard to assess the validity of claims such as that for Minkowski "the visualization of nature's laws through geometry enters as the primary motivation for the creation of a new physical and metaphysical outlook" (p. 117), or similar ones throughout his paper.

69. See Corry 1997, 6-9.

Elucidating the specific nature of Minkowski's conception of geometry becomes particularly important if we are to understand why, once he decided to undertake the axiomatic clarification of the role of the principle of relativity in physics, Minkowski came forward with a space-time geometry as an essential part of his analysis. Of primary interest in any discussion of this issue must be the connection between groups of transformations and geometry, which in "Space and Time", as was seen above, becomes a focal point of Minkowski's analysis. Felix Klein was evidently very excited about this particular feature, and as early as April 1909 he declared that the ideas behind Minkowski's study of the Lorentz group had in fact been anticipated by himself, in the framework of his Erlanger program dating back to 1872.[70] On the other hand, when lecturing in 1917 on the history of mathematics in the nineteenth century, Klein remarked that among Minkowski's four papers he liked the first one most. Klein stressed the invariant-theoretic spirit of this paper as the faithful manifestation of Minkowski's way of thought.[71] Minkowski, for his part, did not mention Klein's ideas at all in his own articles, and one wonders what would have been his reaction to Klein's assessments, had he lived to read them. Although the connections suggested by Klein between the Erlanger program and the group-theoretical aspects of relativity in Minkowski's work may seem in retrospect clearly visible, it is important to examine them with some care, since the actual historical influence of the Erlanger program was slighter than is sometimes assumed and than Klein would have had others believe.[72] There is no direct evidence that Minkowski was thinking literally in terms of

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70. Klein expressed these views in a meeting of the Göttingen Mathematical Society, and they were published as Klein 1910.

71. Klein 1926-7 Vol. 2, 74-75. Klein contrasted this paper with the *Grundgleichungen* in which—in order not to demand previous mathematical knowledge from his audience—Minkowski had adopted a more concise, but somewhat ad-hoc, matricial approach. The latter, Klein thought, was perhaps more technically accessible, but also less appropriate for expressing the essence of Minkowski's thoughts.

72. As Hawkins 1984 has convincingly argued, the general question of the relationship between groups of motions and geometry was of interest to a relatively wide circle of mathematicians during the last quarter of the past century, and Klein's ideas were a part of this more general trend. The Erlanger program began to attract real interest as a program for actual research in the late 1880s with the work of the Italian school, and more so after 1892 mainly through the work of Poincaré and of Sophus Lie's students.

the Erlanger program when elaborating his own ideas on space and time;[73] on the other hand, the more general idea that geometries can be characterized in terms of their groups of motions was by then widely accepted, and was certainly part and parcel of Hilbert's and Minkowski's most basic mathematical conceptions.

The first to establish the explicit connection between the terminology and the ideas of group theory and the Lorentz covariance of the equations of electrodynamics was Poincaré, in his 1905 article. Remarkably, he had also been the first to use four-dimensional coordinates in connection with electrodynamics and the principle of relativity. Minkowski, on the other hand, was the first to combine all these elements into the new conception of the four-dimensional manifold of space-time, a conception that emerged fully-fledged only in his 1908 Köln lecture. What was the background against which Minkowski was led to take a step beyond the point that Poincaré had reached in his own work, and thus to introduce the idea of space-time as the underlying concept that embodies the new conception of physics? It is here that a more detailed analysis of Minkowski's geometry and of the role of the ideas associated with the Erlanger program has to be brought to bear. At the same time, such an analysis would also have to make reference to two ideas that have been mentioned in the foregoing pages. First, it is perhaps at this particular point that the specific impact of Einstein's work on Minkowski may have been decisive. As I said above, Minkowski was very much impressed by Einstein's contribution to modifying the traditional concept of time; following this train of thought Minkowski proposed to do something similar for the concept of space, replacing it by a four-dimensional geometry of space-time. Second, ideas like those put forward by Hilbert in his lectures on physics may have afforded Minkowski a conceptual framework within which to combine the various mathematical elements manifest in his work. As I said in discussing

73. Remarkably, although Klein's assessments appear in widely-known sources, to the best of my knowledge the connection between Minkowski's space-time and the ideas associated with Klein's Erlanger program has only been explicitly noticed in the recent secondary literature in Norton 1993, 797. Norton raises an important point when he claims that "the notion of spacetime was introduced into physics almost as a perfunctory by-product of the *Erlangen* program," but it seems to me that this compact formulation does not account for the full complexity of the ideas involved here. In particular, this formulation would seem to imply that the Erlanger program subsumed all the contemporary work on the relations between geometry and groups of transformations, an assumption that needs to be carefully qualified (see the preceding footnote).

the introduction to “Space and Time”, in explaining his motivation for studying kinematics with tools usually applied in geometry Minkowski referred to the separation between these two domains that was assumed both in existing axiomatic analyses and in group-theoretical investigations—a point that had emerged very suggestively in Hilbert’s 1905 lectures.

A final point to be considered in this context is Hilbert’s evaluation of the significance of Minkowski’s work in electrodynamics. Such an evaluation is expressed, in the first place, in Hilbert’s unpublished lecture notes. Hilbert’s course in the summer semester of 1908 dealt with foundational questions of mathematics and the place of the axiomatic method in addressing them. Hilbert described the recent work of Minkowski in electrodynamics as an axiomatic investigation, along the lines of his own work on the foundations of geometry, of the basic principles of this “most difficult domain of mathematical physics” (Hilbert 1908, 5). A more detailed explanation of this view appears in Hilbert’s obituary of his deceased friend. Not surprisingly, perhaps, in the section where he discussed Minkowski’s work on physics, Hilbert focused on the *Grundgleichungen* and emphasized the importance of its axiomatic component and the precise mathematical formulation of the World-postulate. Minkowski’s most significant, positive contribution to electrodynamics, Hilbert claimed, was his derivation of the equations for moving matter from the World-postulate, together with the three axioms appearing there (discussed above). The correct form of these equations had been an extremely controversial issue among physicists, but Minkowski’s equations—Hilbert said in his typically categorical and unqualified fashion—were completely transparent and certain, and, in addition, they fitted all known empirical data.[74] On the other hand, Hilbert did not make any connection between the question of the ultimate nature of physical phenomena and Minkowski’s work, and barely mentioned the geometrical aspects of the latter. As with Klein’s assessment quoted above, one may of course doubt whether Hilbert’s opinion fairly reflected Minkowski’s own evaluation of the significance of the main contributions of his work. In fact, Hilbert had a very marked tendency to reinterpret other people’s thoughts, so as to make them fit his own current picture of the domain in question. The manuscript of his lecture notes of 1905 shows how he did this systematically for works in diverse fields of physics and mathematics. But given Hilbert’s close association with Minkowski, and especially their collabora-

74. See Hilbert 1909, 93-94.

tion during the latter's last years, and given the analysis of Minkowski's work presented in this article, I think that in this case one can take Hilbert's word as representing quite accurately the kind of emphases that Minkowski himself might have adopted if asked to assess his own work.

The subsequent development of the theory of relativity can hardly be told without referring to the enormous influence of Minkowski's contributions. The space-time manifold as well as the four-vector language became inseparable from the fundamental ideas introduced by Lorentz, Poincaré and Einstein. Minkowski's term "World-postulate", however, was not so enthusiastically adopted, and even less so was his insistence on the need to reform all branches of physics by performing a systematic, axiomatic analysis of them in terms of the World-postulate. The main obstacle in actually undertaking such a reform was, of course, gravitation. In general, physicists in later works did not accord any particular importance to Minkowski's specific axioms for the equations of moving matter. Moreover, the significance of an axiomatic analysis of the kind practiced by Minkowski for electrodynamics and the postulate of relativity never became a central issue among most physicists and mathematicians dealing with relativity. The outstanding exception to the latter rule was Hilbert, who over the years following Minkowski's death continued to insist in his lectures, at least at the declarative level, on the need for an axiomatic treatment of physical theories, and to point out the importance of Minkowski's contribution in this regard. Eventually, when in 1915 Hilbert dedicated his efforts to finding generally covariant field-equations of gravitation, he certainly saw himself as following in the footsteps of Minkowski's earlier work.

But Hilbert's work on physics did not gain the widespread acceptance accorded by physicists to that of his friend. For instance, it is well known that Einstein, in a letter to Hermann Weyl, judged Hilbert's approach to the general theory of relativity to be "childish, just like an infant who is unaware of the pitfalls of the real world."[75] Einstein believed that Hilbert had correctly addressed many of the central open issues of the theory, but that the axiomatic method had been of little help in this. Weyl himself considered that Hilbert's

75. In a letter of November 23, 1916. Quoted in Seelig 1954, 200.

work in physics—and especially his application of the axiomatic method—was of rather limited value compared to that in pure mathematics. A valuable contribution to physics, Weyl thought, required skills other than those in which Hilbert excelled. In an obituary of Hilbert, Weyl wrote:

> The maze of experimental facts which the physicist has to take into account is too manifold, their expansion too fast, and their aspect and relative weight too changeable for the axiomatic method to find a firm enough foothold, except in the thoroughly consolidated parts of our physical knowledge. Men like Einstein and Niels Bohr grope their way in the dark toward their conceptions of general relativity or atomic structure by another type of experience and imagination than those of the mathematician, although no doubt mathematics is an essential ingredient.[76]

Yet in spite of the different receptions generally given to Minkowski's and to Hilbert's contributions to physics, it is interesting that Weyl's retrospective judgment of Minkowski's contributions stressed the same kind of limitations that, in his opinion, affected the Hilbert's also. Writing in 1947 to Minkowski's sister, Fanny, Weyl said:

> Someone who contributes to a field foreign to himself is easily inclined, in the pride of also having mastered something foreign and lacking an overall view, to make an exaggerated assessment of his contribution. The lecture ["Space and Time"] suffers also from the fact that he wanted to fix or immortalize a transitional phase in physics.[77]

Thus, from a distance of seven years separating them, the contributions of the two Göttingen mathematicians to these fundamental issues in physics appear to be connected—by their similar motivations, by their conceptions of the role of mathematics in science and of the role of the axiomatic analysis of mathematical theories, and perhaps even by their limitations.

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76. Quoted in Sigurdsson 1994, 363.

77. Weyl's letter is quoted in Sigurdsson 1994, 365.

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