'Axiomatics, Empiricism, and Anschauung in Hilbert's Conception of Geometry: Between Arithmetic and General Relativity'

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To what extent the philosophy of mathematics of any individual mathematician is relevant to historically understanding his mathematical work, and to what extent his mathematical work has any bearing in understanding philosophical issues related with mathematics, are questions that have different meanings and have to be approached differently when they refer to different mathematicians. Take, for example, Descartes and Frege. These two thinkers can be considered philosophers in the strict sense of the word, with philosophical interests going well beyond the strict scope of mathematics, each of them in his own way. They devoted much of their time and efforts to develop coherent, well-elaborated philosophical systems, and their writings turned them into philosophers in the eyes of the philosophical community. Their philosophical systems are directly relevant to addressing central questions pertaining to the nature of mathematical knowledge, but they were not intended exclusively as answers to specific problems in the philosophy of mathematics. And besides their intense involvement with philosophical questions, both Descartes and Frege contributed positive mathematical results of various kinds, albeit of different overall impact on mathematics at large, and while working under quite different professional circumstances. A natural question that the historian may be easily led to ask in relation to these two thinkers concerns the mutual relationship between the philosophical systems they developed and the mathematics that each of them produced. One way to answer this question is by investigating, separately, the philosophy and the mathematics of each of them, and then trying to articulate the said relationship.

Descartes and Frege, however, are far from representative of the mainstream mathematician in any given period. Most mathematicians devote little or no effort to philosophical questions in general, and, in particular, they devote little time to formulate coherent philosophical systems. A historian investigating the work of an individual mathematician of the more mainstream kind may attempt to reconstruct her putative philosophy by analysing her mathematical work, and by trying to illuminate the philosophical preconceptions underlying it. The historian may likewise be led to ask for the roots and the background of the philosophical views thus embodied in the mathematician's work. In cases where the mathematician in question has also left some philosophical or quasiphilosophical texts, the historian may try to assess their value and their relationship with her actual mathematical practice. This exercise may be more or less interesting according to the individual mathematician involved, and in many cases it may be of rather limited consequence.

The case of David Hilbert is particularly appealing when seen from the perspective of the spectrum whose two extreme points I outlined above. His contributions to the foundations of geometry led to momentous changes in the most basic conceptions about the nature of this discipline and of mathematics in general. He developed close connections and meaningful intellectual interchanges with influential philosophers in Göttingen, such as Leonard Nelson and Edmund Husserl. He made important contributions to the foundations of logic and of arithmetic, and the finitist program he initiated in this context turns him into a natural focus of philosophical interest. From the point of view of the philosophical discourse about mathematics in the twentieth century Hilbert's name remained intimately linked to the idea of a formalist conception of mathematics as a mainstream interpretation of the nature of this discipline. Hilbert was prone to express ideas of a philosophical or quasiphilosophical tenor and a great many of his pronouncements have remained on the written record. These pronouncements are rich in ideas and they are very illuminating when trying to reconstruct the intellectual horizon within which he produced his mathematics. Still, the picture that arises from this abundance of activities and sources is by no means that of a systematic philosopher. Nor is there any solid reason to expect it to be so. After all, Hilbert was a working mathematician continually involved in many threads of research activity in various fields of mathematics, pure and applied, and he had neither the time nor, apparently, the patience and the kind of specifically focused interest, to devote himself to the kind of tasks pursued by philosophers.

Rather than trying to construe a fully coherent picture of what would be a putative philosophy of mathematics of Hilbert—similar to what one could do for Descartes or Frege, for instance—that would allow analysing his entire mathematical horizon from a single, encompassing perspective, in the present chapter I will suggest that it is more convenient to speak in terms of his 'images of mathematics' and their development throughout the years, and to analyse—in terms of the latter—separate aspects of the enormous body of scientific knowledge that can be attributed to him. In this chapter I will focus the discussion on Hilbert's approach to geometry.

Elsewhere, I have elaborated in greater detail the distinction between 'body' and 'images' of mathematical knowledge,[1] and the possible ways to use these concepts in investigating the history of mathematics. For the purposes of the present discussion it will suffice to point out that this is a flexible, schematic distinction focusing on two interconnected layers of mathematical knowledge. In the body of mathematics I mean to include questions directly related to the subject matter of any given mathematical discipline: theorems, proofs, techniques, open problems. The images of mathematics refer to, and help elucidating, questions arising from the body of knowledge but which in general are not part of, and cannot be settled within, the body of knowledge itself. This includes, for instance, the preference of a mathematician to declare, based on his professional expertise, that a certain open problem is the most important one in the given discipline, and that the way to solve it should follow a certain approach and apply a certain technique, rather than any other one available or yet to be developed. The images of mathematics also include the internal organization of mathematics into subdisciplines accepted at a certain point in time and the perceived interrelation and interaction among these. Likewise, it includes the perceived relationship between mathematics and its neighbouring disciplines, and the methodological, philosophical, quasiphilosophical, and even ideological conceptions that guide, consciously or unconsciously, declared or not, the work of any mathematician or group of mathematicians.

Examining a mathematician's work in terms of the body and the images of mathematical knowledge allows us to focus on the role played by philosophical ideas in his work, without thereby assuming that these ideas must be part of a well-elaborated system that dictates a strict framework of intellectual activity. Rather, one may consider these images as a historically conditioned, flexible background of ideas, in a constant process of change, and in mutual interaction with the contents of the body of mathematics, on the one hand, and with external factors, on the other hand. The images of mathematics of a certain mathematician may contain tensions and even contradictions, they may evolve in time and they may eventually change to a considerable extent, contradicting at times earlier views held by her. The mathematician in question may be either aware or unaware of the essence of these images and the changes affecting them.

The body/images scheme turns out to be useful for analysing Hilbert's conceptions, especially concerning his putative, 'formalist' views on mathematics. As already said, to the extent that Hilbert's name is associated with any particular philosophical approach in mathematics, that approach is formalism. This association, however, is rather misleading on various counts. For one thing, it very often conflates two different meanings of the term 'formalism'. Thus, from about 1920 Hilbert was indeed involved in a program for proving the consistency of arithmetic

[1] Corry 2001; 2003.

based on the use of strictly finitist arguments. This program was eventually called the 'formalist' approach to the foundations of mathematics, and it gained much resonance as it became a main contender in a well-known and unusually heated debate known as 'the crisis of foundations' in mathematics. Associating Hilbert with this sense of the word 'formalism' is essentially correct, but it says very little about Hilbert's images of mathematics. The term formalism, at any rate, was not used by Hilbert himself in this context, and it is somewhat misleading. 'Hilbert's Programme' and 'Finitism' have become accepted, and much more appropriate, alternatives.[2] 'Formalism', however, is far from accurately describing Hilbert's *images* of mathematics.

Indeed, a second meaning of the word formalism is associated with the general attitude towards the practice of mathematics and the understanding of the essence of mathematical knowledge that gained widespread acceptance in the twentieth century, especially under the aegis of the Bourbaki group. As these two meanings came to be conflated in an interesting historical process, Hilbert, the formalist in the more reduced sense of the term, came to be associated also with this second sense, not the least because of Bourbaki's efforts to present themselves as the 'true heirs of Hilbert'. Thus Jean Dieudonné explained the essence of Hilbert's mathematical conceptions in a well-known text where he referred to the analogy with a game of chess. In the latter, he said, one does not speak about truths but rather about following correctly a set of stipulated rules. If we translate this into mathematics we thus obtain the conception of Hilbert: 'mathematics becomes a *game*, whose pieces are graphical *signs* that are distinguished from one another by their form.'[3]

On the face of it, one should not be too surprised to realize how widespread the image of Hilbert the formalist became. It is not only the dominance of formalist approaches in twentieth-century mathematics, and more specifically of the Bourbakist approach. It is also that, given this dominance, Hilbert's important early work on the foundations of geometry could easily be misread in retrospect as a foremost representative of this trend in mathematics. And yet, the historical record contains as many important contributions of Hilbert that could hardly be seen as embodying any kind of formalist approach. This is especially—but not exclusively—the case when one looks at his contributions to physics. It is thus remarkable that this side of Hilbert's works was in many cases systematically overlooked as it did not fit his widespread image as the paradigmatic twentieth-century mathematical formalist.

This view of Hilbert as a formalist, in the more encompassing sense of the word, has been consistently criticized for a few years now. Likewise, it is only relatively recently that the real extent and depth of Hilbert's involvement with

[2] Detlefsen 1986.

[3] Dieudonné 1962, 551. A similar view is presented in Kleene 1952. Also Weyl (1925–27, 127) refers to the chess metaphor in describing Hilbert's quest to '*formalize* mathematics', but he clearly states that Hilbert followed this approach in order to secure 'not the *truth*, but the *consistency* of the old analysis.'

physics became well known (Corry 2004). Nevertheless, some of his specific contributions to physics were never a secret. This is particularly the case with his solution of the Boltzmann equation, on the one hand, and, on the other, with the formulation of the field equations of general relativity (GTR) followed by a continued involvement with this discipline. Curiously enough, the well-known, enormous impact of GTR on the perceived relationship between geometry and physics has consistently been described in the literature, not infrequently coupled to the claim that GTR turned geometry into a branch of physics. How could then a mathematician like Hilbert be intensely involved in the decisive stages of the development of this discipline around 1915 and at the same time hold a purely formalistic view of geometry from at least as early as 1900? This tension was more easily ignored than explained by anyone who accepted at face value the description of Hilbert as the ultimate formalist.

As will be seen below, and contrary to that view, Hilbert's actual conceptions about the essence of geometry throughout his career fitted very naturally the kind of intimate association postulated by GTR between this discipline and physics, and as a matter of fact this is what led him to be among the first to focus on some of the important issues arising from GTR in regard with this association. It is noteworthy, however, that Einstein himself took for granted the kind of separation between mathematics (particularly geometry) and the physical world that formalist views of geometry tend to favour, and he was prone to attribute such a view to Hilbert. In his talk, 'Geometry and Experience', presented at the Berlin Academy of Sciences on January 27, 1920, Einstein famously asserted that:

> Insofar as the theorems of mathematics are related to reality, they are not certain; and insofar as they are certain, they are not related to reality. (Einstein 1921, 4)

In his view, this relatively recent conception 'first became widespread through that trend in recent mathematics which is known by the name of "Axiomatics." Thus, even though Einstein did not say it explicitly, the context makes it clear the he was referring here to the axiomatic approach developed by Hilbert, as he understood it. For very different reasons both Einstein and Dieudonné coincided in associating Hilbert's conception of geometry with mathematical formalism.

In the present chapter I discuss some of the central images of mathematics, and particularly of geometry, espoused by Hilbert throughout his career. Amid significant changes at several levels, these images never envisaged formalist considerations as a possible way to explain the essence of geometry. In fact, perceptual experience and intuition (*Anschauung*) in the Kantian sense of the term (as Hilbert understood it) are the two main motives of Hilbert's images of elementary geometry. The axiomatic analysis of scientific theories, which provides the methodological backbone and the main unifying thread of Hilbert's overall images of science, was not meant as a substitute for these two main components. Rather, it embodied Hilbert's preferred way to organically combine and articulate them in the framework of a regulative 'network of concepts' (*Fachwerk von Begriffe*) that helps clarify their logical interrelationship.

Perceptual experience and intuition appear in constant interaction in Hilbert's writings and lectures, with their relative importance and the kind of interplay affecting them undergoing subtle changes along the years. One of the interesting consequences of Hilbert's involvement with GTR was that the delicate balance that existed between experience and *Anschauung* in Hilbert's images of geometry was finally disrupted in favour of experience, and decidedly away from intuition.

Before entering into the details of this discussion, and in order to conclude the introduction, it is pertinent to introduce here a quote of Hilbert from around 1919, the time when he began to work out in collaboration with Bernays his finitist program for the foundations of arithmetic. Even if the formalist aspects of this program (in the more restricted sense of the term) may have already begun to emerge at this stage in his work, they were certainly circumscribed to the question of the proof of consistency for arithmetic. As for the essence of mathematical knowledge in general, Hilbert stated a view totally opposed to that attributed to him many years later by Dieudonné. Thus Hilbert said:

> We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise. (Hilbert 1919–20, 14)

Roots and Early Stages

Hilbert's work in geometry, its contents, its methodology and the conceptions associated with the discipline, has as its focal point the introduction and further development of the new axiomatic approach that came to be associated with his name. For all of its innovative aspects, this approach had deep roots in a complex network of ideas developed in the second half of the nineteenth century in research on the foundations of fields as diverse as geometry, analysis, and physics, which deeply influenced him. Moreover, the essentially algebraic outlook that permeated all of Hilbert's work (including his work in fields like analysis and the foundations of physics) played a major role in shaping his axiomatic approach.

Prior to his arrival in Göttingen in 1895, Hilbert had lectured on geometry at Königsberg, and it is interesting to notice how some of the topics that will eventually become central to his mathematical discourse already emerge at this early stage. Thus, for instance, in a lecture course on Euclidean geometry taught in 1891 Hilbert said:

> Geometry is the science dealing with the properties of space. It differs essentially from pure mathematical domains such as the theory of numbers, algebra, or the theory of functions. The results of the latter are obtained through pure thinking … The situation is completely different

> in the case of geometry. I can never penetrate the properties of space by pure reflection, much the same as I can never recognize the basic laws of mechanics, the law of gravitation or any other physical law in this way. Space is not a product of my reflections. Rather, it is given to me through the senses. (Quoted in Hallet and Majer (eds.) 2004, 22)

Separating mathematical fields into two different classes, one having its origin in experience and one in pure thinking, is a motive that had traditionally found a natural place in German mathematical discourse for many generations now. It had been particularly debated, and adopted with varying degrees of commitment, among the Göttingen mathematicians since the time of Gauss (Ferreirós 2005?). In the passage just quoted, Hilbert fully endorsed the separation, and thus geometry and all physical domains appear here together on the same side of a divide that leaves in the second side those purely mathematical disciplines for which 'pure thinking', provides the main foundation. Thus, while Hilbert's views images geometry reflect strongly empiricistic conceptions, concerning arithmetic he adopted in the early years of his career an essentially logicist point of view, strongly influenced by Dedekind (Ferreirós 2005??).

Within a strongly empiricistic conception of geometry such as expressed here, the axioms play a clearly defined role that is not different from the role they might play for any other *physical* discipline. Thus, it should come as no surprise that, when Hilbert read Hertz's book on the principles of mechanics very soon after its publication in 1893, he found it highly congenial to his own conceptions about the role of axioms in geometry and became strongly influenced by it. The axioms of geometry and of physical disciplines, Hilbert said in a course of 1893, 'express observations of facts of experience, which are so simple that they need no additional confirmation by physicists in the laboratory.'[4]

The empiricist images characteristic of his early courses, especially concerning the status of the axiom of parallels, is also manifest in his consistent references to Gauss's experimental measurement of the sum of angles of a triangle formed by three mountain peaks in Hannover.[5] Hilbert found Gauss's measurements convincing enough to indicate the correctness of Euclidean geometry as a true description of physical space. Nevertheless, he envisaged the possibility that some future measurement would yield a different result. The example of Gauss's measurement would arise very frequently in his lectures on physics in years to come, as an example of how the axiomatic method should be applied in physics, where new empirical facts are often found by experiment. Hilbert stressed that the axiom of parallels is likely to be the one to be modified in geometry if new experimental discoveries would necessitate so. Geometry was especially amenable to a full axiomatic analysis only because of its very advanced stage of development and elaboration, and not because of any other specific, essential trait concerning

4 Quoted in Hallet and Majer (eds.) 2004, 74.

5 See, for instance Hallet and Majer (eds.) 2004, 119–120.

its nature that would set it apart from other disciplines of physics.[6] Hilbert's empiricist image of geometry is epitomized in the following quotation taken from a 1898–99 lecture course on the foundations of Euclidean geometry:

> We must acknowledge that geometry is a *natural science*, but one whose theory can be *described as perfect*, and that also provides an example to be followed in the *theoretical treatment* of other natural sciences. (Quoted in Hallet and Majer (eds.) 2004, 221. Emphasis in the original)[7]

Grundlagen der Geometrie

A main topic in Hilbert's involvement with geometry between 1893 and 1899 was a detailed enquiry of the mutual relations between the main theorems of projective geometry and, specifically, of the precise role played by continuity considerations in possible definitions of purely projective co-ordinates and a purely projective metric. Foundational questions of this kind had been thoroughly investigated throughout the century by mathematicians such as Klein, Lie, Veronese, and, more recently, Ludwig Wiener (to mention just a few). The role of continuity considerations in the foundations of analysis and arithmetic had been systematically investigated by Dedekind in various works that Hilbert's was well aware of. As Dedekind developed a distinctly axiomatic way to pursue his own analysis of this question, there can be little doubt that his works provided an additional catalyst for Hilbert's own ideas that reached final consolidation by 1899.[8]

That was the year of publication of *Grundlagen der Geometrie*, the text of which elaborated on a course just taught by Hilbert. In the notes to a different course taught the same year, this one on mechanics, we find a balanced and interesting combination of the various topics that inform the basis of Hilbert's views on geometry. In the first place, there is the role of full axiomatization as a means for the proper mathematization of any branch of *empirical* knowledge:

> Geometry also [like mechanics] emerges from the observation of nature, from experience. To this extent, it is an *experimental science*. . . . But its experimental foundations are so irrefutably and so *generally acknowledged,* they have been confirmed to such a degree, that no further proof of them is deemed necessary. Moreover, all that is needed is to derive these foundations from a minimal set of *independent axioms* and thus to construct the whole edifice of geometry by *purely logical means*. In this way [i.e., by means of the axiomatic treatment] geometry is turned into a *pure mathematical* science. In mechanics it is also the case that all physicists recognize its most *basic facts*. But the *arrangement* of the basic concepts is still subject to

6 Hallet and Majer (eds.) 2004, 72.

7 See also, on p. 302: 'Geometry is the most perfect of (*vollkommenste*) the natural sciences.'

8 For details, see Corry 2004, 37–40.

> changes in perception . . . and therefore mechanics cannot yet be described today as a *pure mathematical* discipline, at least to the same extent that geometry is. (Quoted in Corry 2004, 90)

At the same time, however, the choice of axioms is also guided by the pervasive and subtle concept of 'Anschauung', whose actual nature, however, is preferably left out of the discussion:

> Finally we could describe our task as a logical analysis of our faculty of intuition (*Anschauungvermögen*). The question if our space intuition has a-priori or empirical origins remains nevertheless beyond our discussion.

This dilemma, whether the origin of the axioms of mathematics is empirical or is related to some kind of Kantian a-priori intuition, is never fully resolved in Hilbert's early lectures. A strong connection with other natural sciences is a main image of Hilbert's conception of geometry, and it continually strengthens his inclination to emphasize, at the epistemological level, the origins of axioms in perceptual experience. On the other hand, the unity of mathematics, a second main pillar of Hilbert's images of the discipline, underlies the stress on the connection with arithmetic and with a-priori intuition. It thus seems as if having a properly axiomatized version of geometry relieves Hilbert from the need to decide between these two alternatives: axiomatized geometry may equally well serve a thoroughly empiricistic or an aprioristic account of the essence of this discipline.

When Hilbert published his full-fledged axiomatized analysis of the foundations of geometry in *Grundlagen der Geometrie*, in the framework of a *Festschrift* to celebrate the unveiling of the Gauss–Weber monument in Göttingen, it was more than appropriate to open with a festive quotation from Kant. This famous quote states:

> 'All human knowledge thus begins with intuitions, proceeds thence to concepts and ends with ideas'.

One might attempt to analyse in detail the philosophical reasons for Hilbert' choice of this sentence and how the various terms (intuitions, concepts, ideas) relate, or perhaps do not relate, to Hilbert's own conceptions, to his declared views, and to his practice. This would require a thorough discussion of the relevant Kantian texts, and, more importantly, of how Hilbert's contemporaries (and possibly Hilbert himself) understood these texts. I think, however, that this complex exercise would not justify the effort and would not, in itself, shed much light on Hilbert's views. Hilbert clearly wanted, in the first place, to pay a due tribute to the towering figure of his fellow Königsberger at this very festive event. It remains a matter of debate, what this sentence exactly meant for Hilbert and whether or not it faithfully describes his 'true motivations' or the philosophical underpinnings of his work. In fact, there is some irony in the specific sentence that Hilbert chose to use from the Kantian corpus, and in which, of all terms, 'experience' is not mentioned in any way. Intuitions, concepts and ideas – all of these appear in varying degrees of importance in Hilbert's philosophical discourse about mathematics and in his images of geometrical knowledge. But 'perceptual experience' is the one whose paramount epistemological significance was never called into question by Hilbert. It was also the one that his involvement with GTR was to reinforce even more.

In his thoroughgoing exploration of the foundations of Euclidean geometry and of the fundamental theorems of projective geometry and their interdependences, Hilbert saw the culmination of a process whereby geometry turns into a 'purely mathematical' discipline. The above-mentioned, traditional divide with number theory and analysis could thus be overcome and Hilbert's continued quest for unity in mathematics and in the sciences gained additional strength. A fully axiomatized version of geometry thus embodied a network of concepts preserving meaningful connections with intuition and experience, rather than a formal game with empty symbols.

At the technical level, Hilbert undertook in *GdG* several tasks that became cornerstones of all foundational activities in mathematics for decades to come. Thus, Hilbert presented a completely new system of axioms for geometry, composed of five separate groups, and put forth a list of concrete requirements that his system should satisfy: simplicity, completeness, independence, and consistency. We briefly look now at each of these requirements.

Unlike the other requirements, simplicity is one that did not become standard as part of the important mathematical ideas to which GdG eventually led. Through this requirement Hilbert wanted to express the desideratum that an axiom should contain 'no more than a single idea.' However, he did not provide any formal criterion to decide when an axiom is simple. Rather, this requirement remained implicitly present in GdG, as well as in later works of Hilbert, as a merely aesthetic guideline that could not be transformed into a mathematically controllable feature.

The idea of a complete axiomatic system became pivotal to logic after 1930 following the works of Gödel, and in connection with the finitist program for the foundations of arithmetic launched by Hilbert and his collaborators around 1920. This is not, however, what Hilbert had in mind in 1899, when he included a requirement under this name in the analysis presented in GdG. Rather, he was thinking of a kind of 'pragmatic' completeness. In fact, what Hilbert was demanding here is that an adequate axiomatization of a mathematical discipline should allow for a derivation of *all* the theorems already known in that discipline. This was, Hilbert claimed, what the totality of his system of axioms did for Euclidean geometry or, if the axiom of parallels is ignored, for the so-called absolute geometry, namely that which is valid independently of the latter .

Also, the requirement of consistency was to become of paramount importance thereafter. Still, as part of GdG, Hilbert devoted much less attention to it. For one, he did not even mention this task explicitly in the introduction to the book. For another, he devoted just two pages to discussing the consistency of his system in the body of the book. In fact, it is clear that Hilbert did not intend to give a direct proof of consistency of geometry here, but even an indirect proof of this fact does not explicitly appear in *GdG*, since a systematic treatment of the question implied a full discussion of the structure of the system of real numbers, which was not included. Rather, Hilbert suggested that it would suffice to show that the specific kind of synthetic geometry derivable from his axioms could be translated into the standard Cartesian geometry, if the axes are taken as representing the entire field of real numbers. Only in the second edition of *GdG*, published in 1903, Hilbert added an additional axiom, the so-called 'axiom of completeness' (*Vollständigkeitsaxiom*), meant to ensure that, although infinitely many incomplete models satisfy all the other axioms, there is only one complete model that satisfies this last axiom as well, namely, the usual Cartesian geometry.

The requirement on which I want to focus in the context of the present discussion is the requirement of independence, and in particular, the fact that Hilbert analysed the mutual independence of the groups of axioms rather than the mutual independence of individual axioms. The reason for this was that for Hilbert each of these groups expresses one way in which our intuition of space is manifest, and he intended to prove that these are independent of each other. Of course, he paid special attention to the role of continuity considerations and the possibility of proving that continuity is not a necessary feature of geometry. This latter fact was previously known, of course, from the works of Veronese, but Hilbert's study of non-Archimedean geometries appeared here as part of a more systematic and thorough approach.

The focus of Hilbert on the groups of axioms as expression of our spatial capacities or intuitions stresses the non-formalistic essence of the views underlying his entire research. Although the clear, formal building of geometry that emerges from his study is in itself an important mathematical achievement with broad consequences, it by no means indicates an interest in presenting geometry as a purely formal game devoid of inherent meaning. The opposite is true: this successful mathematical exercise was meant to provide conceptual support to the centrality he attributed to empirical and intuitive experience as a basis for geometry. Hilbert did not really elaborate, however, a clear philosophical analysis of space and geometry around these elements. Rather this presentation of geometry successfully embodied a set of images of mathematics where the two elements, empirical experience and some version of Kantian *Anschauung*, could be effectively accommodated.

Roughly simultaneously with his detailed treatment of geometry, Hilbert also advanced a cursory discussion of the foundations of arithmetic, in a talk delivered in 1899 under the title of 'On the Concept of Number'.[9] Hilbert opened his discussion by stating that in arithmetic one is used mostly to the 'genetic' approach for defining the various systems of numbers. He was evidently referring to Dedekind's stepwise construction, starting from the naturals, and successively adding those new numbers that allow extending the operations so that they become universally applicable. The last step in this process is the definition of the real numbers as cuts

[9] Hilbert 1900.

of rationals.[10] Hilbert praised the advantages of this approach, but at the same time he intended now to propose that also arithmetic, like geometry, could and should be axiomatically built.

The axiomatic system he proposed for the real numbers is based on ideas he also used in *GdG* and by means of which the real numbers are characterized as an ordered, Archimedean field. To this characterization, however, Hilbert added now, under the heading of 'axioms of continuity', a new condition, namely, the already mentioned axiom of completeness (*Vollständigkeitsaxiom*). Like in geometry, the completeness of his axiom system (not to conflate with the axiom of completeness) was not a property he would know exactly how to handle, and he thus remained silent in relation to it after having mentioned it in the opening passages. Concerning the proof of consistency he simply stated that 'one needs only a suitable modification of familiar methods of inference',[11] but he did not provide further details about the kind of modification that he had in mind.

This talk of Hilbert has been repeatedly mentioned as a harbinger of the views that he would develop later concerning the foundations of arithmetic and of logic.[12] It would be beyond the scope of the present chapter to discuss that point. The lecture is relevant for the present discussion for the contrast it presents between the genetic and the axiomatic points of view. Hilbert found both of them to be legitimate, and as playing important, different roles. At the same time, however, he clearly stated that the logical soundness and the foundational stability of arithmetic is provided, above all *but not exclusively*, by the axiomatic method. At this stage of his career, Hilbert's views on arithmetic were still strongly influenced by Dedekind's logicistic attitudes, and this influence is clearly felt in this talk. However, also this aspect of his conceptions was to change, and in lecture courses he would teach in Göttingen over the next years, he preferred to stress the foundational contribution of intuition, in the sense of *Anschauung*, as part of the stability and soundness that the genetic method provided to arithmetic via the axiomatic method.

Lectures on the Axiomatic Method – 1905

In the period immediately following the publication of GdG Hilbert occupied himself briefly with research on the foundations of geometry, and so did some of his students, prominent among whom were Max Dehn and Georg Hamel. At the same time, Ernst Zermelo, who had arrived in Göttingen in 1897 in order to complete his *Habilitation* in mathematical physics, started now to address questions

[10] Dedekind 1888. Cf. Ferreirós 1999, 218–224.

[11] Hilbert 1900 (1996), 1095.

[12] See, for instance, Ewald (ed.) 1999, 1090–1092.

pertaining to the foundations of arithmetic and set theory. It was only with the publication of Russell's paradox in 1903 that these latter topics started to receive serious attention in Göttingen. It seems as if Hilbert had initially expected that the difficulty in completing the full picture of his approach to the foundations of geometry would lie in dealing with assumptions such as the *Vollständigkeitsaxiom*, but he now realized that the actual problems lay in arithmetic and even perhaps in logic. It was at this point that he started to seriously consider the possible use of the axiomatic method as a way to establishing the consistency of arithmetic.[13] Still, significant work was done only by Zermelo, who had just started working more specifically on open problems of the theory of sets, such as the well ordering of the real numbers and the continuum hypothesis, and whose famous papers on well ordering would be published in 1904 and 1908.[14]

Hilbert's direct involvement with foundational questions of this kind became increasingly reduced, and after 1905 he devoted very little time to them for many years to come.[15] One of the few, but well-known, instances of what he did in this period of time is a talk presented at the International Congress of Mathematicians held in Heidelberg in 1904, later published under the title of 'On the Foundations of Logic and Arithmetic'. Hilbert outlined here a program for addressing the problem of the consistency of arithmetic as he then conceived it. Hilbert cursorily reviewed several prior approaches to the foundations of arithmetic and declared that the solution to this problem would finally be found in the correct application of the axiomatic method.[16] A somewhat elaborate discussion of the ideas he outlined in Heidelberg appears also in the notes to an introductory course taught in Göttingen in 1905, devoted to 'The Logical Principles of Mathematical Thinking' (Hilbert 1905). These notes are highly interesting since they provide a rather detailed and broad overview of Hilbert's current views on the axiomatic method as applied to arithmetic, to geometry and to physics at large. In particular, and as part of that overview, the notes allow a significant glimpse into the inherent tension among the various elements that inform Hilbert's images of mathematics and his views about the roles of *Anschauung*, empirical experience, and axioms.

An adequate appreciation of how these elements and their interrelations appear in the course notes and, more generally, of Hilbert's conception of the essence and role of the axiomatic method, must pay due attention to the following, illuminating passage:

> The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces

[13] Peckhaus 1990, 56–57.

[14] Zermelo 1908.

[15] Hilbert's gradual return to this field, starting in a limited way in 1914 and then increasingly expanding towards 1918, until it came to dominate his activities after 1922, is described in Sieg 1999 and Zach 1999.

[16] Hilbert 1905, 131.

> to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development. (Hilbert 1905, 102)

This process of fortifying the 'loose foundations' is attained, of course, by means of an axiomatic analysis of the discipline in question. Thus, the very idea of investigating the foundations of mathematics, and indeed of science in general, is seen by Hilbert as an essentially pragmatic exercise meant to allow the healthy development of any discipline. It is a necessity that arises occasionally, only in case of real necessity. Axiomatic analysis is not a starting point of research in any field of mathematics (certainly not in geometry), and in fact it should not and cannot be done at the early stages of development of any discipline. Rather, it may be of great help only later on, when the theory has reached a considerable degree of maturity.

Of course, the paradigmatic example, but by no means the only one, of correctly and fruitfully applying the axiomatic method is geometry. Hilbert's own GdG could be evidently seen as the paramount successful instance of this, whereas the situation with arithmetic was much less clear at this stage. But what is the image of geometry and of arithmetic and of the philosophical underpinnings of these two disciplines that emerged in Hilbert's eyes in view of this situation? Here is what the notes to his 1905 course tell us about that:

> We arrive now to the construction of geometry, in which axiomatics was fully implemented for the first time. In the construction of arithmetic, our real point of departure was in its intuitive (*anschaulischen*) foundation, namely the concept of natural number (*Anzahlbegriff*) which was also the starting point of the genetic method. *After all, the number system was not given to us as a network of concepts* (*Fachwerk von Begriffen*) defined by 18 axioms. It was *intuition* that led us in establishing the latter. As we have started from the concept of natural number and its genetic extensions, the task is and naturally remains to attain a system of numbers which is as clear and as easily applicable as possible. This task will evidently be better achieved by means of a clearly formulated system of axioms, than by any other kind of definition. Thus it is the task of every science to establish on the axioms, in the first place, a network of concepts, for which formulation we let *intuition and experience* naturally serve as our guides. The ideal is, then, that in this network all the phenomena of the domain in question will find a natural place and that, at the same time, every proposition derivable from the axioms will find some application. (Hilbert 1905, 35–36)

Thus, for *both* geometry and arithmetic, the axiomatic analysis is meant to allow a systematic and thorough analysis of what in the final account provide the fundamental guide, source and justification, namely, intuition and experience. Hilbert is not very clear about the specific contribution of each of these two main components to every separate discipline. In earlier lectures we have seen him clearly separating between arithmetic and geometry. He then emphasized that while the former is a product of pure thought the latter is based on experience. It is indeed not untypical for Hilbert to change views with time. But as I said above, one should not try to extract a completely coherent and systematic philosophical system from his mathematical practice and from his various pronouncements, but rather attempt to see the elements that conform, sometimes in changing interrelations, his images of mathematical knowledge. And in this regard the above quotation is very revealing and typical, since it brings together very clearly all those important elements: intuition and experience as a starting point, and axiomatic analysis as a clarification tool. A successful axiomatic analysis implies a complete mathematization of the discipline in question, and in this sense logic is granted a main, foundational role for mathematics and for science at large. But this role is not autonomous and fundamental, since logic operates after the basic ideas are already in place creating a 'network of concepts'. In fact, Hilbert indicated an important difference between arithmetic and geometry in relation to the interaction between these various elements take place within them:

> Thus, if we want to erect a system of axioms for geometry, the starting point must be given to us by the intuitive facts of geometry and these must be made to correspond with the network that must be constructed. The concepts obtained in this way, however, must be considered as completely detached from both *experience and intuition.* In the case of arithmetic this demand is relatively evident. To a certain extent, this is already aimed at by the genetic method. In the case of geometry, however, the indispensability of this process [i.e. detachment from both intuition and experience (L.C.)] was acknowledged much later. On the other hand, the axiomatic treatment was attempted here earlier than in arithmetic where the genetic method was always the dominant one. (Hilbert 1905, 36–37)

This detachment from intuition and experience explains the equal mathematical validity and value that has to be attributed to all the possible, axiomatically defined, geometries. And yet, in spite of this, Hilbert explicitly and consistently expressed an inclination to grant a preferred status to Euclidean geometry from among all possible ones. What is the basis of this preference, if from the purely mathematical point of view all geometries are equally legitimate and valid? Hilbert was definitely puzzled about this, and this is no doubt one of the main reasons, as will be seen below, that he welcomed so strongly the rise of GTR with its momentous implications for the relations between geometry and physics. But in 1905, this is what he told his students in Göttingen:

> The question how is it that in nature only the Euclidean geometry, namely the one determined by all the axioms taken together, is used, or why our experience accommodates itself precisely with this system of axioms, does not belong to our logico-mathematical inquiry. (Hilbert 1905, 67)

Thus, five years after the publication of *GdG* and the flurry of activity that followed it both in Göttingen and outside, Hilbert had no doubts concerning the validity of Euclidean geometry as the most adequate description of physical space, but he definitely believed that mathematics itself could not explain the reason for this.

In spite of the successful application of the axiomatic method in geometry, the evidence from the 1905 lecture course clearly indicates that Hilbert did by no means adopt a formalistic view of mathematics in general and of geometry in particular, and did not bar from his lexicon the word *Anschauung* in connection with the foundations of geometry. This does not mean, however, that a thoroughly formalist view could not be derived from the new perspectives opened by Hilbert's innovations. One could find a very different and illuminating example of such a view in the works of a mathematician like Felix Hausdorff, and in the kind of radical views he developed under the explicit influence of *GdG* in a direction initially unintended by Hilbert himself.

Hausdorff indeed postulated the vie of geometry as a fully autonomous discipline, independent of any kind of *Anschauung* or empirical basis.[17] In a manuscript dated around 1904, and properly entitled 'Formalism', Hausdorff praised the full autonomy attained by geometry following Hilbert's work, in the following words:

> In all philosophical debates since Kant, mathematics, or at least geometry, has always been treated as heteronomous, as dependent on some external instance of what we could call, for want of a better term, intuition (*Anschauung*), be it pure or empirical, subjective or scientifically amended, innate or acquired. The most important and fundamental task of modern mathematics has been to set itself free from this dependency, to fight its way through from heteronomy to autonomy.[18]

This autonomy, so fundamental for the new view of mathematics predicated by Hausdorff and widely adopted later on as a central image of twentieth-century mathematics, was to be attained precisely by relying on the new conception of axiomatic systems embodied in *GdG*. As he explicitly wrote in a course on 'Time and Space', taught in 1903–04:

> Mathematics totally disregards the actual significance conveyed to its concepts, the actual validity that one can accord to its theorems. Its indefinable concepts are arbitrarily chosen objects of thought and its axioms are arbitrarily, albeit consistently, chosen relations among these objects. Mathematics is a science of pure thought, exactly like logic.[19]

Pure mathematics, under this view, is a 'free' and 'autonomous' discipline of symbols with no determined meaning. Once a specific meaning is accorded to them, we obtain 'applied' mathematics. Intuition plays a very important heuristic and pedagogical role, but it is inexact, limited, misleading and changing, exactly

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[17] Purkert 2002, 50, quotes a letter of Hausdorff expressing an opinion in this spirit as early as October 1900.

[18] Quoted in Purkert 2002, 53–54.

[19] Quoted in Purkert 2002, 54.

the opposite of mathematics.[20] Although one can find in some of Hilbert's texts or lecture notes pronouncements that may be seen as fitting this view, the main thrust of his images of geometry is opposed to it, and this opposition became even stronger after 1915.

GTR and Geometry

In the years immediately after 1905 Hilbert directed most of his energies to the theory of integral equations and to physics, including foundational issues of various kinds in the latter discipline. It may thus have come as a nice and unexpected surprise for him to find out that his new focus of interests would eventually bring him back to the foundations of geometry. Indeed, this happened in 1916, as part of his intensive involvement with GTR, and the novel relationship that this theory uncovered between gravitation and geometry. One might think, on the face of it, that Hilbert's involvement with GTR was directly motivated by the strongly geometric content of this theory, but this is far from being the case. Rather, Hilbert came to be interested in GTR in a very roundabout way. As a matter of fact, until 1912, Hilbert's involvement with physics was essentially limited to topics related to mechanics (including fluid mechanics, statistical mechanics, and mechanics of continua). Only after 1912 did the scope of this involvement with physical disciplines significantly broaden so as to include also kinetic theory, radiation theory and, most significantly, current theories of the structure of matter. As part of his involvement with the latter domain, Hilbert came across the electromagnetic theory of matter of Gustav Mie, and, taking it as a starting point, Hilbert attempted to develop his own unified field theory of matter and gravitation. This is what led him around 1914 to increasingly direct his attention to Einstein's recent attempts to complete his generalized theory of relativity, including a relativistic theory of gravitation.[21]

After his initial involvement with GTR, that included a short-lived tension between him and Einstein around the question of priority in the formulation of the explicit, generally covariant field equations of the theory, Hilbert became a main promoter of the theory, which he explicitly and consistently presented as Einstein's brainchild and as one of the most important creations of the human spirit ever. In particular, Hilbert was among the first to teach a systematic course on the theory in 1916–17 and he continued to give public lectures for many years to come, on the implications of the theory for our understanding of space and

[20] For the precise quotations, see Purkert 2002, 54.

[21] This is described indetail in Corry 2004, especially in Chapters 5 and 6.

time.[22] As part of all this, the subtle balance manifest in Hilbert's early writings between empirical and a-priori intuition as possible sources of geometric knowledge was finally altered, and unmistakably resolved in favour of experience. Moreover, and very importantly, Euclidean geometry had lost its preferred status as the one that naturally accommodates with empirical experience.

Plenty of evidence indicates the strong impact that these developments had on Hilbert. Some of Hilbert's pronouncements to this effect are worth quoting and discussing in some detail. The lecture notes of his 1916–17 course on GTR, for instance, included a section on 'the new physics', in which Hilbert referred to the new relationship between this discipline and geometry. He thus said:

> In the past, physics adopted the conclusions of geometry without further ado. This was justified insofar as not only the rough, but also the finest physical facts confirmed those conclusions. This was also the case when Gauss measured the sum of angles in a triangle and found that it equals two right ones. That is no longer the case for the new physics. *Modern physics must draw geometry into the realm of its investigations.* This is logical and natural: every science grows like a tree, of which not only the branches continually expand, but also the roots penetrate deeper.
>
> Some decades ago one could observe a similar development in mathematics. A theorem was considered according to Weierstrass to have been proved if it could be reduced to relations among natural numbers, whose laws were assumed to be given. Any further dealings with the latter were laid aside and entrusted to the philosophers. . . . That was the case until the logical foundations of this science (arithmetic) began to stagger. The natural numbers turned then into one of the most fruitful research domains of mathematics, and especially of set theory (Dedekind). The mathematician was thus compelled to become a philosopher, for otherwise he ceased to be a mathematician.
>
> The same happens now: the physicist must become a geometer, for otherwise he runs the risk of ceasing to be a physicist and vice versa. The separation of the sciences into professions and faculties is an anthropological one, and it is thus foreign to reality as such. For a natural phenomenon does not ask about itself whether it is the business of a physicist or of a mathematician. On these grounds we should not be allowed to simply accept the axioms of geometry. The latter might be the expression of certain facts of experience that further experiments would contradict. (Hilbert 1916–17, 2–3)

In the course and elsewhere, Hilbert constantly emphasized that both Euclidean geometry and Newtonian physics were theories of 'action-at-a-distance', and that the new physics had indicated the problems underlying such theories. Retrospectively seen, describing Euclidean geometry in these terms may sound somewhat artificial but Hilbert's point was to indicate that the old question of the validity of Euclidean geometry had been rekindled and should be now understood

[22] See Corry 2004, Chapters 7 and 8.

in two different senses. The first sense is the logical one: is Euclidean geometry consistent? From the mathematical point of view, as Hilbert had already stressed in GdG, Euclidean geometry exists if it is free from contradiction. But from the physical point of view, such an answer is unsatisfactory. What we are interested in is the question of the validity of Euclidean geometry *as a description of nature*. This question, of course, 'cannot be decided through pure thinking'. In the past, even though he could not provide a full, satisfactory philosophical explanation for this, Hilbert had no doubts concerning the primacy of Euclidean geometry. Now conditions had changed and physical theory offered strong reasons to abandon that primacy. The need for such a change posed no problem for Hilbert, especially because of his long-professed, essentially empiricist image of geometry. Moreover, the new insights into the connection between gravitation and geometry fitted easily into ideas originally raised by Riemann, a mathematician whose conceptions of geometry Hilbert widely shared.[23]

But Hilbert's images of mathematics not only provided a natural background that allowed for a smooth adoption of the new conception of geometry implied by GTR. These images and the general mathematical background of Hilbert also led him into a direction within GTR that was quite idiosyncratic by that time. Thus, Hilbert was the first to wonder about the solution of the field equations in the absence of matter. Specifically, he asked about the conditions under which the Minkowski metric becomes a unique solution, hoping that this would happen in the absence of matter and radiation.[24] In contrast, for Einstein the main focus of interest in this context was the question of the Newtonian limit, and therefore the existence of empty-space solutions was not a natural, immediate question to be asked.

The status of Euclidean geometry in connection with the axioms of GTR was a topic that Hilbert addressed in the second of his two communications on the foundations of physics, presented to the Göttingen Scientific Society on December 23, 1916. Hilbert focused on what he called the 'Axiom of Space and Time', a postulate he had previously introduced in his first communication, as an attempt to deal correctly with the question of causality in GTR. Also here we find interesting views about the empirical grounding of geometry, as, for instance, in the following passage:

> According to my presentation here physics is a four-dimensional pseudogeometry, whose metric $g_{\mu\nu}$ is connected with the electromagnetic magnitudes .... Having realized this, an old question seems to be ripe for solution, namely, the question if, and in what sense, Euclidean geometry – which from mathematics we only know to be a logically consistent structure – is also valid of reality.

23 Hilbert explicitly mentioned this connection in the opening passages of his 1915 communication (Hilbert 1916, 398) and also in the lecture course (Hilbert 1916–17, 168).

24 See Corry 2004, §8.3.

> The old physics, with its concept of absolute time, borrowed the theorems of Euclidean geometry, and made them the foundation of every particular physical theory. Gauss himself proceeded hardly differently: he hypothetically built a non-Euclidean physics, which, while retaining absolute time, renounced only the axiom of parallels. But the measurement of the angles of a large triangle indicated him the invalidity of this non-Euclidean physics.
>
> The new physics based on Einstein's general relativity takes a completely different approach to geometry. It assumes neither Euclidean nor any other kind of geometry in order to deduce from it the laws of physics . . . .
>
> Euclidean geometry is *a law of action at a distance, foreign to modern physics*. By renouncing Euclidean geometry as a general presupposition of physics, the theory of relativity also teaches us that geometry and physics are similar in kind and, being one and the same science, they rest upon a common foundation. (Hilbert 1917, 63–64. Italics in the original)

Moreover, from the fact that the Minkowski metric, $g_{\mu\nu} = \delta_{\mu\nu}$, cannot be a general solution of the field equations, Hilbert deduced the following important conclusion, that gave additional strength to his empiricist leanings and indicates a possible direction to counter conventionalist or formalist interpretations of geometry:

> This is in my opinion a positive result of the theory, since we can in no way impose Euclidean geometry upon nature by means of a different interpretation of the experiment. Assuming that the fundamental equations of physics that I will develop here are the correct ones, then no other physics is possible, i.e., reality cannot be conceived differently. On the other hand, we will see that under certain, very specialized assumptions—perhaps the absence of matter in space will suffice—the only solution of the differential equations is $g_{\mu\nu} = \delta_{\mu\nu}$. Also this I must take as further support for my theory, since Gauss's angle-measurement experiment in a triangle has shown that *Euclidean geometry is valid in reality as a very good approximation*. (Hilbert 1916–17, 106. Emphasis in the original)

By referring to Gauss's experiment Hilbert was evidently closing a circle that had started way back in his early courses on Geometry. Back then Hilbert had interpreted the outcome of that experiment as the requisite empirical evidence for primacy of Euclidean geometry, but he nevertheless clearly suggested that future experiments could change current views in this regard, and might necessitate correcting our understanding of the role of the parallel axiom. Obviously he had no idea at that time that this assumption would prove correct two decades later, and much less under what circumstances. The new findings of physical science may indeed necessitate a specific choice of the correct geometry of nature, but in order to accommodate these changes in his overall view of mathematics and of science Hilbert could remain true to the empiricist approach that had characterized all of his images of geometry, as well as much of his foundational conceptions of mathematics, from very early on.

The last important phase of Hilbert's career was devoted to the foundations of logic and of arithmetic, and it comprises the years of activity in which the 'formalist' programme for proving the consistency of arithmetic in finitist terms was formulated and initially implemented. As already noted, the presence of Bernays in Göttingen since 1917 was a main factor in rekindling Hilbert's interest in this field. In 1922 Hilbert published his first significant article on the topic: 'New Foundations of Mathematics'.[25] Given his intense involvement during 1916–18 with questions related to GTR, and to the foundations of physics and geometry, one may wonder if, and possibly how, all the significant epistemological ideas developed in this framework played a direct role in the background to the elaboration of ideas related with the finitist program. There seems to be no direct evidence for a positive answer to such a question and to establish a direct, causal connection between Hilbert's activities in the foundation of physics and the transition to the last stage of his career. Nevertheless, it is interesting to examine Hilbert's views on the foundations of geometry after 1920, in order to realize that even at this late point there is no trace of 'formalism' in it, and that, on the contrary, these views become increasingly empiricistic.

One interesting instance of this appears in a series of public lectures given by Hilbert in the winter semester of 1922–23 under the name 'Knowledge and Mathematical Thought'. It is noteworthy that the fourth lecture in this series was entitled 'Geometry and Experience' (*Geometrie und Erfahrung*), exactly like Einstein's 1920 talk in Berlin quoted above. This may have been pure coincidence, but it is nevertheless remarkable that the declared aim of the series of talks was to refute a 'widespread conception of mathematics', and in particular conceptions such as implied by the views alluded to by Einstein in Berlin.[26] In fact, Hilbert thought it necessary to comment on the title of his talk, and he thus said:

> The problem that I want to address here is a very old, difficult and deep-going one. I could also call it: Representation and Reality, Man and Nature, Subjectivity and Objectivity, Theory and Praxis, Thinking and Being. If I have chosen such a title, then I must also stress that I can only treat this problem from a one-sided perspective and within a rather limited scope. . . . The problem stands in front of us like a high mountain peak that no one has yet fully conquered. . . . Perhaps we may succeed at least in reaching some important and beautiful observation points. (Hilbert 1922–23, 78)

Hilbert thought that epistemology, in its current state of development, was not yet ready to cope with the new situation created in view of the insights afforded by general relativity,[27] and in particular one cannot see in his own writings meaningful contributions in this direction. Yet he thought it important to stress

[25] Hilbert 1922.

[26] Two years earlier Hilbert had given another series of public talks where he pursued a similar aim. See Hilbert 1919–20. He would repeat many of the ideas expressed here in his Königsberg lecture of 1930 (Hilbert 1930).

[27] See, e.g., Hilbert 1922–23, 98.

the essence of these insights and to explain how they affected our conception of the connection between geometry and physics, between geometry and intuition. Typical of the kind of ideas that arise in this context are those expressed in the following passage:

> Some philosophers have been of the opinion—and Kant is the most prominent, classical representative of this point of view—that besides logic and experience we have a certain a-priori knowledge of reality. That mathematical knowledge is grounded, in the last account, on some kind of intuitive insight; even that for the construction of the theory of numbers a certain intuitive standpoint (*anschauliche Einstellung*), an a-priori insight, if you wish, is needed; that the applicability of the mathematical way of reflection over the objects of perception is an essential condition for the possibility of an exact knowledge of nature—all this seems to me to be certain.
>
> Furthermore, the general problem of determining the precise conditions of the possibility of empirical knowledge maintains its fundamental importance. And today more than ever, when so many time-honored principles of the study of nature are being abandoned, this question retains an increased interest.
>
> The general basic principles and the leading questions of the Kantian theory of knowledge preserve in this way their full significance. But the boundaries between what we *a-priori* possess and logically conclude, on the one hand, and that for which experience is necessary, on the other hand, we must trace differently than Kant. For, to take just one example, contrary to what was initially assumed, and to what also Kant claimed, the evidence of the basic propositions (*Grundsätze*) is not decisive for ensuring the success of Euclid's method in the real world. (Hilbert 1922–23, 87–88)

The law of inertia and the laws of Maxwell's theory of electromagnetism were two examples of physical laws that no one had expected to be a priori. But the necessity of turning to experience, as opposed to Kantian-like *Anschauung*, Hilbert added, appeared even in places where one would expect the a priori to be essential for the very possibility of science. This was the case for our conceptions of space and Euclidean geometry, the conception of which had radically changed in the wake of GTR. But in this passage Hilbert also pointed to the similar change that had affected the concept of absolute time that Newton and Kant took for granted but that Einstein's theory of relativity, prompted by the result of Michelson's experiment, had by now completely rejected. Thus, concerning the validity of the assumption of an absolute time, Hilbert said in his typically effusive and all-encompassing style:

> Newton actually formulated this as bluntly as possible: absolute, real time flows steadily from itself and by virtue of its nature, and with no relation to any other object. Newton had really given up any compromise in this respect, and Kant, the critical philosopher, proved here to be rather uncritical, because he accepted Newton without further ado. It was first Einstein who freed us definitively from this prejudice and this will always remain as

> one of the most tremendous achievements of the human spirit and thus the all too sweeping a priori theory could not have been driven to absurd more decisively. Of course, a discovery of the magnitude of the relativity of simultaneity caused a drastic upheaval concerning all elementary laws, since now a much closer amalgamation of the spatial and temporal relations holds. We can thus say cum grano salis, that the Pythagorean theorem and Newton's law of attraction are of the same nature, inasmuch as both of them are ruled by the same fundamental physical concept, that of the potential. But one can say more: both laws, so apparently different heretofore and worlds apart from each other—the first one known already in antiquity and taught to everyone in primary school as one of the elementary rules of geometry, the other a law concerning the mutual action of masses on each other—are not simply of the same nature but in fact part of one and the same general law: *Newtonian attraction turned into a property of the world-geometry and the Pythagorean theorem into a special approximated consequence of a physical law*. (Hilbert 1922–3, 90–91. Emphasis in the original)

Anschauung was thus barred from any role in Hilbert's images of geometry and empiricism reigned now alone, as geometry had been definitely turned into a branch of physics. Like Newtonian physics, Euclidean geometry was nothing but a good approximation of truth. Formalism, needless to say, was in no way part of this picture.

Concluding Remarks

We may now return to the questions posed in the opening section, and try to summarize the above discussion by assessing the extent to which the philosophy of mathematics of one individual mathematician, Hilbert, is relevant to historically understanding his mathematical work, and to what extent his mathematical work has any bearing in understanding philosophical issues related with mathematics.

Hilbert's conceptions about geometry, although evidently philosophically well informed, cannot themselves be described as embodying elaborate philosophical views. Rather his changing conceptions are best described as an ongoing dialogue between historically evolving mathematical and scientific theories that Hilbert was involved with, on the one hand, and, on the other hand, a set of fundamental, yet flexible, images of what mathematical knowledge is about, of the relation between mathematics and the empirical sciences, and of the role of empirical perception and *Anschauung* in the various branches of mathematics.

It is indeed certainly necessary to pay due attention to the role played by philosophical conceptions in historically shaping the mathematical work of Hilbert, and its overall intellectual background. But this role needs to be understood in the terms described above: In the case of Hilbert, philosophical ideas provide him, in the first place, with the adequate terms to formulate and understand his own changing conceptions, and also to allow a feeling of continuity amid these changes. An underlying empiricistic drive is highly prominent in Hilbert's images of geometry throughout his career, but this prominence is, to a great extent, an outcome of specific developments in the mathematical disciplines as well as in physics in the relevant period.

Generally speaking—and very conspicuously so in the case of geometry—an overall, consistent philosophical conception does not appear as methodological or epistemological guidelines or as underlying general principles that Hilbert followed in developing mathematical and scientific ideas. Still, it is plausible that in certain, historically localized portions of his scientific career, a more elaborate and consistent philosophical perspective did play a decisive role in shaping his mathematical ideas. On the face of it, a claim in this direction could be made when considering Hilbert's finitistic program for the foundations of arithmetic and the intellectual setting in which it developed. I have not discussed this important part of his career in this chapter, but I suggest that in any such discussion, some of the ideas developed here should also be taken into consideration within the specific circumstances pertaining to the case.

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