Operations Research Letters 38 (2010) 215–217

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

A relaxation-based algorithm for solving the conditional p-center problem

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ARTICLE INFO

Article history:
Received 4 October 2008
Accepted 15 December 2009
Available online 28 December 2009

Keywords:
Conditional p-center
Relaxation

ABSTRACT

We present a new relaxation algorithm for solving the conditional continuous and discrete p-center problems.

1. Introduction

The *unconditional* p-center problem (see, for example, [7]) deals with the optimal location of emergency facilities. The locations of n demand points are given, and we need to locate p service facilities. The *value* of a candidate solution to the p-center problem is the maximum distance between the demand points, each to its nearest service facility. Our objective is to find the solution with the minimal value. It is assumed that all the facilities perform the same kind of service, and that the number of demand points that can get service from a given center is unlimited.

In the *conditional* p-center problem, we are given the locations of q *existing* service facilities. We need to locate p additional service facilities, so as to minimize the maximum distance between the demand points, each to its nearest service facility, whether existing or new.

There are two main variants of the p-center problem. In the *continuous* p-center problem, the location of the service facilities can be anywhere in the two-dimensional Euclidean space. In the *discrete* variant there is a finite set of potential service points to choose from. An analogous representation of the discrete p-center problem is the p-center problem on networks [7]. In the p-center problem on networks, both the demand points and the potential service points are located on a weighted undirected graph, and the distance between any two points is the cost of the shortest path between them.

Berman and Simchi-Levi [2] showed how to solve the conditional p-median and p-center problems by solving a single unconditional $(p+1)$-center problem. Berman and Drezner [1] improved this result by showing how to solve these problems by solving a single unconditional p-center problem. Both papers refer only to problems on networks.

Chen and Handler [5] presented a relaxation algorithm for solving the conditional p-center problem.

Drezner [6] showed how the conditional p-center problem can be solved by solving $O(\log n)$ p-center problems. In this paper we present a new algorithm for solving the conditional p-center problem, which relies on Drezner's observations, as well as a relaxation-based algorithm given by Chen and Chen [3]. This new relaxation algorithm is applicable to both the continuous Euclidean and the discrete conditional p-center problems.

2. Drezner's algorithm

Let $\{X_1, X_2, \dots, X_n\}$ be the set of the demand points. Without loss of generality, let us assume that the demand points are ordered by their distance to the q existing service facilities, from farthest to nearest (recall that the distance of a demand point to a set of service facilities is its distance to the nearest service facility). Let us denote by v the value of the optimal solution to the conditional p-center problem. Since the demand points are ordered by their distance to the existing service facilities, there exists a value r such that all demand points in $\{X_1, X_2, \dots, X_r\}$ are not covered by the existing service facilities, and all demand points in $\{X_{r+1}, X_{r+2}, \dots, X_n\}$ are covered. The basic idea of Drezner's algorithm, which we later explain in more detail, is to perform a binary search in order to find this value of r.

We denote the value of the optimal solution of the unconditional p-center problem for $\{X_1, X_2, \dots, X_k\}$ by F_k. We define $F_0 = 0$.

Let the distance of X_k to its nearest existing service facility be M_k. We define $M_{n+1} = 0$.

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doi:10.1016/j.orl.2009.12.003

Two important observations:

$$M_1 \geq M_2 \geq \cdots \geq M_n \geq M_{n+1}, \tag{1}$$

and

$$F_0 \leq F_1 \leq F_2 \leq \cdots \leq F_n. \tag{2}$$

Expression (1) follows immediately from the ordering of the demand points. Expression (2) follows from the fact that the solution of the p-center problem for a set of demand points is always greater than or equal to the solution of the p-center problem for any subset.

Any optimal solution to the p-center problem on $\{X_1, X_2, \ldots, X_k\}$ induces a feasible (not necessarily optimal) solution to the conditional p-center problem, where each demand point in $\{X_{k+1}, X_{k+2}, \ldots, X_n\}$ is served by its nearest existing service facility, and each demand point in $\{X_1, X_2, \ldots, X_k\}$ is served by the nearest new service facility in the solution of the *unconditional* p-center problem. The value of this candidate solution is $\max\{M_{k+1}, F_k\}$.

Drezner [6] proved the following lemma, for which we present a slightly different proof.

Lemma 1. *The optimal solution to the conditional p-center problem is*

$$\min_{0\leq k\leq n} \max\{M_{k+1}, F_k\}.$$

Proof. Assume to the contrary that there exists a solution to the conditional p-center problem with value $v < \min_{0\leq k\leq n} \max\{M_{k+1}, F_k\}$. Let r be the index satisfying $M_1 \geq M_2 \geq \cdots \geq M_r \geq v \geq M_{r+1} \geq \cdots \geq M_n \geq M_{n+1}$. Therefore only the points $\{X_{r+1}, X_{r+2}, \ldots, X_n\}$ are covered by the existing service facilities. The rest are covered by the new service facilities.

Since $v < \max\{M_{r+1}, F_r\}$, and since $v \geq M_{r+1}$, it follows that $v < F_r$. However, the distance of all points in $\{X_1, X_2, \ldots X_r\}$, each to its nearest new service facility, is less than or equal to v; therefore F_r, the solution of the unconditional p-center problem on $\{X_1, X_2, \ldots X_r\}$, must satisfy $F_r \leq v$, a contradiction to $v < F_r$. □

As Drezner has observed, since $\{F_k\}_{k=0}^{n}$ is monotonically increasing, and $\{M_k\}_{k=1}^{n+1}$ is monotonically decreasing, it immediately follows that the optimal solution to the conditional p-center problem is either M_r or F_r where r is the maximal value satisfying $M_r > F_{r-1}$.

Drezner assumes that one has an algorithm for the solution of the unconditional p-center problem, and proposes to use it to compute the F_k values (the M_k values are easily computed). His algorithm performs a binary search in order to find this r value. In the next section, we propose a different algorithm for finding r.

3. Solving the conditional *p*-center problem using reverse relaxation

Relaxation (in the context of this paper) [4,7] is a method for *optimally* solving a large location problem by solving a succession of small sub-problems. It is an iterative algorithm which updates, at each step, bounds on the optimal solution, until the optimal solution is reached.

The classic relaxation algorithm [4,7] starts with an upper bound of infinity, and keeps updating it until the optimal value is reached (this is similar to Minieka's algorithm [8], which does not involve relaxation). Chen and Chen [3] proposed a new algorithm, *reverse relaxation*, which starts with a lower bound of 0, and constantly updates it (upwards), until the optimal value is reached.

As a fortunate "side-effect", the reverse relaxation algorithm computes the optimal solution of the p-center problem on subsets of the demand points (for a proof, see [3, Section 2.6]). We will use this property to solve the p-center problem, first on $\{X_1\}$, then on $\{X_1, X_2\}$ and so on until $\{X_1, X_2, \ldots, X_r, X_{r+1}\}$ (we halt when $M_{r+1} \leq F_r$).

Algorithm 1 describes the skeleton of a reverse relaxation algorithm for solving the conditional p-center problem. It differs from the reverse relaxation algorithm in [3] in the way the demand-point subsets grow. In [3] we may add more than a single demand point each time, and the demand points may be chosen randomly. In the current algorithm we must add a single demand point, and it should always be the next demand point in order of distance to the existing service facilities. Also, the rules for halting are different in the two algorithms.

Algorithm 1 Skeleton of a reverse relaxation algorithm for the conditional p-center problem.

```
Lower_Bound ← 0
k ← 1
Sub ← {X_1}
while (solution not found)
  Feasible ←FindFeasibleSolution(Sub, Lower_Bound)
  if (feasible solution found for sub-problem)
    Optimal_k = Feasible
    F_k =GetValue(Optimal_k)
    if (M_{k+1} ≤ F_k or k = n)
      if (F_k < M_k or k = n)
        halt and return Optimal_k
      else
        halt and return Optimal_{k-1}
    k ← k + 1
    Sub = Sub ∪ {X_k}
  else
    Lower_Bound ←GetNextBound(Sub, Lower_Bound)
```

In each step of the reverse relaxation algorithm we run a subroutine that we denote as FindFeasibleSolution, which solves a p-center-like problem on a subset of the demand points. Our input is the subset *Sub* and a value r, which is called the *coverage distance*. The subroutine answers the question: "Is there a solution to the sub-problem with value less than r?" (and if so, finds such a solution).

Drezner's algorithm solves $O(\log n)$ p-center sub-problems. While the number of p-center sub-problems that we solve is at most n, a few properties of the reverse relaxation algorithm indicate that the performance of the algorithm will often be very good.

- Reverse relaxation tends to solve small sub-problems, in terms of the number of demand points. Recall that we solve the p-center problem, first on $\{X_1\}$, then on $\{X_1, X_2\}$ and so on until $\{X_1, X_2, \ldots, X_r, X_{r+1}\}$ (we halt when $M_{r+1} \leq F_r$). We do not solve sub-problems with more than $r + 1$ demand points. Although Drezner's algorithm also solves sub-problems, it may need to solve sub-problems with more than $r + 1$ demand points.
- Reverse relaxation tends to solve problems with relatively small coverage distances. The coverage distance has a profound effect on performance; the lower the coverage distance, the better the performance [3, Section 3.3.3].
- We can use the solution of the current sub-problem as a lower bound on the solution of the next sub-problem, which contains one additional demand point.

It is also important to note that the performance of the reverse relaxation algorithm for solving the conditional p-center problem is no worse than the performance of a single run of the reverse relaxation algorithm for solving the unconditional p-center problem (for the case where we add only a single demand point whenever the demand-point subsets grow, in order of distance to the existing service facilities). In [3], experimental results show that the reverse relaxation algorithm for solving the unconditional problem is very efficient.

Drezner assumed that one has an efficient algorithm for solving the p-center problem. We, on the other hand, make two other assumptions.

- We assume that there is a finite number of values for the optimal solution of an unconditional p-center problem. We use this assumption to implement the subroutine GetNextBound($Lower\_Bound$) which returns the smallest value, among the possible values for the optimal solution, which is greater than $Lower\_Bound$.
- We assume that we have a subroutine FindFeasibleSolution(Sub, r), which answers the question: "Is there a solution to the sub-problem with value less than r?" (and if so, finds such a solution).

Both of our assumptions hold true for the continuous Euclidean p-center problem, as well as for the discrete p-center problem. FindFeasibleSolution(Sub, r) can be implemented using set-covering algorithms (see, for example, [4]).

References

[1] O. Berman, Z. Drezner, A new formulation for the conditional p-median and p-center problems, Operations Research Letters 36 (2008) 481–483.
[2] O. Berman, D. Simchi-Levi, Conditional location problems on networks, Transportation Science 24 (1990) 77–78.
[3] D. Chen, R. Chen, New relaxation-based algorithms for the optimal solution of the continuous and discrete p-center problems, Computers and Operations Research 36 (2009) 1646–1655.
[4] R. Chen, G.Y. Handler, Relaxation method for the solution of the minimax location–allocation problem in Euclidean space, Naval Research Logistics 34 (1987) 775–787.
[5] R. Chen, G.Y. Handler, The conditional p-center problem in the plane, Naval Research Logistics 40 (1993) 117–127.
[6] Z. Drezner, Conditional p-center problems, Transportation Science 23 (1989) 51–53.
[7] G.Y. Handler, P.B. Mirchandani, Location on networks: Theory and algorithms, MIT Press, Cambridge, MA, 1979.
[8] E. Minieka, The m-center problem, SIAM Review 12 (1970) 138–139.