Extended Picard complexes and homogeneous spaces

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Joint work with Joost van Hamel (1969–2008)

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X is a right homogeneous space of G:

X is an algebraic variety over k,
we have a map $X \times_k G \to X$,
and G acts transitively on X.

$\mathbb{X}(G) := \mathrm{Hom}(G, \mathbb{G}_{m,k})$, the character group of G.

Let $x \in X(k)$ and $H = \mathrm{Stab}_G(x)$, then $X = H\backslash G$, and we consider the character group $\mathbb{X}(H)$.

We have a restriction map

$$\mathrm{res}\colon \mathbb{X}(G) \to \mathbb{X}(H).$$

I always assume that $\mathrm{Pic}(G) = 0$. I explain what it means.

G^{u} is the unipotent radical of G.
$G^{\mathrm{red}} = G/G^{\mathrm{u}}$, it is reductive.
$G^{\mathrm{ss}} = [G^{\mathrm{red}}, G^{\mathrm{red}}]$, it is semisimple.

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Then $\mathrm{Pic}(G) = 0$ if and only if G^{ss} is simply connected.

If $\mathrm{Pic}(G) \neq 0$, one can find an epimorphism

$$G' \to G$$

with $\mathrm{Pic}(G') = 0$.

Now our X is a homogeneous space of the new group G' with $\mathrm{Pic}(G') = 0$.

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Step 2. Prove that for *a homogeneous space* X of G with $\mathrm{Pic}(G) = 0$, the complex $\mathrm{UPic}(X)$ is indeed "the same" as $\mathbb{X}(G) \to \mathbb{X}(H)$.

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Step 3. Apply Steps 1 and 2 to compute the algebraic Brauer group of a homogeneous space over a not algebraically closed field.

Actually I assume that X and G are defined over some not algebraically closed subfield $k_0 \subset k$, and k is an algebraic closure of k_0.

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The Galois group $\mathrm{Gal}(k/k_0)$ acts on $\mathbb{X}(G)$ and $\mathbb{X}(H)$, and so we obtain a complex of Galois modules $\mathbb{X}(G) \to \mathbb{X}(H)$.

I want to constuct a complex of Galois modules $\mathrm{UPic}(X)$, generalizing $\mathbb{X}(G) \to \mathbb{X}(H)$, for any smooth variety X, defined over k_0 (and irreducible over k).

Step 1: Generalize the complex $\mathbb{X}(G) \to \mathbb{X}(H)$ to *any* smooth irreducible variety X.

Seems crazy: a general variety X has neither $\mathbb{X}(G)$ nor $\mathbb{X}(H)$. But it does have

$$\boxed{\ker[\mathbb{X}(G) \to \mathbb{X}(H)]} \quad \text{and} \quad \boxed{\operatorname{coker}[\mathbb{X}(G) \to \mathbb{X}(H)]}.$$

Notation:
$\mathscr{O}(X)$ is the ring of regular functions on X,
$\mathscr{K}(X)$ is the field of rational functions on X.

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Rosenlicht's Lemma. $U(G) \cong \mathbb{X}(G)$.

The map: $\mathbb{X}(G) \hookrightarrow \mathscr{O}(G)^\times \to \mathscr{O}(G)^\times/k^\times = U(G)$.

Corollary. For a homogeneous space X we have
$U(X) = U(H\backslash G) \cong \ker[\mathbb{X}(G) \to \mathbb{X}(H)]$.

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Corollary. For a homogeneous space X we have
$U(X) = U(H\backslash G) \cong \ker[\mathbb{X}(G) \to \mathbb{X}(H)]$.

Popov's theorem. For a homogeneous space X,
when $\mathrm{Pic}(G) = 0$, we have
$\mathrm{Pic}(X) = \mathrm{Pic}(H\backslash G) \cong \mathrm{coker}[\mathbb{X}(G) \to \mathbb{X}(H)]$.

Definition of UPic(X)

We can define $U(X)$ and $\mathrm{Pic}(X)$ for any variety X. We must glue them to a complex!

Recall that $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X, and there is a canonical exact sequence

$$\mathscr{K}(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0.$$

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$\mathrm{UPic}(X) = \mathscr{K}(X)^\times / k^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X)$.

Then $\ker \mathrm{UPic}(X) = U(X)$ and $\mathrm{coker}\, \mathrm{UPic}(X) = \mathrm{Pic}(X)$ (this explains the notation U-Pic).

We have done Step 1: defined $\mathrm{UPic}(X)$ for any X.

Note the complex $\mathrm{UPic}(X)$ was independently introduced by David Harari and Tamás Szamuely.

Step 2

Step 2. For a homogeneous space X with $\mathrm{Pic}(G) = 0$, the complex

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We claim that these complexes are *isomorphic in the derived category* of Galois modules.

Quasi-isomorphisms

By definition, a morphism of complexes

$$\varphi\colon C^\bullet = (C^0 \to C^1) \to D^\bullet = (D^0 \to D^1)$$

is a commutative diagram

$$\begin{array}{ccc} C^0 & \longrightarrow & C^1 \\ {\scriptstyle \varphi^0}\downarrow & & \downarrow{\scriptstyle \varphi^1} \\ D^0 & \longrightarrow & D^1 \end{array}$$

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Such a morphism defines morphisms on the cohomology $\varphi_{\mathrm{ker}}\colon \ker C^\bullet \to \ker D^\bullet$ and $\varphi_{\mathrm{coker}}\colon \operatorname{coker} C^\bullet \to \operatorname{coker} D^\bullet$.

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Definition

A morphism of complexes $\varphi\colon C^\bullet \to D^\bullet$ is called a *quasi-isomorphism* if φ_{ker} and φ_{coker} are isomorphisms.

Example of a quasi-isomorphism

Example.

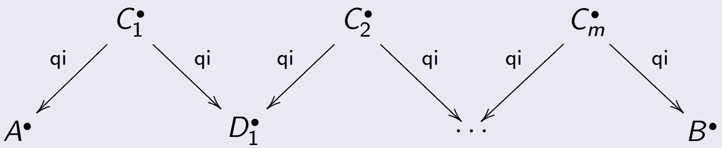
$$\begin{array}{ccc} C^0 & \hookrightarrow & C^1 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C^1/C^0 \end{array}$$

This is a quasi-isomorphism which is not an isomorphism of complexes.

Isomorphism in the derived category

Definition

If there exists a diagram



where all the arrows are quasi-isomorphisms, we say that complexes $A^\bullet = (A^0 \to A^1)$ and $B^\bullet = (B^0 \to B^1)$ are *isomorphic in the derived category*.

This is not the standard definition.

Later I shall construct a diagram

$$\mathrm{UPic}(X) \xleftarrow{\text{qi}} D^\bullet \xrightarrow{\text{qi}} \mathbb{X}(G) \to \mathbb{X}(H)$$

for a homogeneous space X of G with $\mathrm{Pic}(G) = 0$.

This will prove the following theorem:

Main Theorem

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For a homogeneous space X with stabilizer H, of a connected linear group G over a field k_0 of characteristic 0, with $\mathrm{Pic}(G) = 0$, we have a canonical isomorphism in the derived category of Galois modules

$$\mathrm{UPic}(X) \xrightarrow{\sim} (\,\mathbb{X}(G) \to \mathbb{X}(H)\,).$$

Equivariant Picard group $\mathrm{Pic}_G(X)$

Equivariant Picard group

A *G-line bundle* on a G-variety X is a pair (L, α), where L is a line bundle, and α is a *G-linearization of L*, i.e. an action of G on L compatible with the action on X.

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Definition. The *equivariant Picard group* $\mathrm{Pic}_G(X)$ is the group of classes of G-line bundles on X.

We have a canonical homomorphism

$$\mathrm{Pic}_G(X) \to \mathrm{Pic}(X), \quad [L, \alpha] \mapsto [L]$$

Proposition (Popov 1974). *If X is a homogeneous space and* $\mathrm{Pic}(G) = 0$, *then the canonical homomorphism*

$$\mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$$

is surjective, i.e. any line bundle on X admits a G-linearization.

Popov's theorem

A fundamental observation:

Theorem (Popov, 1974)

Let X be a homogeneous space of a connected linear algebraic group G with stabilizer H. Then

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The stabilizer H of x acts on the 1-dimensional fibre L_x of L over x, giving us a character of H.

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The stabilizer H of x acts on the 1-dimensional fibre L_x of L over x, giving us a character of H.

Now we see that $\mathrm{Gal}(k/k_0)$ indeed acts on $\mathbb{X}(H)$, because it acts on $\mathrm{Pic}_G(X)$.

Extended equivariant Picard complex $\mathrm{UPic}_G(X)$

Joost van Hamel:

If there is an equivariant Picard group $\mathrm{Pic}_G(X)$,
then there must be
an *extended equivariant Picard complex*

$$\mathrm{UPic}_G(X),$$

a complex in degrees 0 and 1 with

$$\operatorname{coker} \mathrm{UPic}_G(X) = \mathrm{Pic}_G(X).$$

Double complex

In order to define the complex $\mathrm{UPic}_G(X)$, we consider the following double complex for any irreducible G-variety X of a connected linear k-group G:

$$\begin{array}{ccc}
\cdots & & \cdots \\
\uparrow d^2_{\mathscr{K}} & & \uparrow d^2_{\mathrm{Div}} \\
\mathscr{K}(X \times G \times G)^\times & \xrightarrow{\mathrm{div}^2} & \mathrm{Div}(X \times G \times G) \\
\uparrow d^1_{\mathscr{K}} & & \uparrow d^1_{\mathrm{Div}} \\
\mathscr{K}(X \times G)^\times & \xrightarrow{\mathrm{div}^1} & \mathrm{Div}(X \times G) \\
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\mathscr{K}(X)^\times & \xrightarrow{\mathrm{div}^0} & \mathrm{Div}(X).
\end{array}$$

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\mathscr{K}(X)^\times & \xrightarrow{\mathrm{div}^0} & \mathrm{Div}(X).
\end{array}$$

We interprete the groups in this diagram as *algebraic cochains of G*, using an idea of a paper by Knop, Kraft, and Vust (1989).

$$
\begin{array}{ccc}
\mathscr{K}(X \times G \times G)^{\times} & \xrightarrow{\mathrm{div}^2} & \mathrm{Div}(X \times G \times G) \\
\uparrow {\scriptstyle d^1_{\mathscr{K}}} & & \uparrow {\scriptstyle d^1_{\mathrm{Div}}} \\
\mathscr{K}(X \times G)^{\times} & \xrightarrow{\mathrm{div}^1} & \mathrm{Div}(X \times G) \\
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\mathscr{K}(X)^{\times} & \xrightarrow{\mathrm{div}^0} & \mathrm{Div}(X).
\end{array}
$$

Here $\mathscr{K}(X)^{\times}$ is in bidegree $(0, 0)$.

The horizontal arrows div^0, div^1, $\dots$, associate to a rational finction its divisor.

$$
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\end{array}
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The vertical differentials are given by the usual formulas:

$$d^0_{\mathscr{K}}(f)_g(x) = ({}^g f/f)(x) = f(xg)/f(x) \text{ for } f \in \mathscr{K}(X),$$

$$d^1_{\mathscr{K}}(c)_{g_1,g_2}(x) = ({}^{g_1}c_{g_2} \cdot (c_{g_1g_2})^{-1} \cdot c_{g_1})(x) = \frac{c_{g_2}(xg_1)c_{g_1}(x)}{c_{g_1g_2}(x)}$$

for $c \in \mathscr{K}(X \times G)^\times$, and similar for d^0_{Div} and d^1_{Div}.

$$\begin{array}{ccc}
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\end{array}$$

We denote by $C^\bullet$ the total complex of this double complex, and we set:

$$\mathrm{UPic}_G(X) = \tau_{\leq 1} C^\bullet / k^\times .$$

This means the following. Set

$$Z^1_{\mathrm{alg}}(G, \mathscr{K}(X)^\times) := \{z \in \mathscr{K}(X \times G)^\times \mid z_{g_1 g_2}(x) = z_{g_1}(x) \cdot z_{g_2}(xg_1)\}$$

Then

$$\mathrm{UPic}_G(X) = \ \mathrm{UPic}_G(X)^0 \xrightarrow{\partial} \mathrm{UPic}_G(X)^1,$$

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where

$$\mathrm{UPic}_G(X)^0 = \mathscr{K}(X)^\times / k^\times,$$

$$\mathrm{UPic}_G(X)^1 =$$

$$\{(z, D) \in Z^1_{\mathrm{alg}}(G, \mathscr{K}(X)^\times) \oplus \mathrm{Div}(X) \mid \mathrm{div}(z) = d^0_{\mathrm{Div}}(D)\}$$

$$\partial([f]) = (d^0_{\mathscr{K}}(f), \mathrm{div}(f))$$

.

Lemma. $\mathscr{H}^0(C^\bullet) = (\mathscr{O}(X)^\times)^G$.

Corollary.
$\ker \mathrm{UPic}_G(X) := \mathscr{H}^0(\,\mathrm{UPic}_G(X)\,) = (\mathscr{O}(X)^\times)^G/k^\times.$

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Corollary.
$\ker \mathrm{UPic}_G(X) \;:=\; \mathscr{H}^0(\,\mathrm{UPic}_G(X)\,) = (\mathscr{O}(X)^\times)^G/k^\times.$

Corollary. When X is a *homogeneous space*,
$\ker \mathrm{UPic}_G(X) \;=\; 0.$

Theorem

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The cokernel of our $\mathrm{UPic}_G(X)$ is indeed $\mathrm{Pic}_G(X)$!

Using $\mathrm{UPic}_G(X)$, we define an isomorphism in the derived category, see the diagram below, assuming that X is a homogeneous space and $\mathrm{Pic}(G) = 0$.

Isomorphism in the derived category

Both rectangles in the following diagram are quasi-isomorphisms:

$$\begin{array}{ccccc} \mathbb{X}(G) & \longleftarrow & \mathbb{X}(G) \oplus \mathscr{K}(X)^\times/k^\times & \longrightarrow & \mathscr{K}(X)^\times/k^\times \\ \downarrow \sigma & & \downarrow & & \downarrow \mathrm{div} \\ \mathrm{Pic}_G(X) & & & & \mathrm{Div}(X) \end{array}$$

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\downarrow \sigma & \text{qi} & \downarrow \psi & \text{qi} & \downarrow \text{div} \\
\mathrm{Pic}_G(X) & \longleftarrow & \mathrm{UPic}_G(X)^1 & \longrightarrow & \mathrm{Div}(X)
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where the arrow σ takes a character $\chi \in \mathbb{X}(G)$ to the class of the trivial line bundle $A^1 \times X$ over X with the G-action given by χ,
and the arrow ψ is given by

$$\psi(\chi, [f]) = (\chi \cdot d^0_{\mathscr{K}}(f), \mathrm{div}(f)) \in \mathrm{UPic}_G(X)^1 \subset Z^1_{\mathrm{alg}}(G, \mathscr{K}(X)^\times) \oplus \mathrm{Div}(X).$$

Since by Popov's theorem we have a commutative diagram

$$\begin{array}{ccc} \mathbb{X}(G) & = & \mathbb{X}(G) \\ {\scriptstyle \text{res}}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathbb{X}(H) & \xrightarrow{\cong} & \mathrm{Pic}_G(X), \end{array}$$

where $\mathbb{X}(H) \cong \mathrm{Pic}_G(X)$,
the diagram in the previous screen gives a canonical isomorphism in the derived category

$$(\,\mathbb{X}(G) \to \mathbb{X}(H)\,) \xrightarrow{\sim} \mathrm{UPic}(X):$$

$$\begin{array}{ccccc}
\mathbb{X}(G) & \longleftarrow & \mathbb{X}(G) \oplus \mathscr{K}(X)^\times / k^\times & \longrightarrow & \mathscr{K}(X)^\times / k^\times \\
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\mathbb{X}(H) & \longleftarrow & \mathrm{UPic}_G(X)^1 & \longrightarrow & \mathrm{Div}(X)
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which completes the proof of the Main Theorem.

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be the cohomological Brauer-Grothendieck group of X_0.

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Let

$$\mathrm{Br}_a(X_0) = \ker[\mathrm{Br}(X_0) \to \mathrm{Br}(X)]/\mathrm{im}[\mathrm{Br}(k_0) \to \mathrm{Br}(X_0)]$$

be the algebraic Brauer group of X_0,
it is a subquotient of $\mathrm{Br}(X_0)$.

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$$\mathrm{Br}_a(X_0) = \ker[\mathrm{Br}(X_0) \to \mathrm{Br}(X)]/\mathrm{im}[\mathrm{Br}(k_0) \to \mathrm{Br}(X_0)]$$

be the algebraic Brauer group of X_0,
it is a subquotient of $\mathrm{Br}(X_0)$.

One needs $\mathrm{Br}_a(X_0)$ when working with the Brauer-Manin obstruction to the Hasse principle, when k_0 is a number field.

UPic(X) and $\mathrm{Br}_a(X_0)$

Proposition

Let X_0 be a smooth geometrically integral variety over a field k_0 of characteristik 0. If $H^3(k_0, \mathbb{G}_m) = 0$ (e.g. when k_0 is a number field, or a p-adic field, or $\mathbb{R}$), then there is a canonical isomorphism

$$\mathrm{Br}_a(X_0) \cong \mathbb{H}^2(k_0, \mathrm{UPic}(X)).$$

UPic(X) and $Br_a(X_0)$

Proposition

Let X_0 be a smooth geometrically integral variety over a field k_0 of characteristik 0. If $H^3(k_0, \mathbb{G}_m) = 0$ (e.g. when k_0 is a number field, or a p-adic field, or $\mathbb{R}$), then there is a canonical isomorphism

$$\mathrm{Br}_a(X_0) \cong \mathbb{H}^2(k_0, \mathrm{UPic}(X)).$$

Here $\mathbb{H}^2(k_0, \mathrm{UPic}(X))$ denotes the second Galois hypercohomology with coefficients in the complex of Galois modules $\mathrm{UPic}(X)$.

$Br_a(X_0)$

From the above results we obtain:

Theorem

Let X_0 be a homogeneous space with geometric stabilizer H of a connected linear k_0-group G_0 with $\mathrm{Pic}(G) = 0$.
If $H^3(k_0, \mathbb{G}_m) = 0$
(e.g. k_0 is a number field, or a p-adic field, or $\mathbb{R}$),
we have a canonical isomorphism

$$\mathrm{Br}_a(X_0) \cong \mathbb{H}^2(k_0, \mathbb{X}(G) \to \mathbb{X}(H)).$$

Proof. By the proposition, $\mathrm{Br}_a(X_0) \cong \mathbb{H}^2(k, \mathrm{UPic}(X))$. By the Main Theorem we have a canonical isomorphism in the derived category of Galois modules

$$\mathrm{UPic}(X) \xrightarrow{\sim} (\mathbb{X}(G) \to \mathbb{X}(H)),$$

hence a canonical isomorphism

$$\mathbb{H}^2(k_0, \mathrm{UPic}(X)) \xrightarrow{\sim} \mathbb{H}^2(k_0, \mathbb{X}(G) \to \mathbb{X}(H)),$$

which gives us the required isomorphism

$$\mathrm{Br}_a(X_0) \xrightarrow{\sim} \mathbb{H}^2(k_0, \mathbb{X}(G) \to \mathbb{X}(H)).$$

We have computed $\mathrm{Br}_a(X_0)$ in terms of $\mathbb{X}(G)$ and $\mathbb{X}(H)$, when k_0 is a number field.

This result was recently generalized by Cyril Demarche, who, for a homogeneous space X_0 of a connected group G_0 over k_0, computed the group $\mathrm{Br}_a(X_0, G_0)$ defined by

$$\mathrm{Br}_a(X_0, G_0) = \ker[\mathrm{Br}(X_0) \to \mathrm{Br}(X) \to \mathrm{Br}(G)]/\mathrm{im}[\mathrm{Br}(k_0) \to \mathrm{Br}(X_0)]$$

when the geometric stabilizer H is connected. One needs this group for the Brauer-Manin obstruction to *strong approximation* for X_0.