Number Theory Homework #5

1. Let g be a primitive root modulo m. Prove that g^k is a primitive root modulo m if and only if $\gcd(k, \varphi(m)) = 1$. Deduce that if there exists a primitive root modulo m, then the number of primitive roots modulo m is $\varphi(\varphi(m))$.

2. (a) Show that 2 is a primitive root modulo 29.
(b) Compute all primitive roots for $p = 11,\ 13,\ 17$, and 19.

3. (a) Find the four primitive roots modulo 26 and the eight primitive roots modulo 25.
(b) Determine all the primitive roots modulo 3^2, 3^3, and 3^4.

4. (a) Prove that 3 is a primitive root for all integers of the form 7^k and $2 \cdot 7^k$.
(b) Find a primitive root for any integer of the form 17^k.

5. Prove that if p and $q = 2p + 1$ are both odd primes (for example $p = 5$ and $q = 11$), then -4 is a primitive root mod q.

6. Show that 4 is not a primitive root modulo n for any $n \geq 2$.

7. Let $p \geq 3$ be a prime number, let $r \in \mathbb{N}$, and let x be a primitive root modulo p^r. Show that x is a primitive root modulo p.