Lagrangian Quantum Structures I
Algebraic Structures and Topological Rigidity

Conference on Symplectic Geometry and Physics

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Joint work with Octav Cornea, University of Montreal

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Our goal is NOT Lagrangian intersections!!!

Quick review of HF and QH

The pearl complex (suggested by Fukaya, Oh).

- $L \subset M$ monotone.
  $\omega(A) > 0$ iff $\mu(A) > 0$, $A \in \pi_2(M, L)$.
  $N_L =$ min. Maslov number ≥ 2.
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- $f : L \to \mathbb{R}$ Morse. $\mathcal{C}_*(f) = \mathbb{Z}_2\langle \text{Crit}(f)\rangle \otimes \Lambda_*$.
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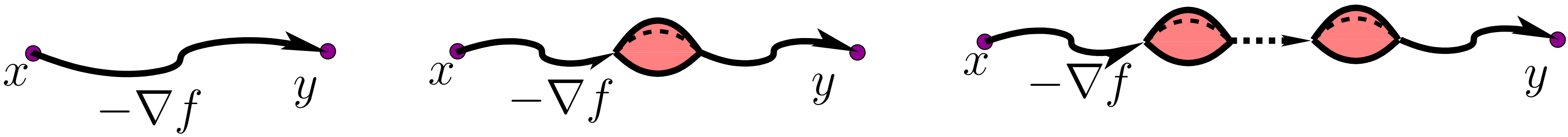

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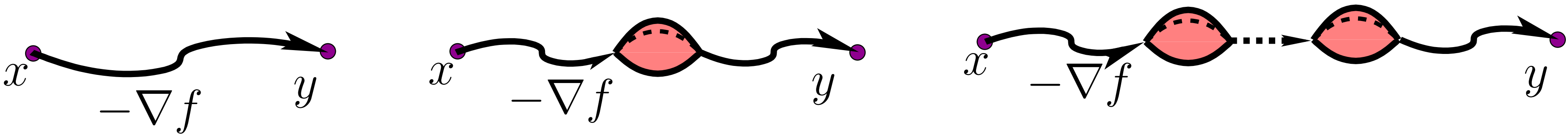

- PSS-type argument $\Longrightarrow H_*(\mathcal{C}(f), d) \cong HF_*(L, L)$.

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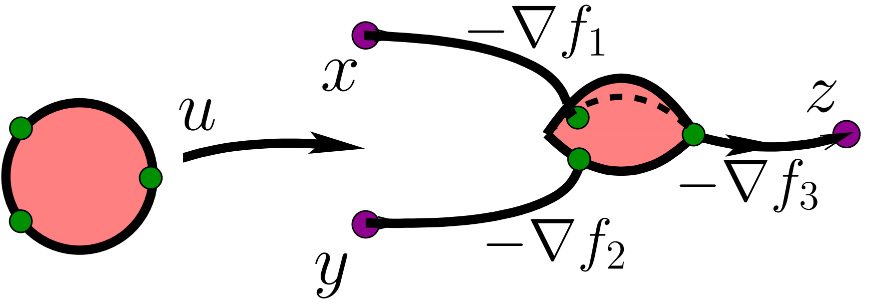

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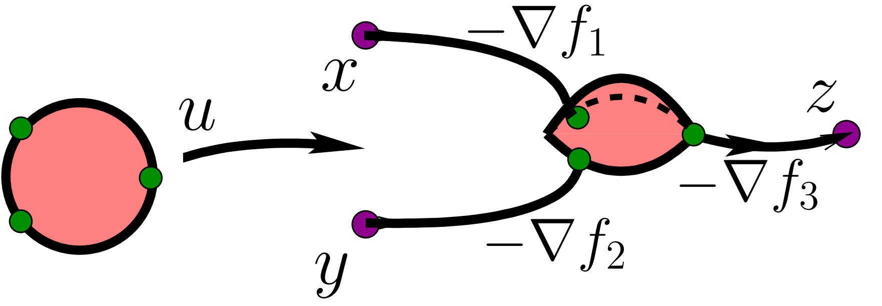

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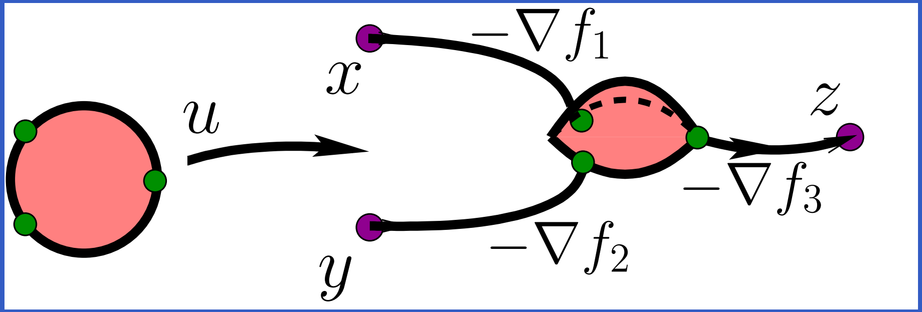

- Possible to work with 2 functions (i.e. $f_3 = f_1$).
- $HF(L)$ becomes a (NON COMMUTATIVE) ring with a unity $w \in HF_n(L)$. Actually, $w = [\text{max}]$.

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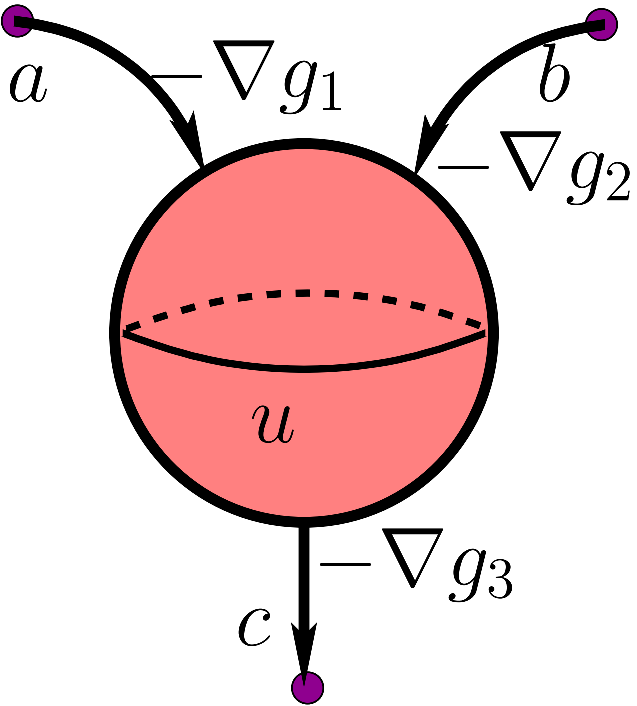

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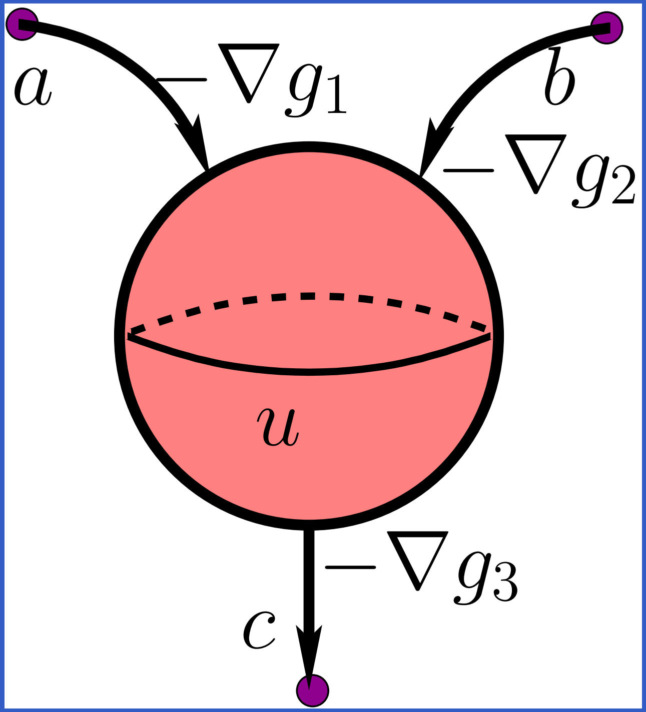

- $QH_*(M)$ becomes a (commutative) ring.
  Unity = fundamental class $u = [M] \in QH_{2n}(M)$.

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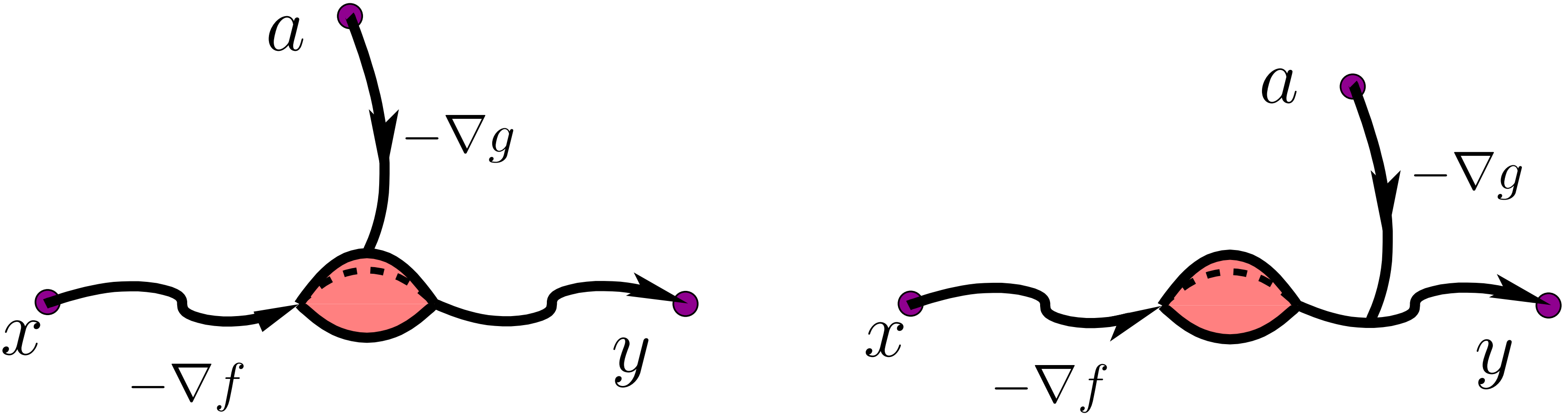

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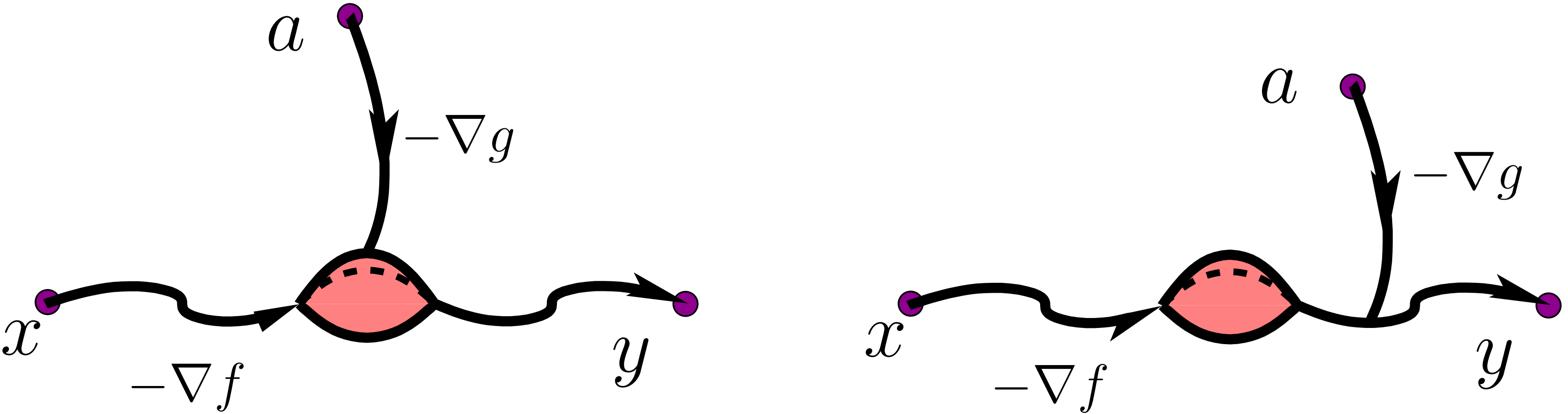
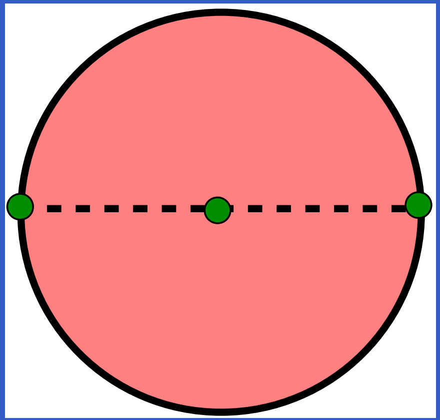


IMPORTANT: The 3 points on the disk marked by the $-\nabla g$ and $-\nabla f$ trajectories are fixed! E.g we can take them to be -1, 0, 1.



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$\forall \;\; a, b \in QH(M)$, $\gamma, \delta \in HF(L)$:

$$a * (b * \gamma) = (a * b) * \gamma,$$

$$a * (\gamma * \delta) = (a * \gamma) * \delta = \gamma * (a * \delta),$$

$u * \gamma = \gamma$ etc.

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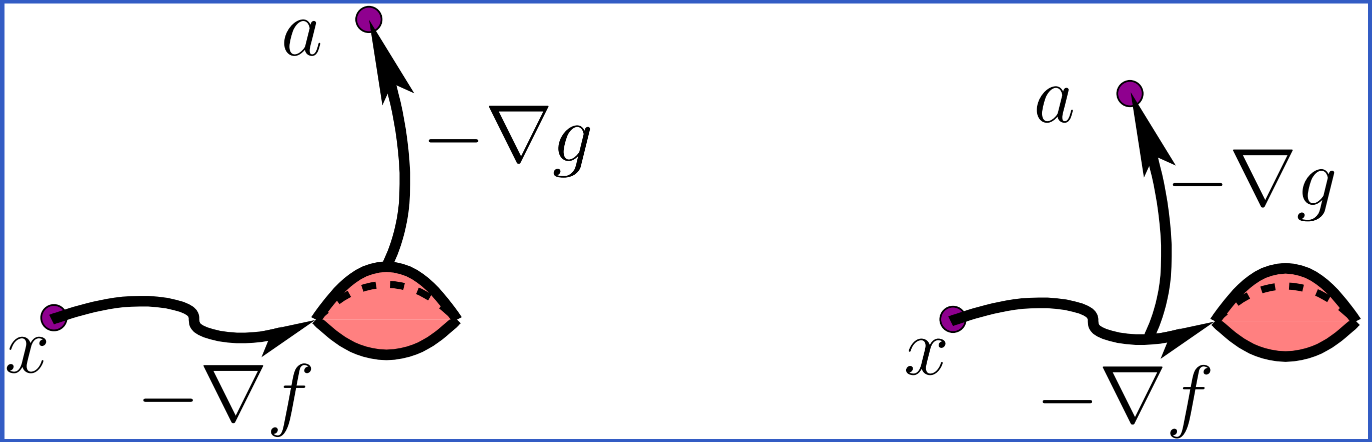

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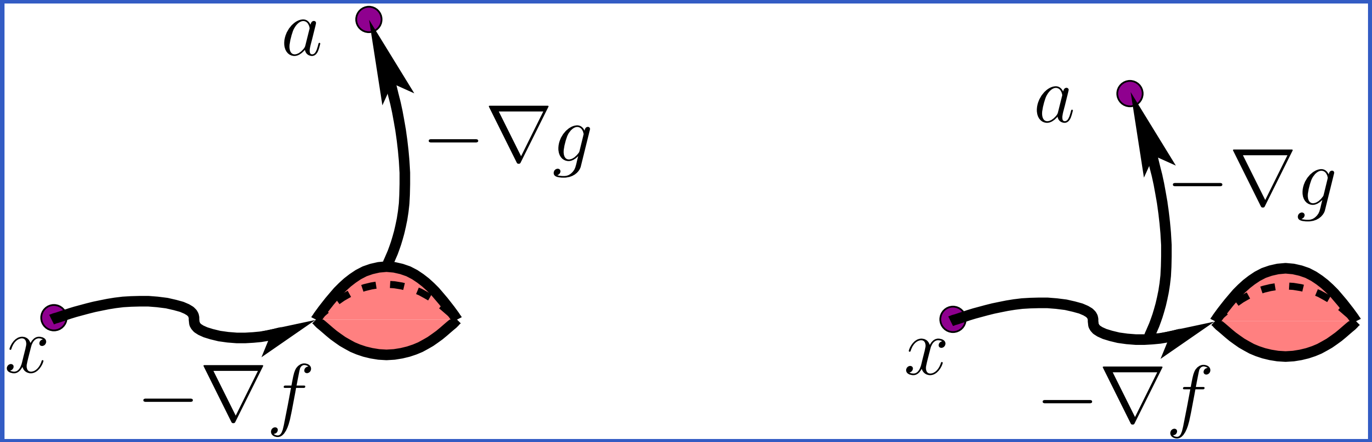

Everything is compatible with duality:
$\forall\, h \in H_*(M)$, $\alpha \in HF_*(L)$: $\langle PD(h), i_L(\alpha)\rangle = \epsilon_L(h * \alpha)$.

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(Compatibility with the quantum product was previously noticed by Buhovsky and by Fukaya-Oh-Ohta-Ono).

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The restriction of λ to $Symp_0(M) \cap Symp(M, L)$ gives automorphisms of $HF_*(L)$ as an algebra over $QH_*(M)$.

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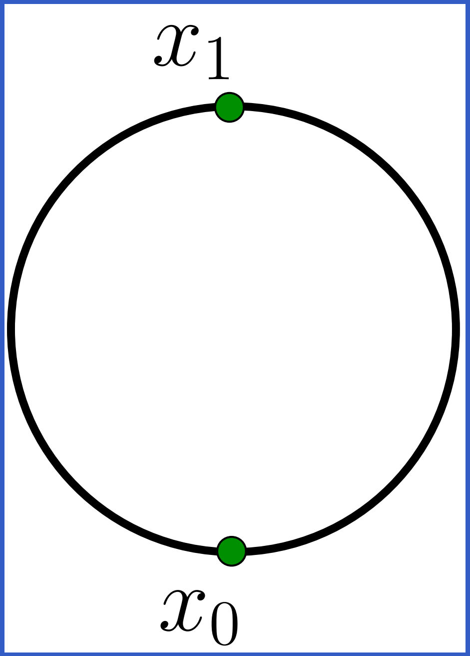
(A similar object has been studied in the context of Lagrangian intersections by Fukaya-Oh-Ohta-Ono).

An example: $L = S^1 \subset \mathbb{R}^2$



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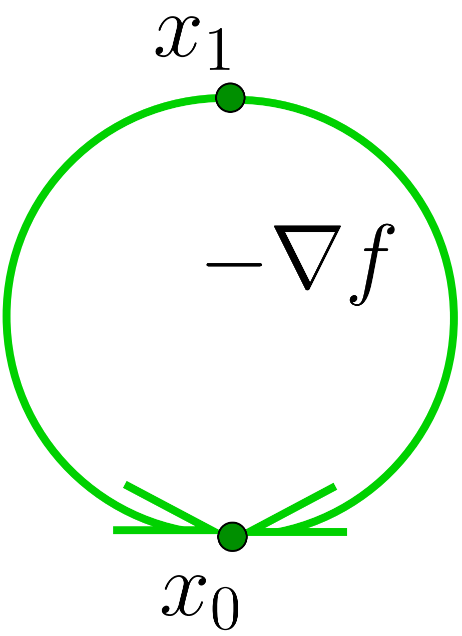

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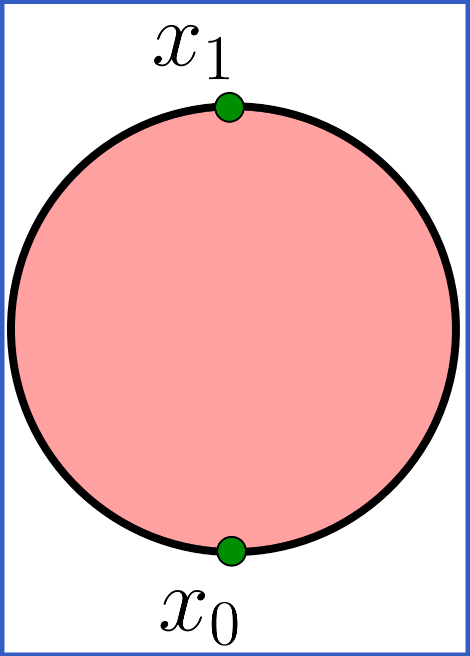

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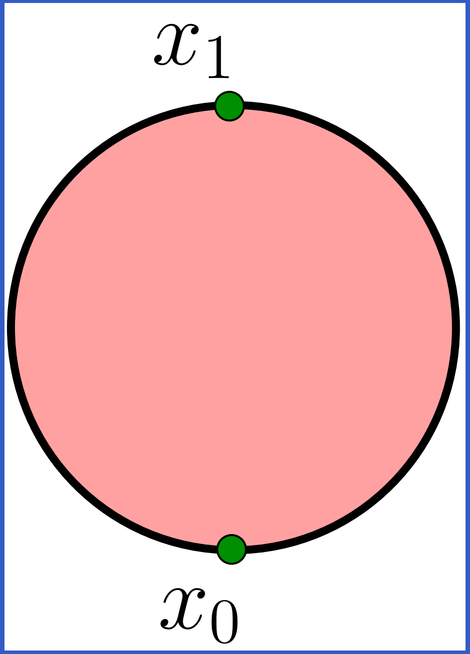

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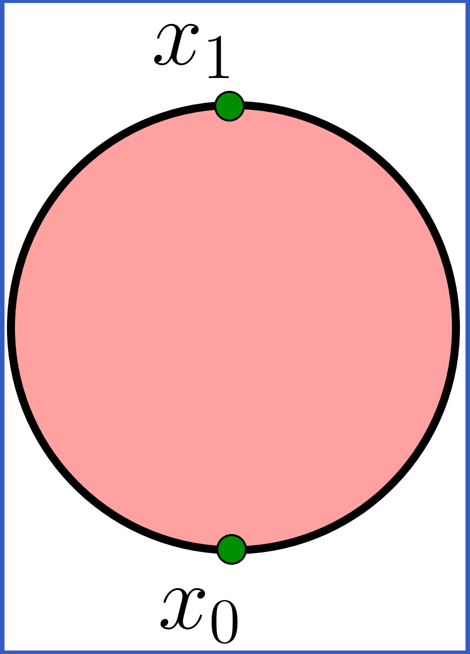

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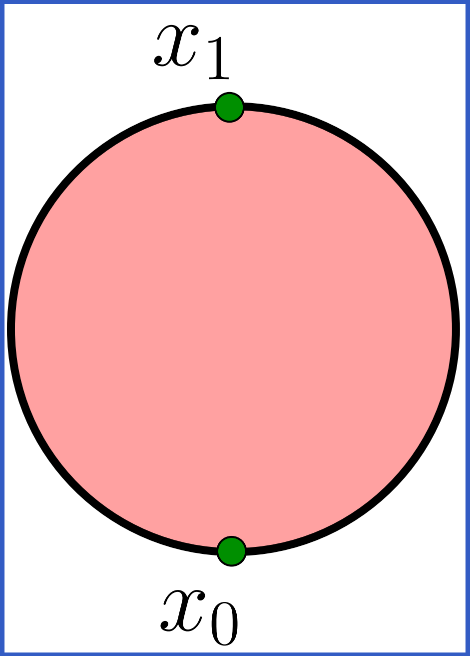

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$HF(L) = 0$ (in fact L is displaceable).

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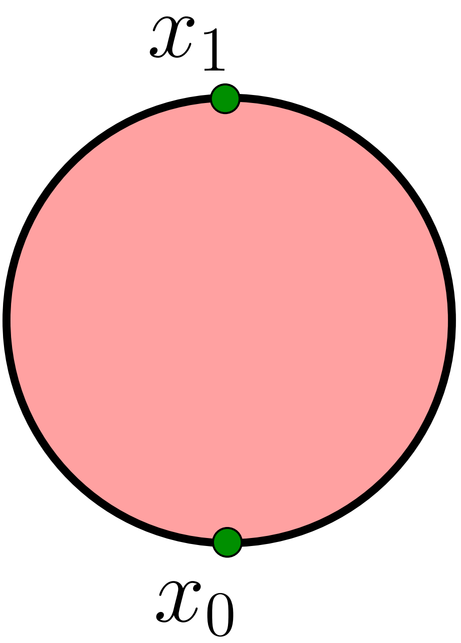

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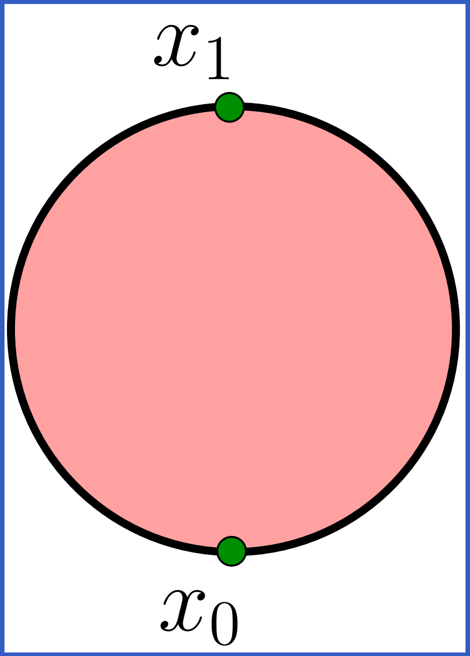

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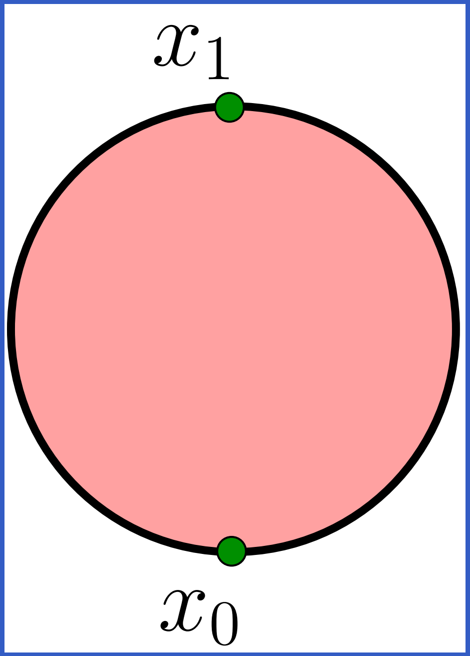

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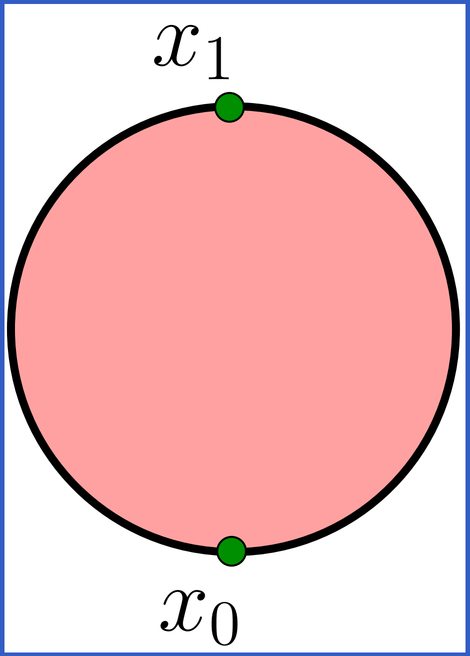

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In this example x_1 is torsion: $\quad t \cdot x_1 = 0.$

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- Transversality.

  We need all 0-dim & 1-dim moduli spaces of pearly trajectories to be smooth and of expected dimensions.
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- Compactification of the 1-dim moduli spaces of pearls.
- Gluing.
  Existence: we followed Fukaya-Oh-Ohta-Ono.
  Uniqueness: we proved surjectivity of the gluing map for 0 and 1-dim moduli spaces.

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Very useful if we know that $HF_i(L) \neq 0$ for some i.

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Statement 1 was proved before by Seidel by other methods. An alternative proof by B.

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In other words for L as before we have:

i	...	-1	0	1	...	n-1	n	n+1	...
HF_i	...	$\mathbb{Z}_2\alpha_n t$	$\mathbb{Z}_2\alpha_0$	$\mathbb{Z}_2\alpha_1$	...	$\mathbb{Z}_2\alpha_{n-1}$	$\mathbb{Z}_2\alpha_n$	$\mathbb{Z}_2\alpha_0 t^{-1}$	...

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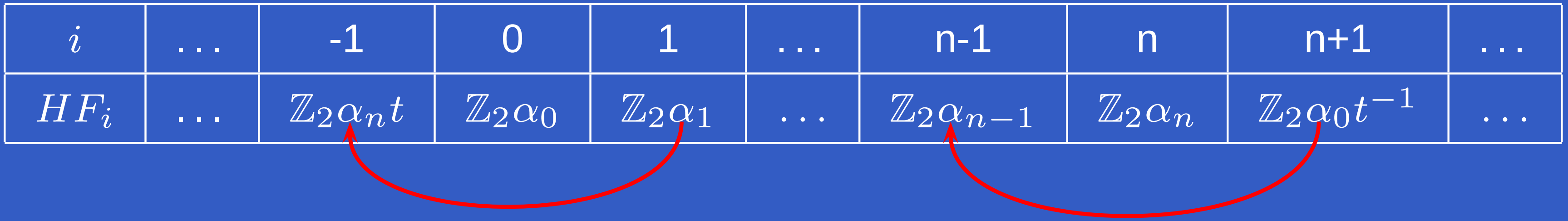

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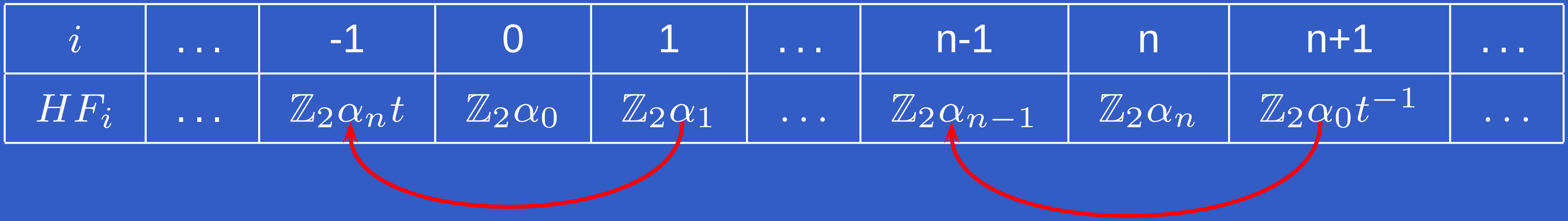

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Thm: If $n =$ even or $L \approx \mathbb{R}P^n$ then

$$\alpha_j * \alpha_k = \alpha_{j+k-n} \quad \forall k, j \in \mathbb{Z}.$$

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1) If $n =$ even then:

$$i_L(\alpha_{2k}) = a_{2k}, \quad \forall\, 0 \leq 2k \leq n,$$
$$i_L(\alpha_{2k+1}) = a_{2k+n+2}t, \quad \forall\, 1 \leq 2k+1 \leq n-1.$$

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Quantum inclusion map $i_L : HF_*(L) \to QH_*(\mathbb{C}P^n)$.
Denote by $a_j \in H_j(\mathbb{C}P^n; \mathbb{Z}_2)$ the generator, $0 \leq j \leq 2n$.

Thm: Let $L \subset \mathbb{C}P^n$ as above.
1) If $n =$ even then:

$$i_L(\alpha_{2k}) = a_{2k}, \quad \forall\, 0 \leq 2k \leq n,$$
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2) If $n =$ odd then:

$$i_L(\alpha_{2k}) = a_{2k} + a_{2k+n+1}t, \quad \forall\, 0 \leq 2k \leq n,$$
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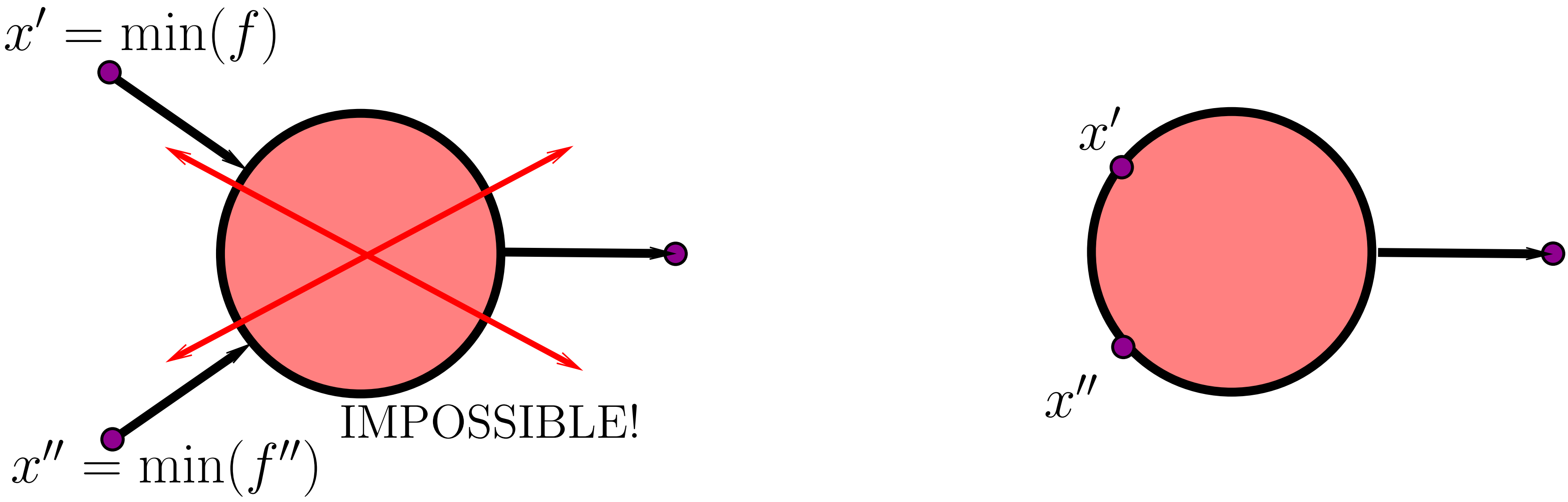

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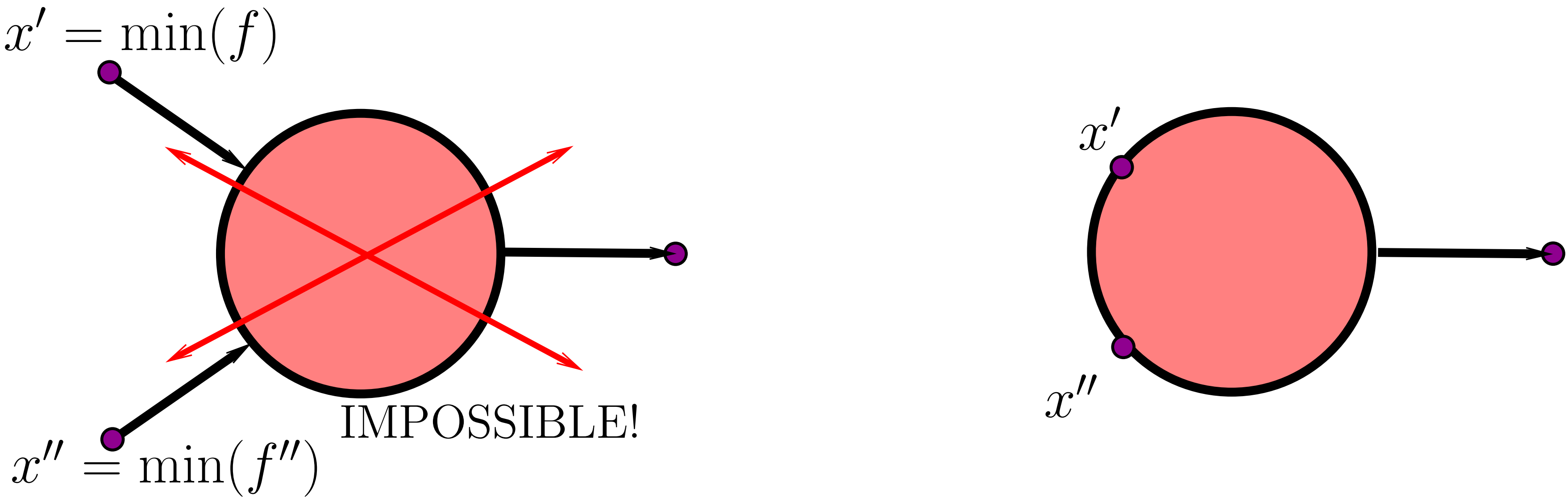

... more on that, in part II by Octav Cornea ...

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Results in the same spirit hold for Fano complete intersections.
(The point is that we know QH by work of Beauville.)

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Remark: This notion of semi-simplicity is somewhat different than semi-simplicity in the sense of Dubrovin (we work with different coefficient ring $\mathbb{F}$).

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Proof. A criterion of Abrams says that $QH_*^{ev}(M;\mathbb{F})$ is semi-simple iff the *quantum Euler class* $\mathcal{E}$ is invertible. But $\mathcal{E} \in QH_0(M)$. Let $S^n \approx L \subset M$. Under above assumptions, $HF_*(L) = H_*(L) \otimes \mathbb{F}$. Now use the module structure to deduce that $\mathcal{E} * (-)$ gives iso's $HF_*(L) \cong HF_{*-2n}(L)$ ... contradiction. $\square$

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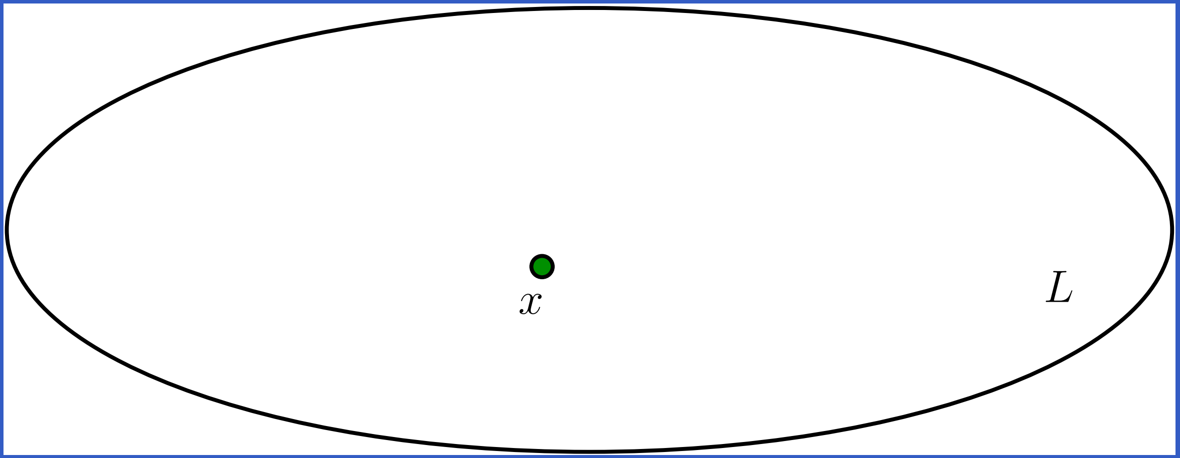

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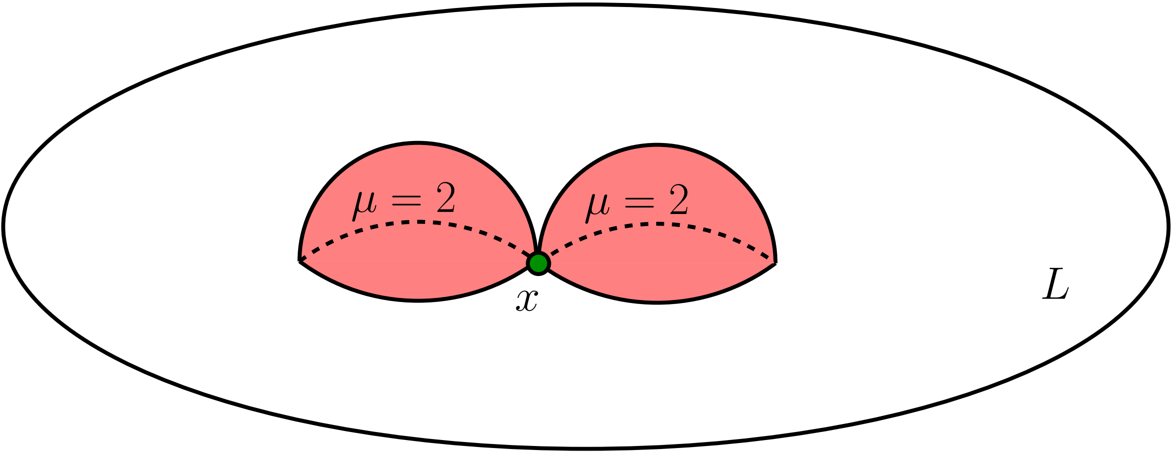

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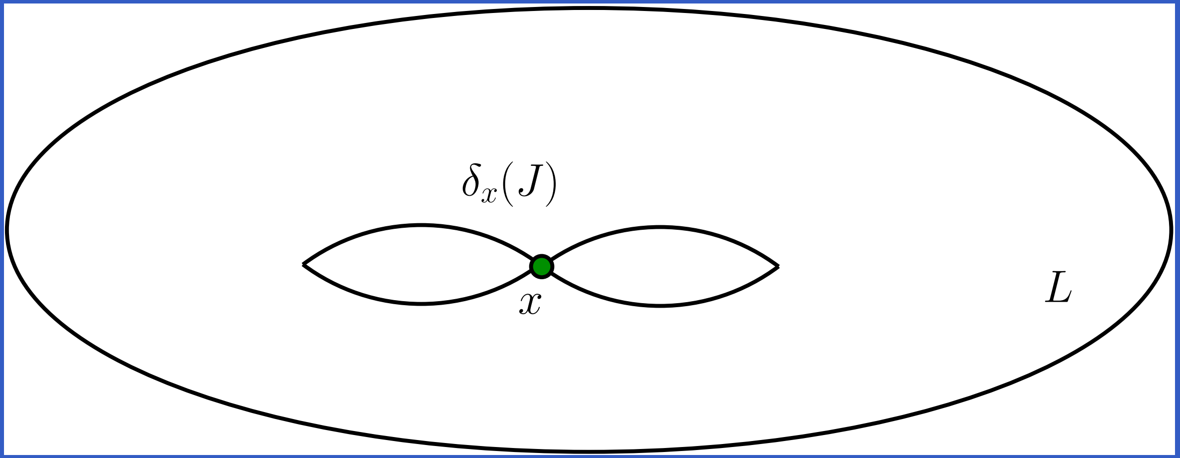

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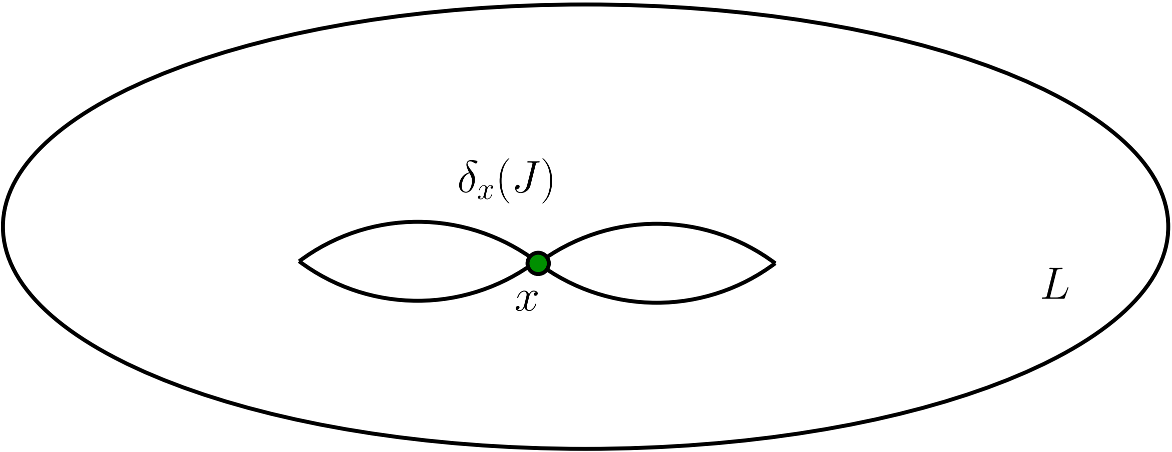

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$D = [\delta_x(J)] \in H_1(L;\mathbb{Z}_2)$. Doesn't depend on x and on J.
If $N_L \geq 3 \Longrightarrow D = 0$.

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Proof $f : \mathbb{T} \to \mathbb{R}$ perfect Morse function. $x_n = \max_f$.

$\implies H_*(\mathbb{T}) = \mathbb{Z}_2\langle \text{Crit}(f) \rangle$.

Assume $N_L = 2$. $\implies d : \mathbb{Z}_2\langle \text{Crit}_{n-1}(f) \rangle \to \mathbb{Z}_2 x_n t$.

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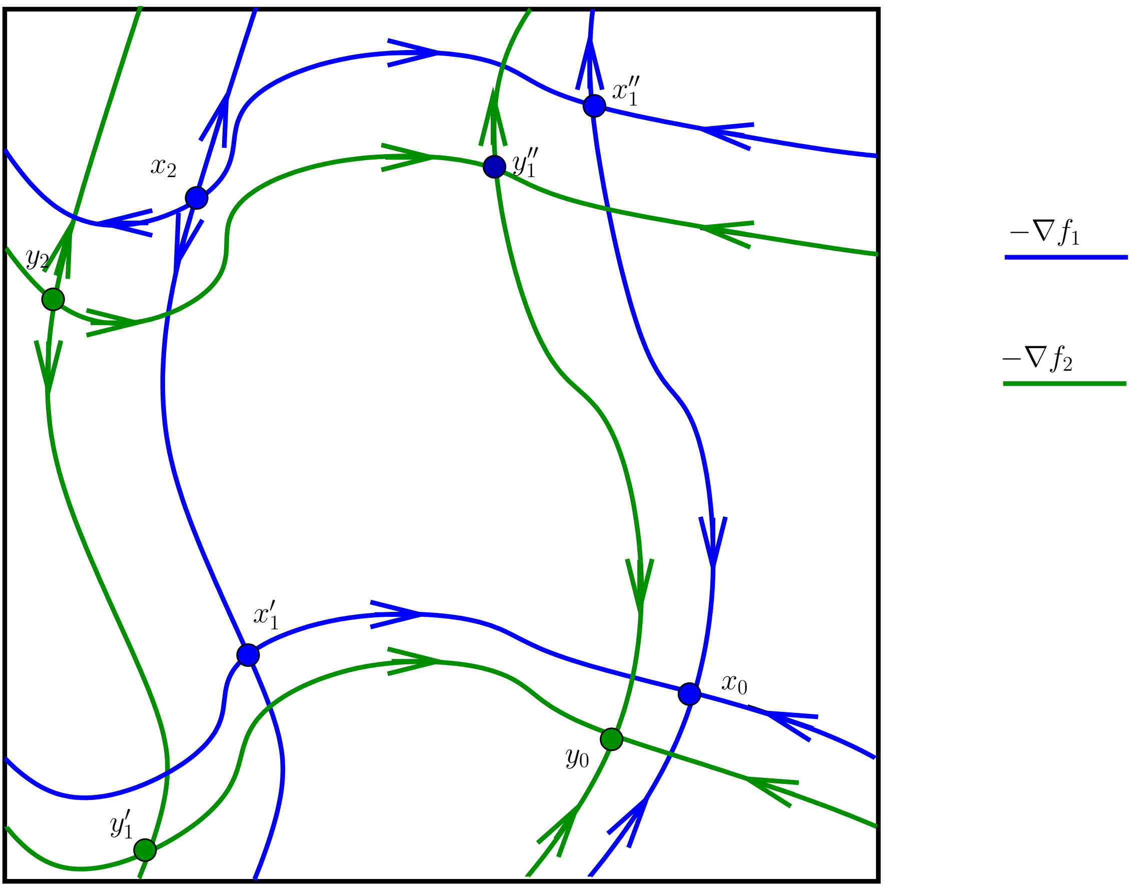

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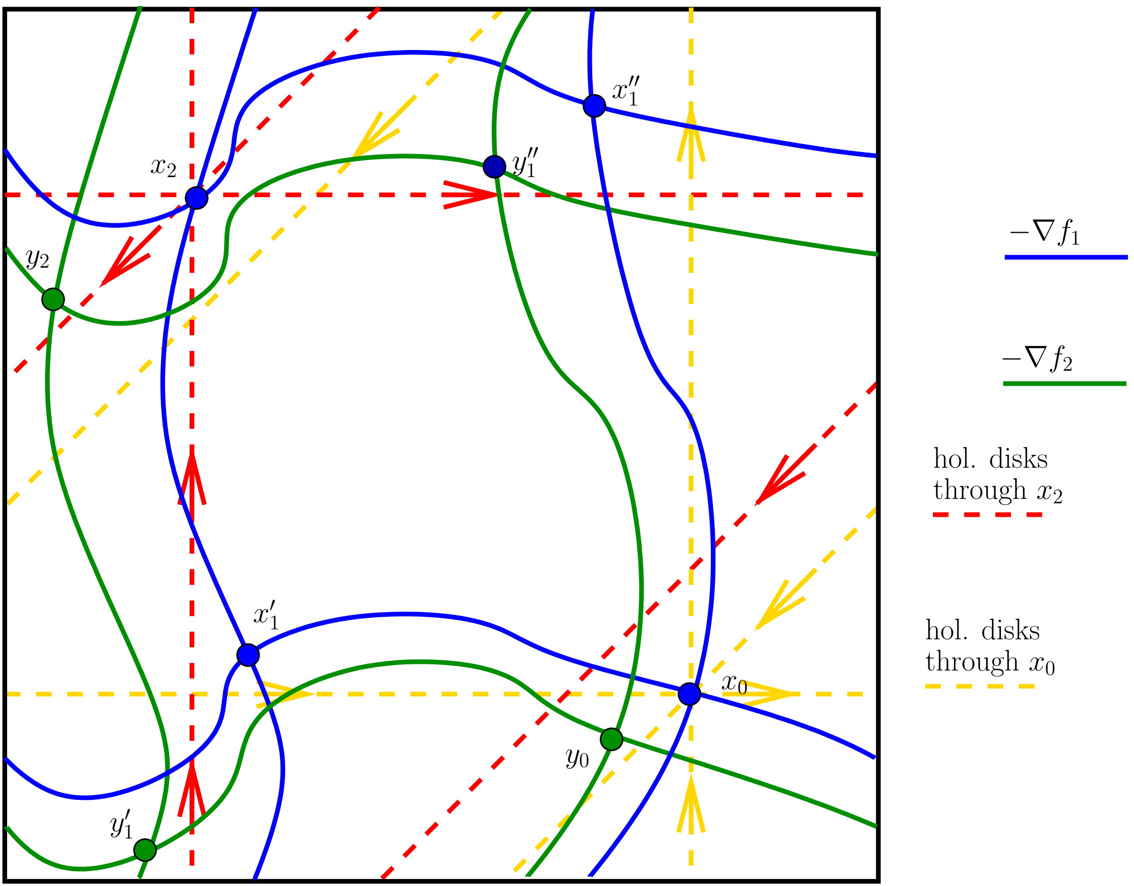

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Computation of $m * m$ follows from associativity.