

FROM MORSE TO FLOER & BEYOND

ZURICH COLLOQUIUM

October 2007

Paul Biran

Tel-Aviv University

Classical Morse Theory



Marston Morse, 1892–1977

Classical Morse Theory



Marston Morse, 1892–1977

- 1925: “Relations between the critical points of a real function of n independent variables”.
- 1934: “Calculus of variations in the large”.

Classical Morse Theory

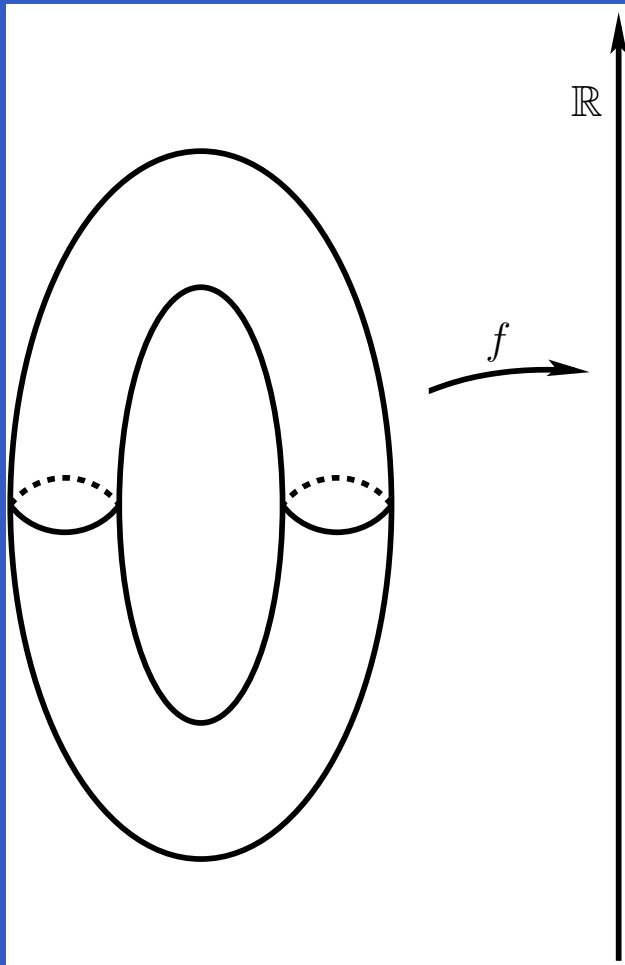


Marston Morse, 1892–1977

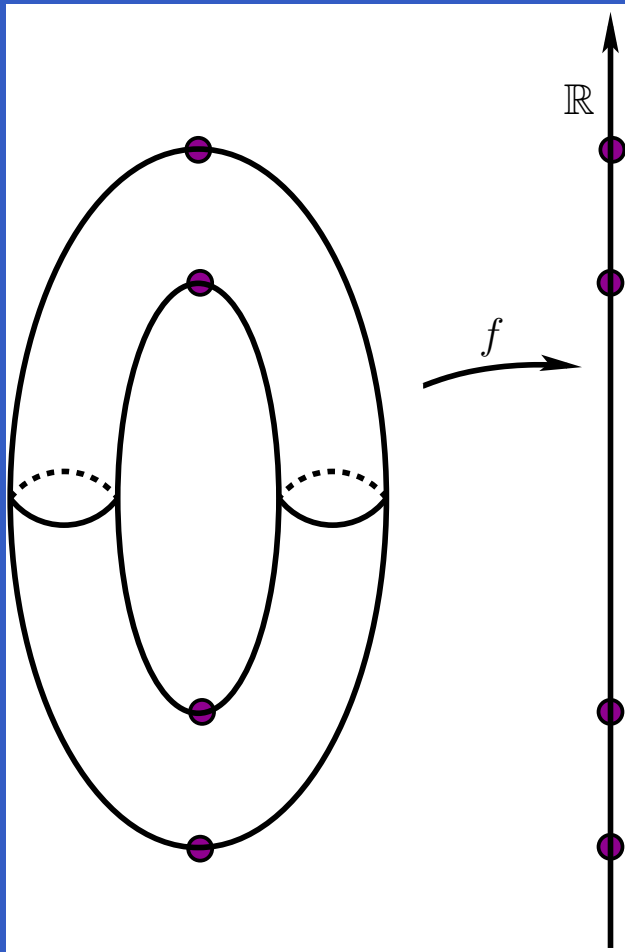
- 1925: “Relations between the critical points of a real function of n independent variables”.
- 1934: “Calculus of variations in the large”.

M manifold. What’s the relation between topology of M and smooth functions $f : M \rightarrow \mathbb{R}$?

Critical points



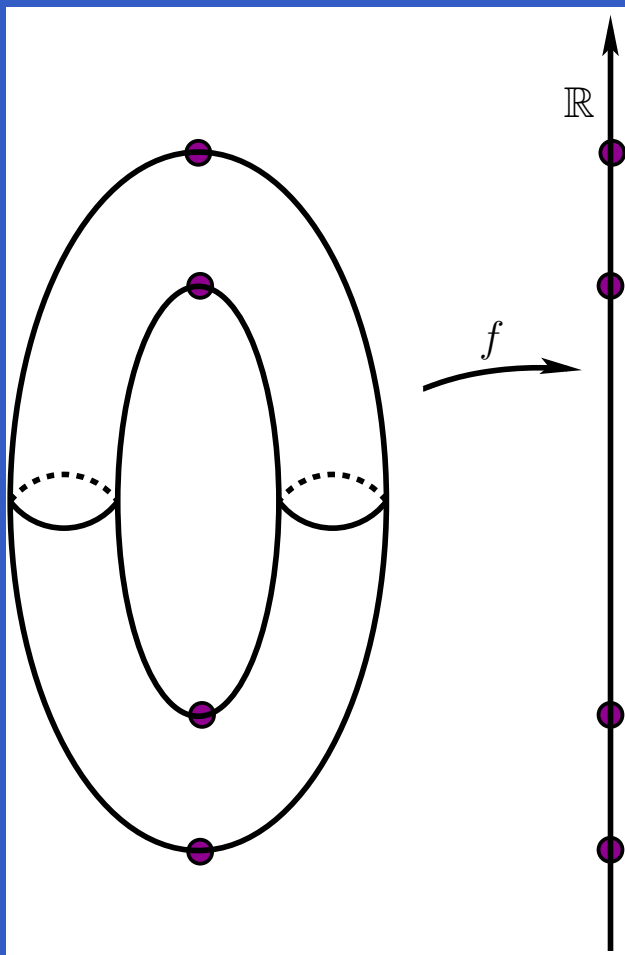
Critical points



- **Critical points:**

$$p \in \text{Crit}(f) \Leftrightarrow df_p = 0.$$

Critical points



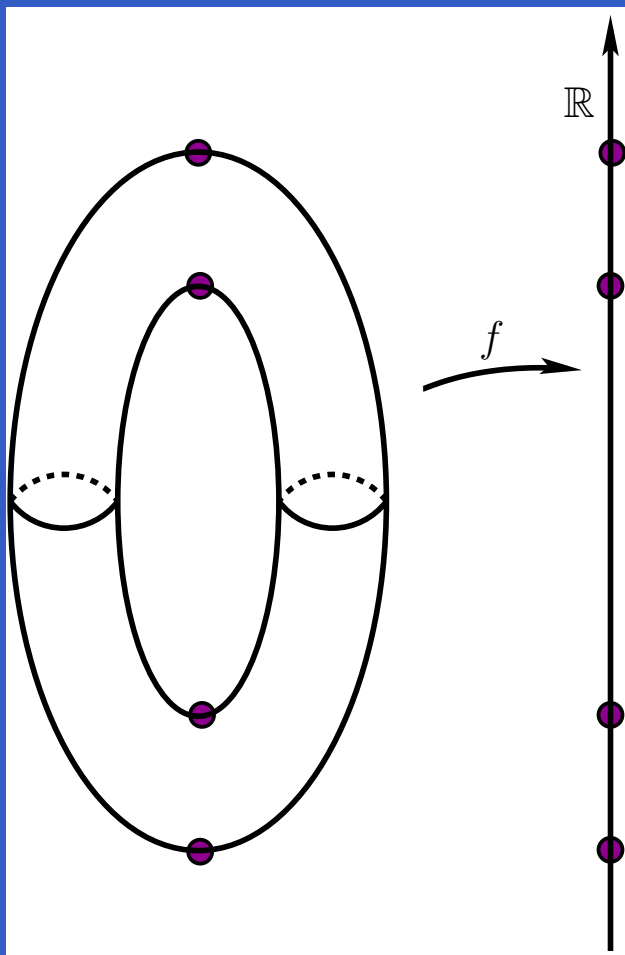
- **Critical points:**

$$p \in \text{Crit}(f) \Leftrightarrow df_p = 0.$$

- **Hessian:** $p \in \text{Crit}(f)$,

$$\text{Hess}_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j}.$$

Critical points



- **Critical points:**

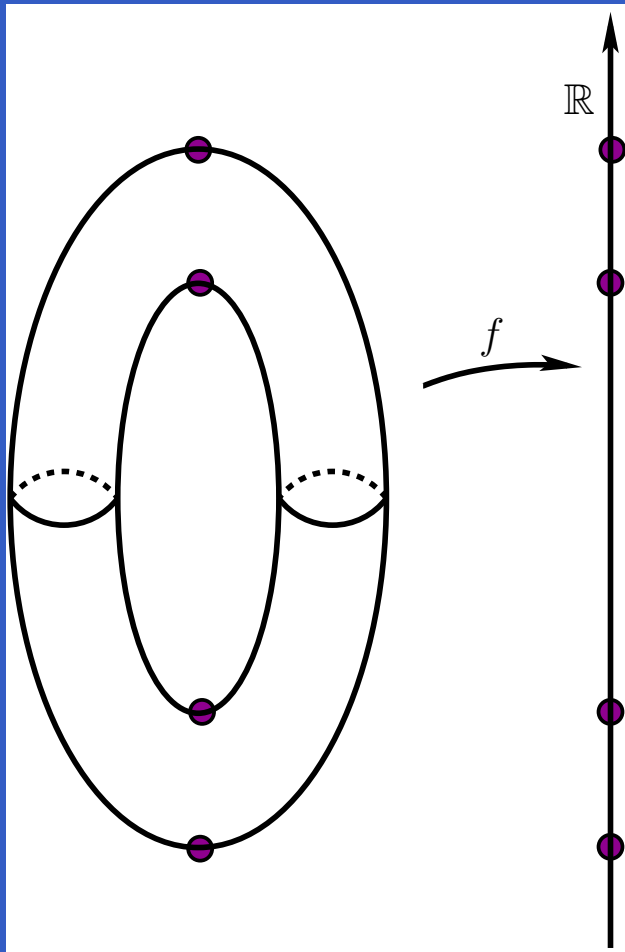
$$p \in \text{Crit}(f) \Leftrightarrow df_p = 0.$$

- **Hessian:** $p \in \text{Crit}(f)$,

$$\text{Hess}_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j}.$$

- $f : M \rightarrow \mathbb{R}$ is **Morse function** if $\forall p \in \text{Crit}(f)$, $\text{Hess}_p(f)$ is non-degenerate.

Critical points



- **Critical points:**

$$p \in \text{Crit}(f) \Leftrightarrow df_p = 0.$$

- **Hessian:** $p \in \text{Crit}(f)$,

$$\text{Hess}_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j}.$$

- $f : M \rightarrow \mathbb{R}$ is **Morse function** if $\forall p \in \text{Crit}(f)$, $\text{Hess}_p(f)$ is non-degenerate.

- **Index:** $p \in \text{Crit} f$,

$i_p := \#$ negative eigenvalues of $\text{Hess}_p(f)$.

- **Local model:** p critical point with $i_p = k \implies$ near p

$$f(x_1, \dots, x_n) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

- **Local model:** p critical point with $i_p = k \implies$ near p

$$f(x_1, \dots, x_n) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

- $p = \max \implies i_p = n, \quad p = \min \implies i_p = 0.$

- **Local model:** p critical point with $i_p = k \implies$ near p

$$f(x_1, \dots, x_n) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

- $p = \max \implies i_p = n, \quad p = \min \implies i_p = 0.$

Do Morse functions exist?

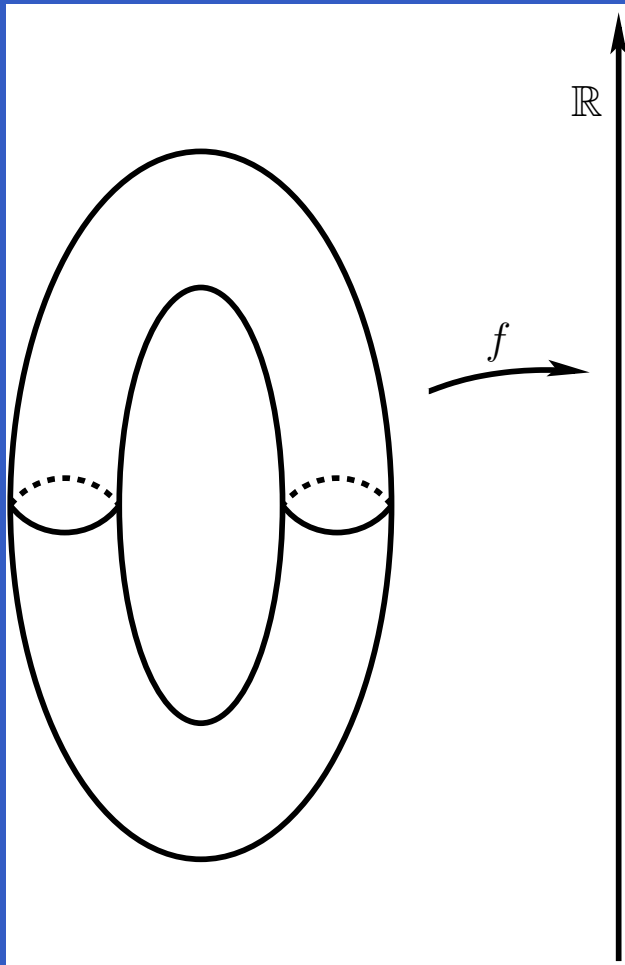
- **Local model:** p critical point with $i_p = k \implies$ near p

$$f(x_1, \dots, x_n) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

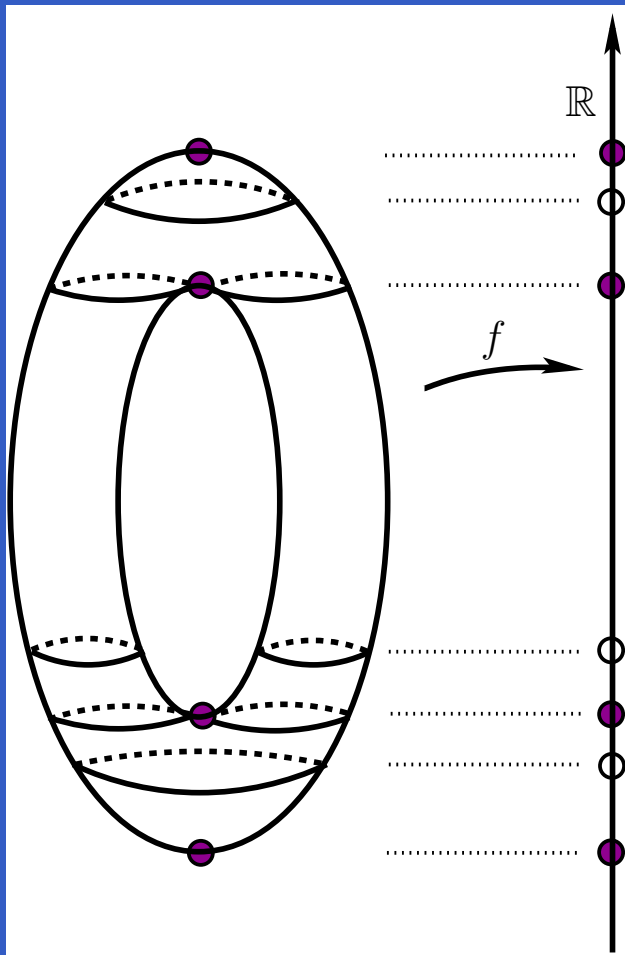
- $p = \max \implies i_p = n, \quad p = \min \implies i_p = 0.$

Do Morse functions exist? ...Almost every function $f : M \rightarrow \mathbb{R}$ is Morse !

What happens when we pass through a crit point ?

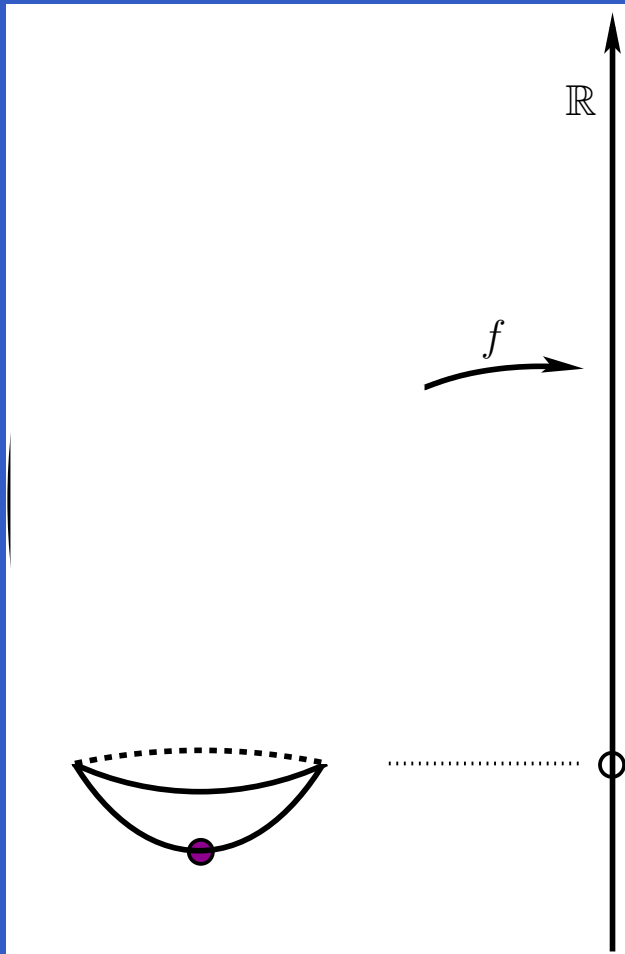


What happens when we pass through a crit point ?



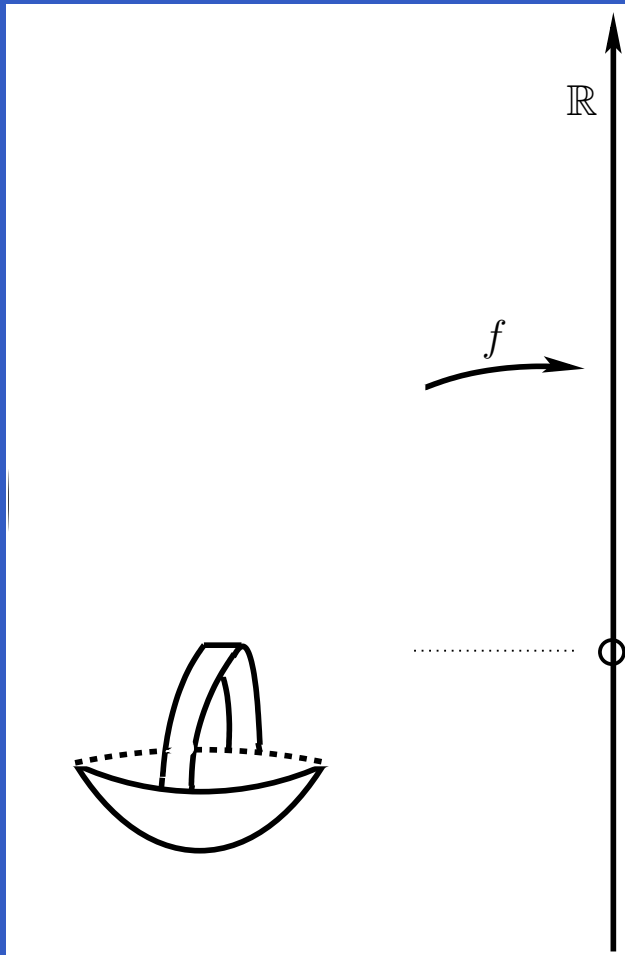
Topology of level sets changes

What happens when we pass through a crit point ?



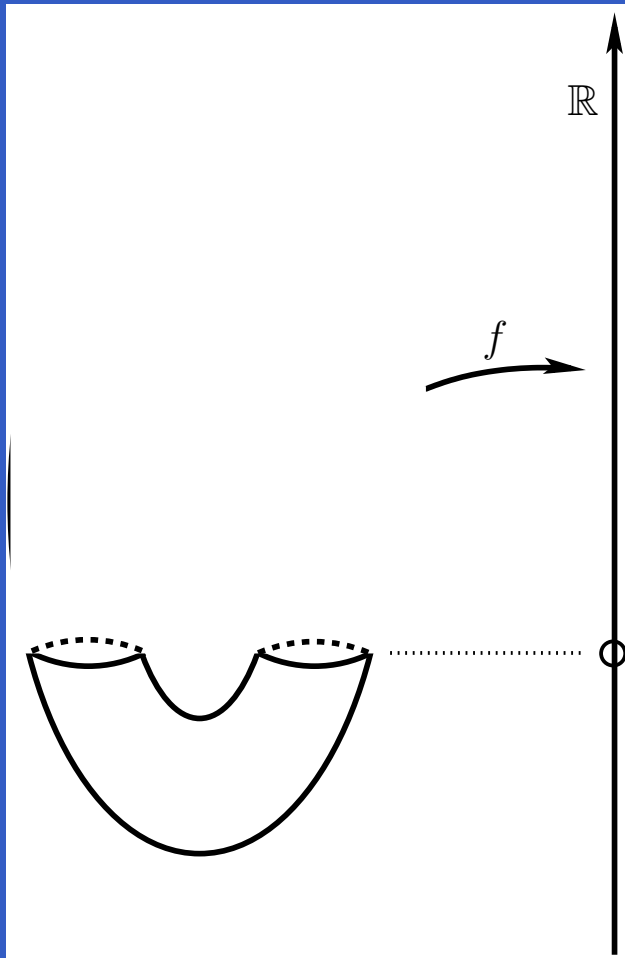
After min. manifold looks like a disk

What happens when we pass through a crit point ?



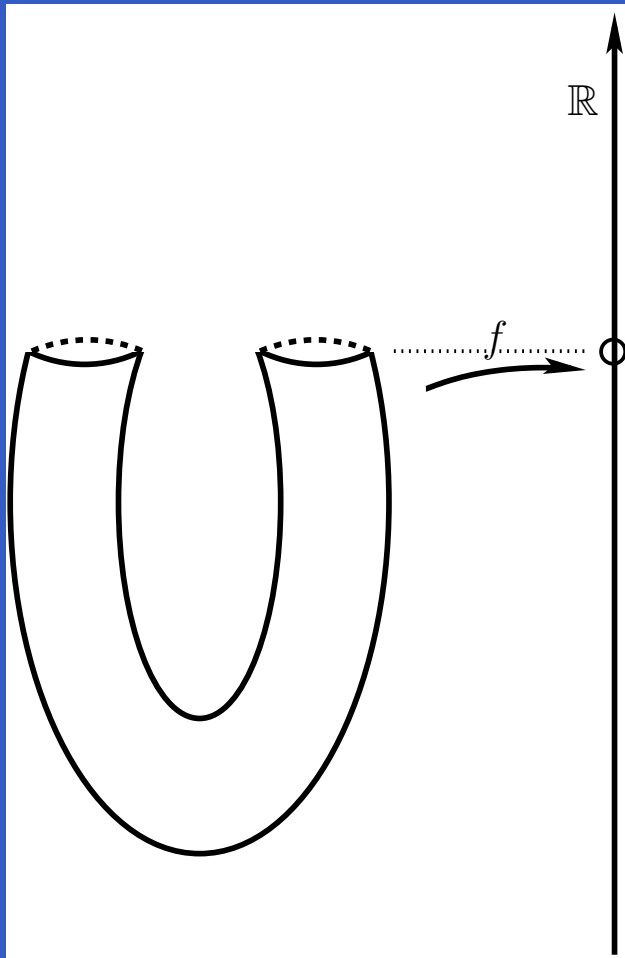
Then we attach a handle of index
 $i_p = 1$

What happens when we pass through a crit point ?



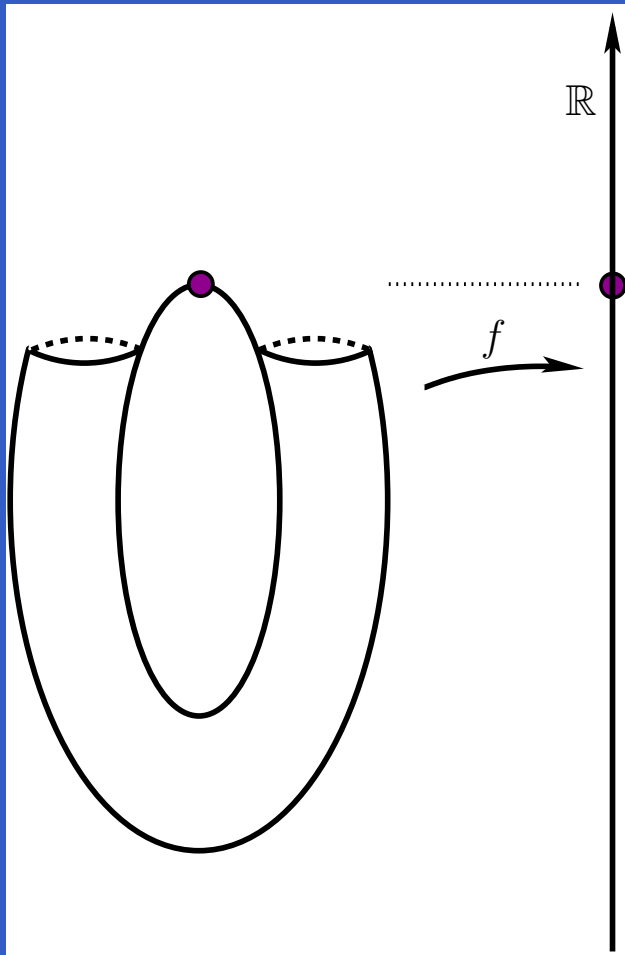
...diffeomorphic picture

What happens when we pass through a crit point ?



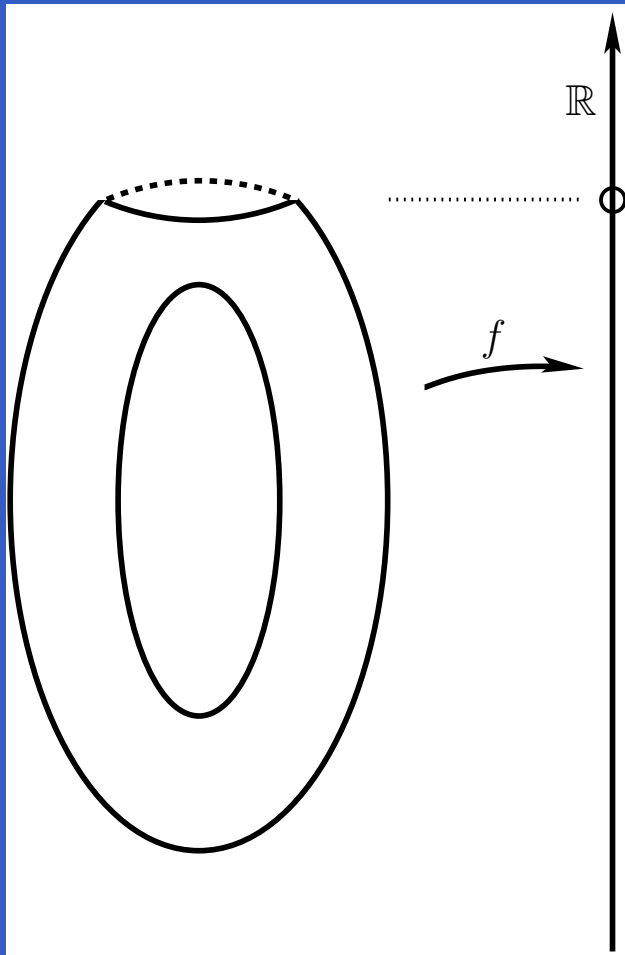
...for a while nothing changes

What happens when we pass through a crit point ?



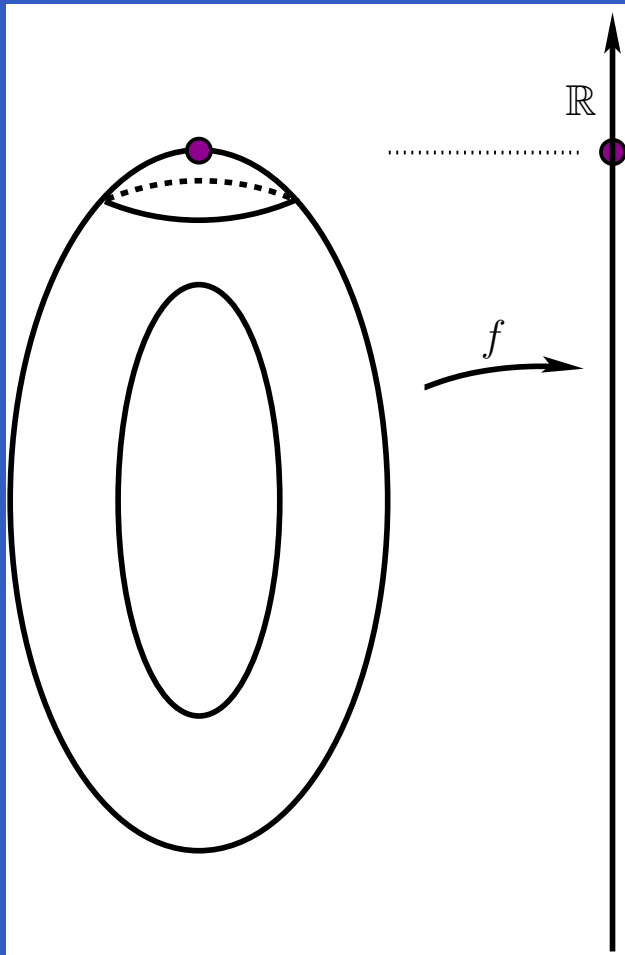
Then we attach another handle of index $i_p = 1$

What happens when we pass through a crit point ?



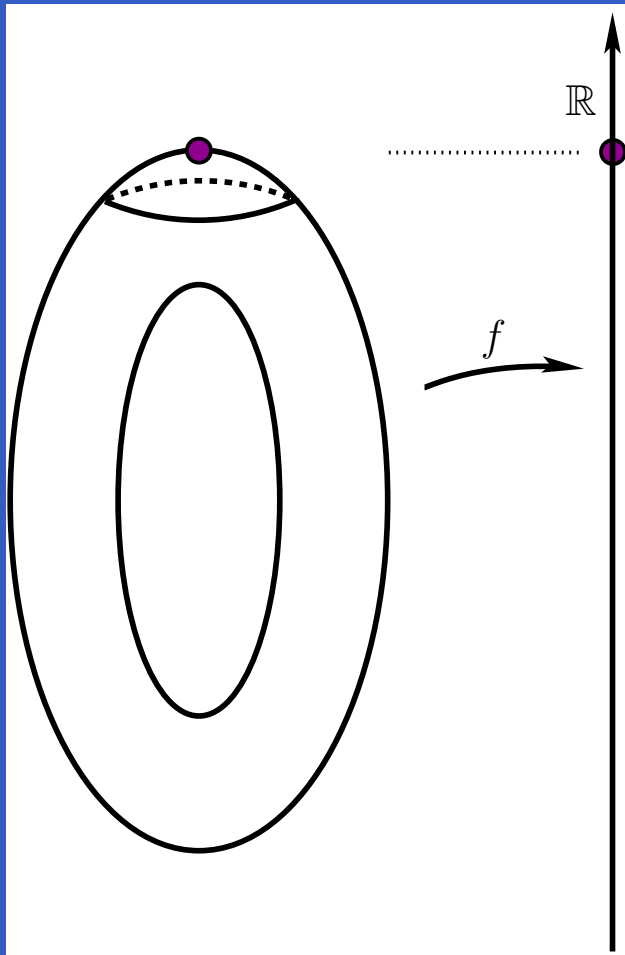
...diffeomorphic picture

What happens when we pass through a crit point ?



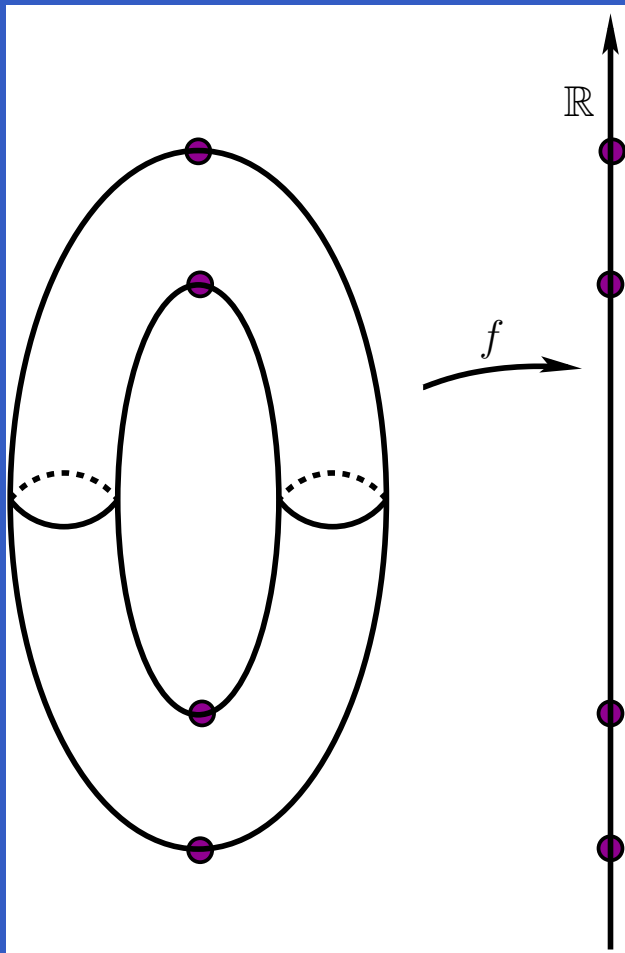
Finally we attach a handle of index
 $i_p = 2$

What happens when we pass through a crit point ?



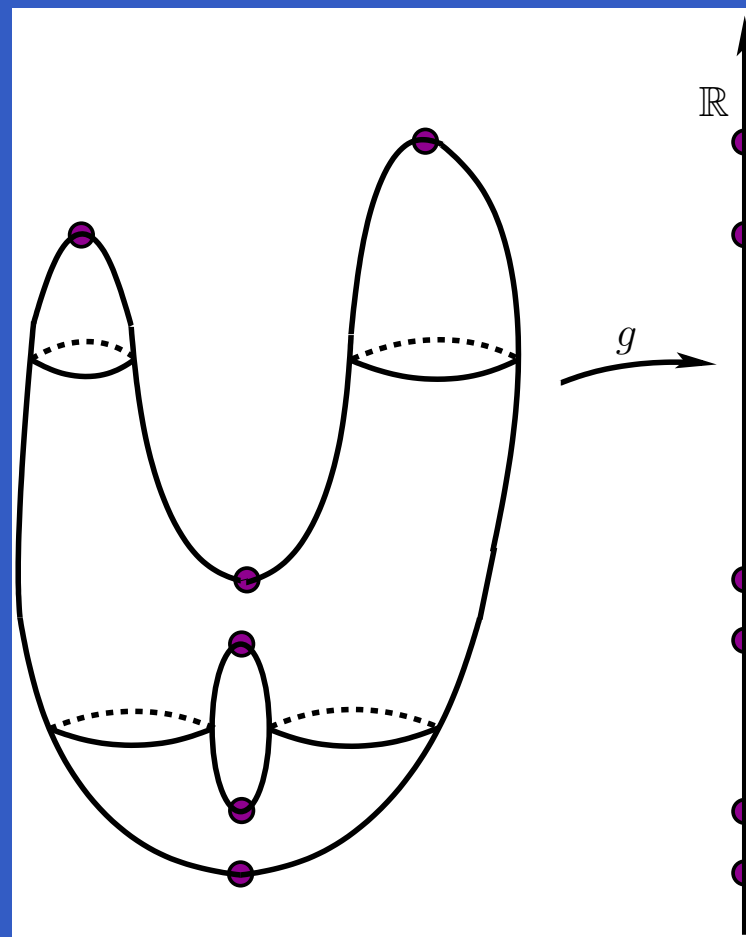
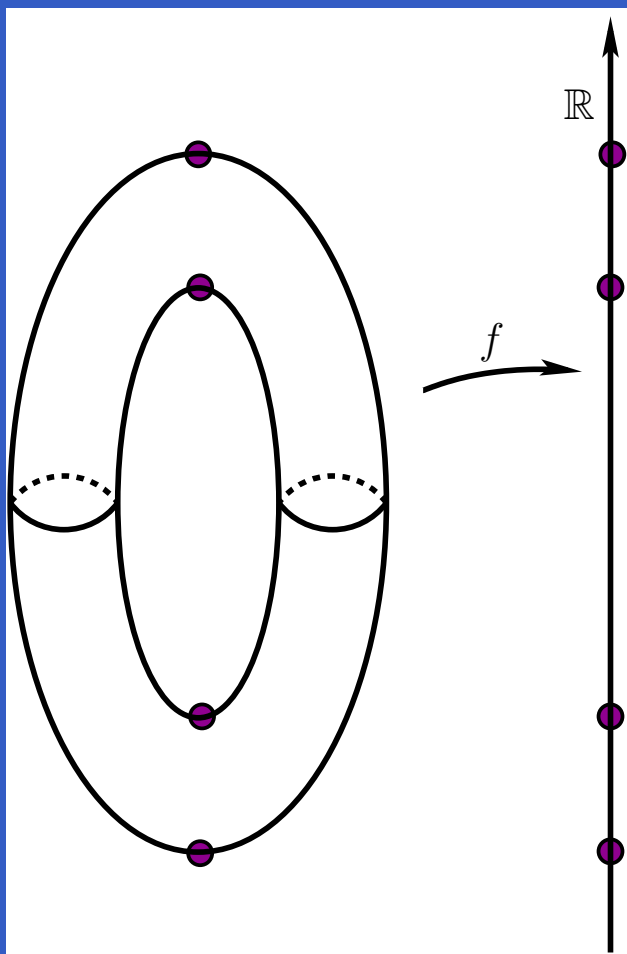
Summary: after crit point of index k , we attach a handle of index k .
Topology changes in a standard way!

What happens when we pass through a crit point ?



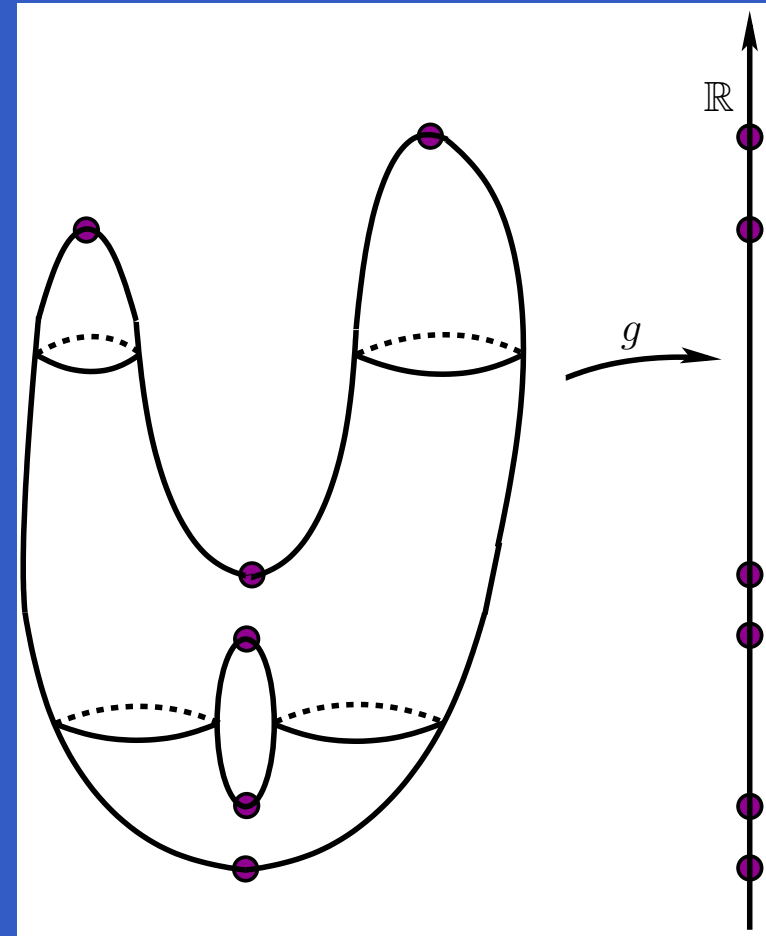
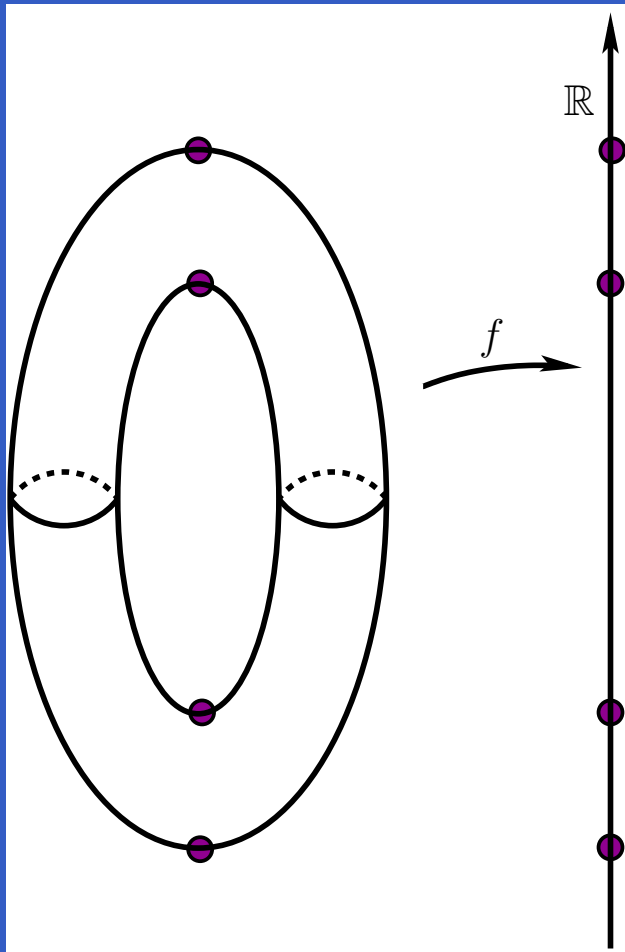
The same happens in higher dimensions too!

What happens when we pass through a crit point ?



Different Morse functions give different ways to build M .

What happens when we pass through a crit point ?



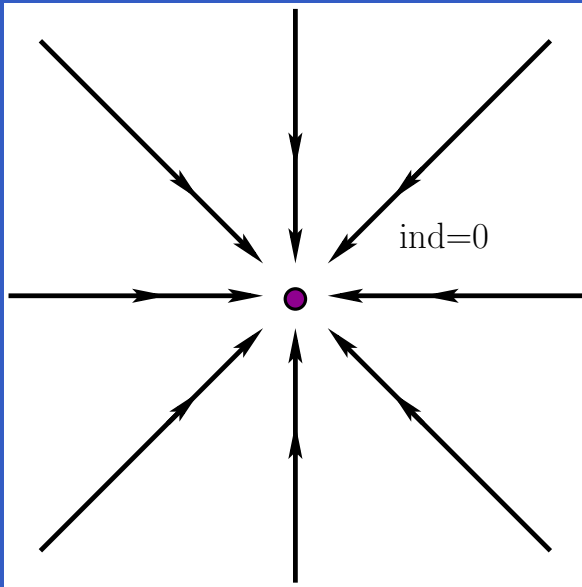
Different Morse functions give different ways to build M .

Corollary: A generic smooth function $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ must have at least 4 critical points.

Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t))$, $t \in \mathbb{R}$.

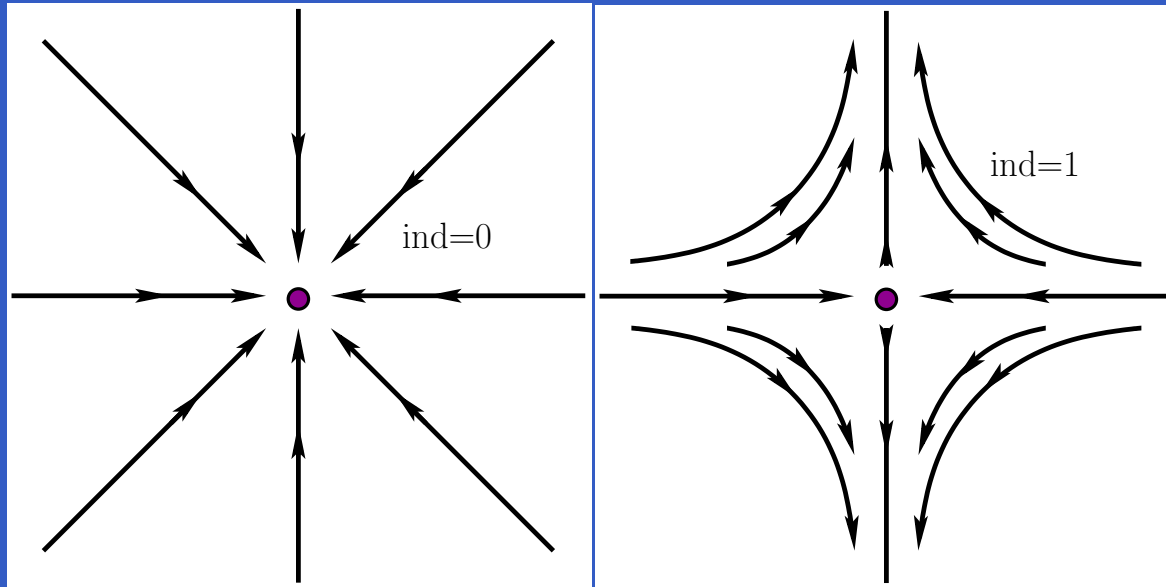
Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t)), t \in \mathbb{R}$.

Local pictures of flow lines near crit points:



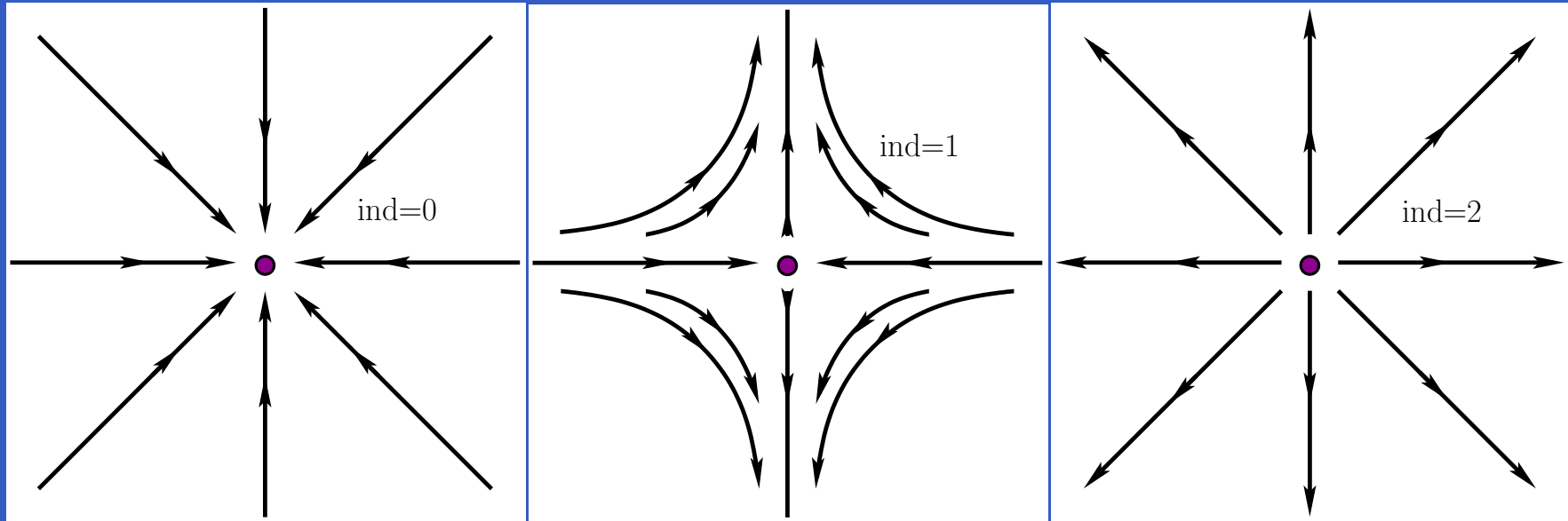
Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t))$, $t \in \mathbb{R}$.

Local pictures of flow lines near crit points:



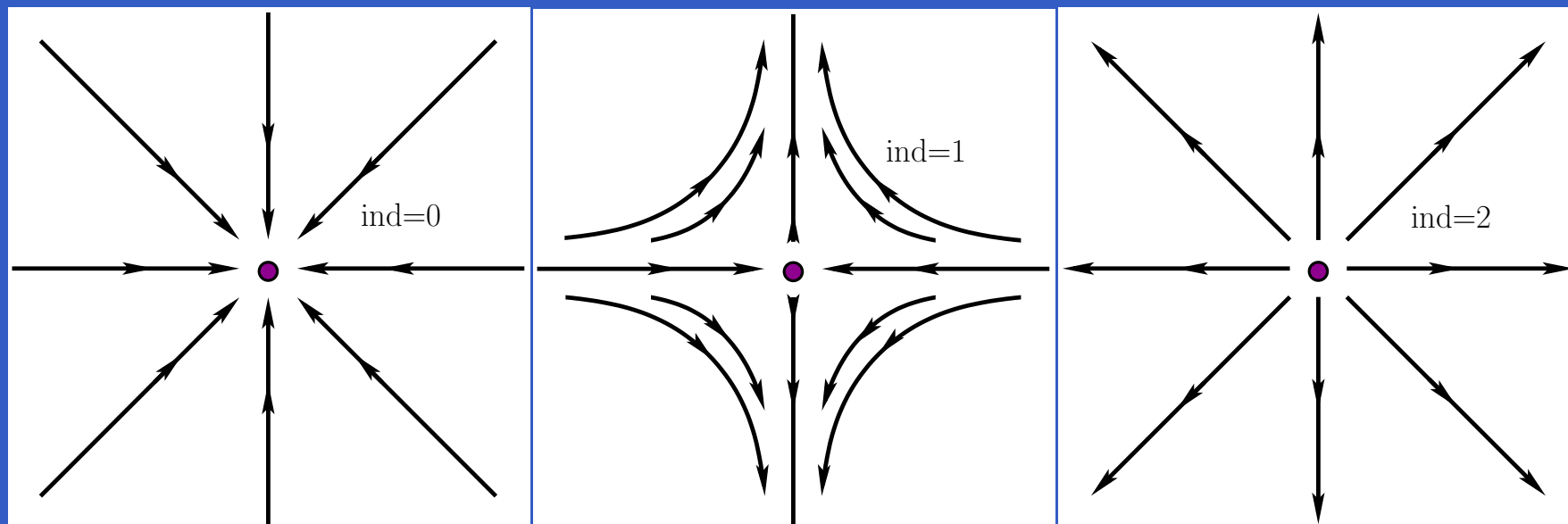
Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t))$, $t \in \mathbb{R}$.

Local pictures of flow lines near crit points:

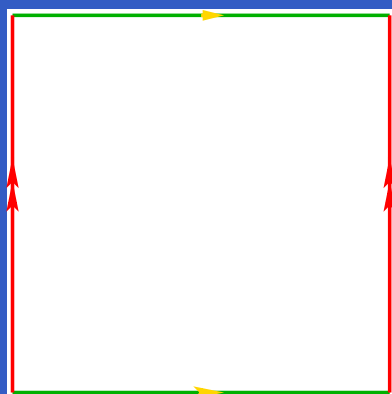


Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t))$, $t \in \mathbb{R}$.

Local pictures of flow lines near crit points:

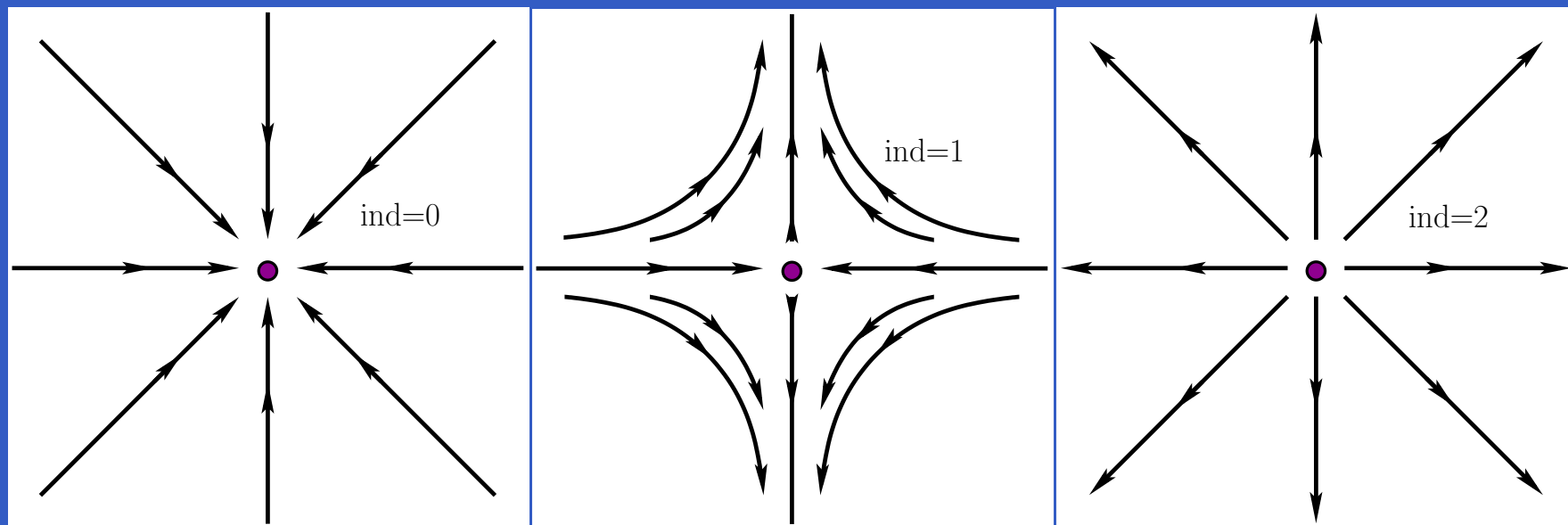


Example on the torus:

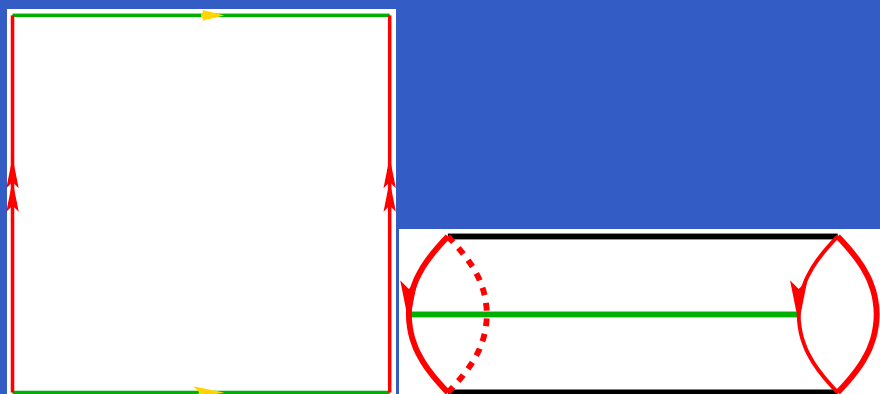


Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t))$, $t \in \mathbb{R}$.

Local pictures of flow lines near crit points:

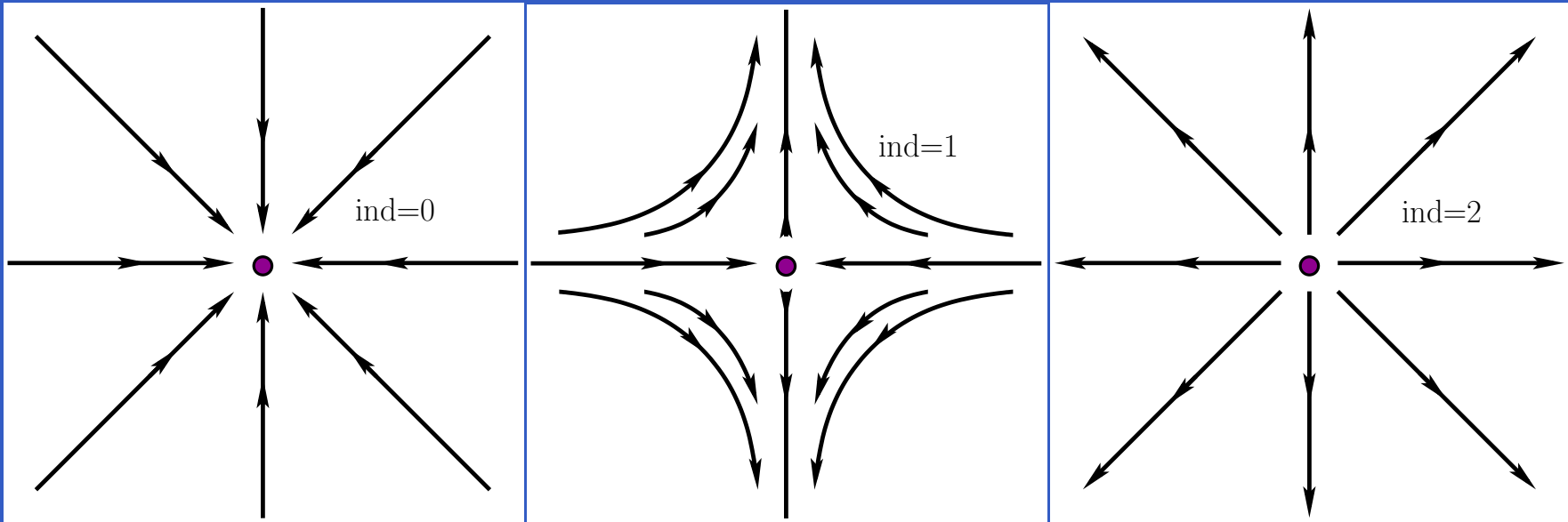


Example on the torus:

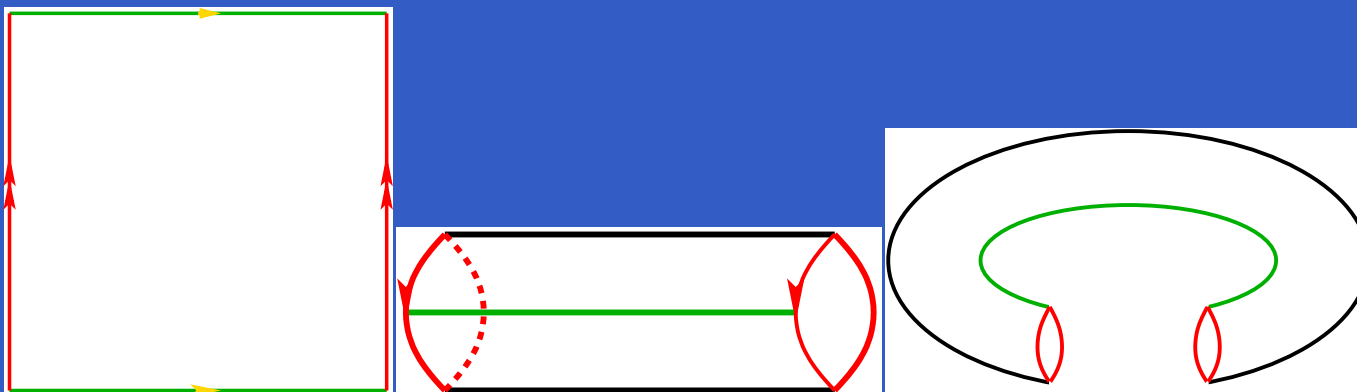


Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t))$, $t \in \mathbb{R}$.

Local pictures of flow lines near crit points:

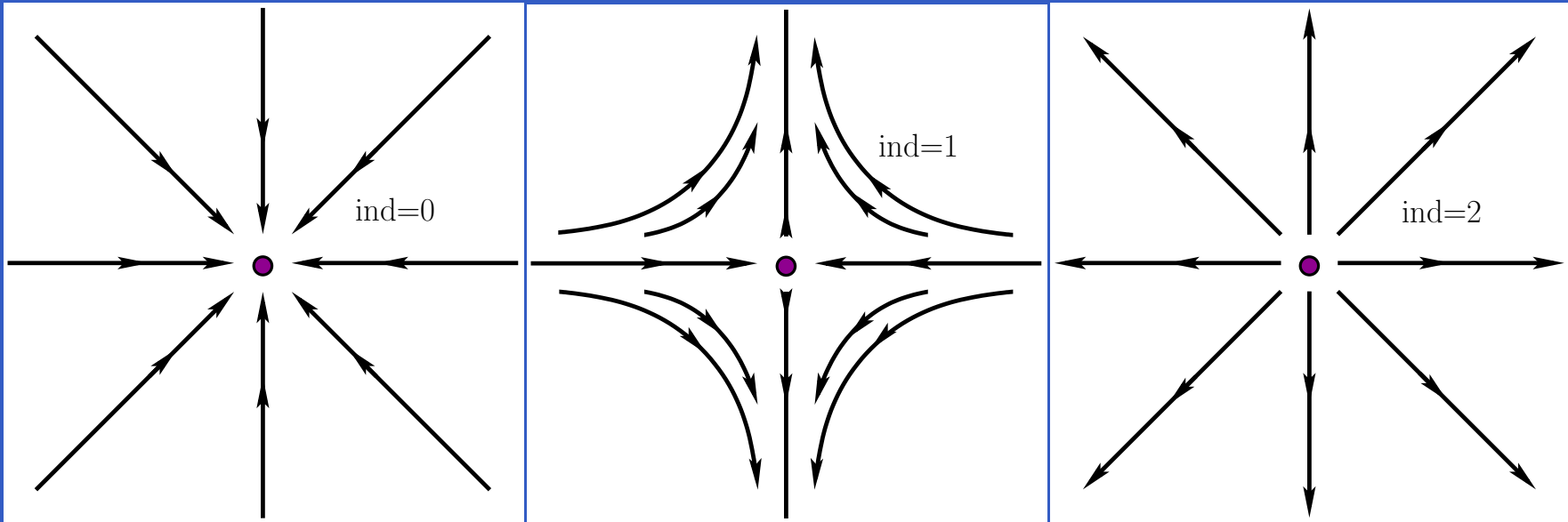


Example on the torus:

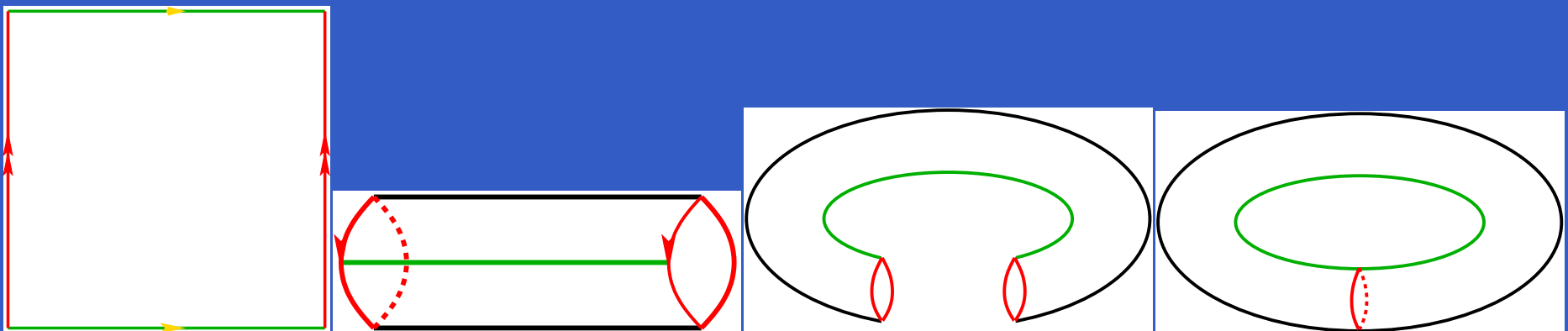


Gradient flow: $\dot{\gamma}(t) = -\nabla f(\gamma(t))$, $t \in \mathbb{R}$.

Local pictures of flow lines near crit points:



Example on the torus:



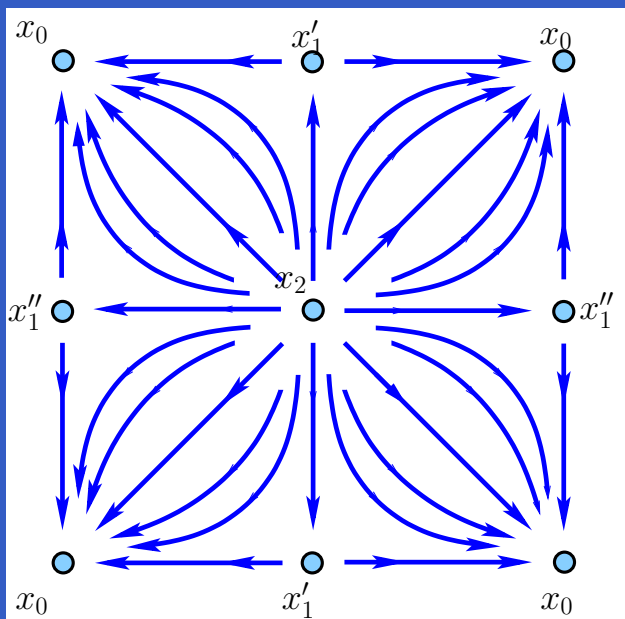
$$f : \mathbb{T}^2 = S^1 \times S^1 \rightarrow \mathbb{R}, \quad f(\theta_1, \theta_2) = \cos(\theta_1) + \sin(\theta_2).$$

$$f : \mathbb{T}^2 = S^1 \times S^1 \rightarrow \mathbb{R}, \quad f(\theta_1, \theta_2) = \cos(\theta_1) + \sin(\theta_2).$$

Crit points: x_0 (ind = 0), x'_1, x''_1 (ind = 1), x_2 (ind = 2).

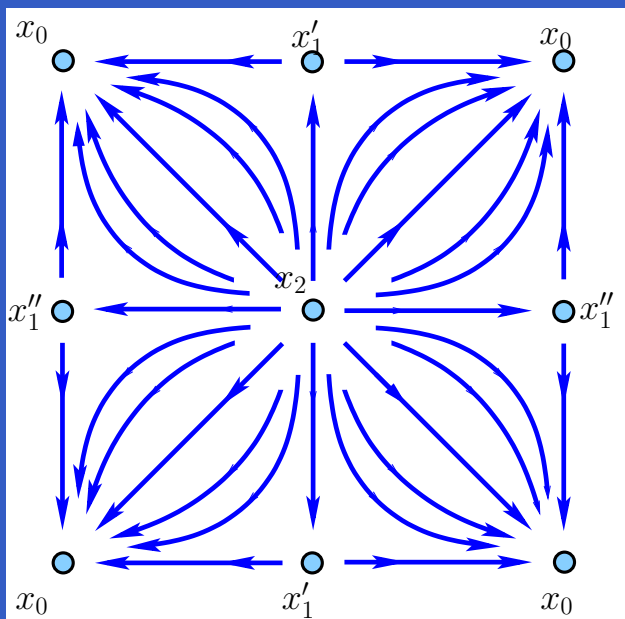
$$f : \mathbb{T}^2 = S^1 \times S^1 \rightarrow \mathbb{R}, \quad f(\theta_1, \theta_2) = \cos(\theta_1) + \sin(\theta_2).$$

Crit points: x_0 (ind = 0), x'_1, x''_1 (ind = 1), x_2 (ind = 2).



$$f : \mathbb{T}^2 = S^1 \times S^1 \rightarrow \mathbb{R}, \quad f(\theta_1, \theta_2) = \cos(\theta_1) + \sin(\theta_2).$$

Crit points: x_0 (ind = 0), x'_1, x''_1 (ind = 1), x_2 (ind = 2).



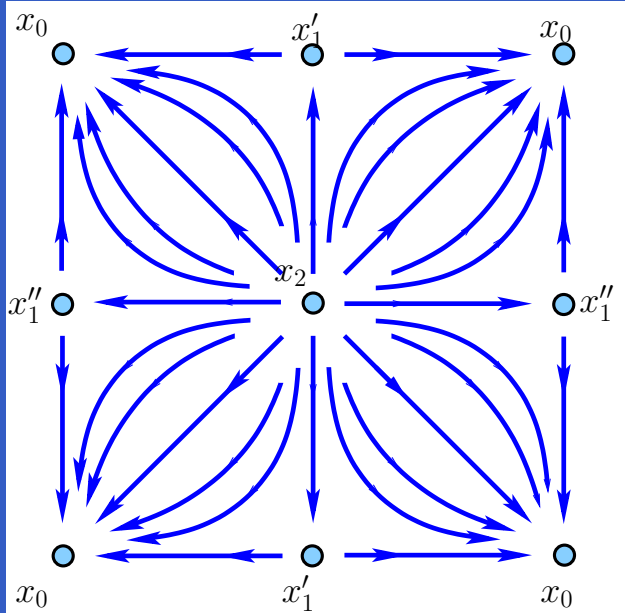
Graph Γ_f

Vertices: crit points $x \in \text{Crit}(f)$.

Edges: If $i_y = i_x - 1$ draw edge $x \rightarrow y \forall$ trajectory from x to y .

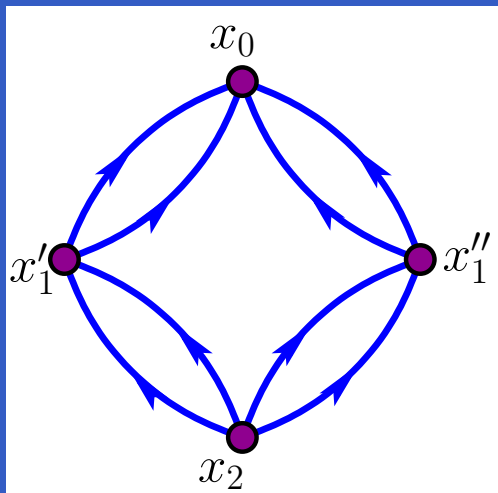
$$f : \mathbb{T}^2 = S^1 \times S^1 \rightarrow \mathbb{R}, \quad f(\theta_1, \theta_2) = \cos(\theta_1) + \sin(\theta_2).$$

Crit points: x_0 (ind = 0), x'_1, x''_1 (ind = 1), x_2 (ind = 2).



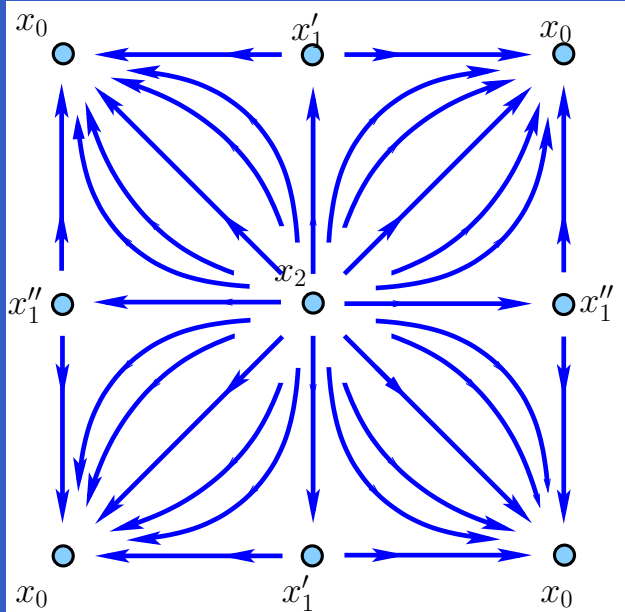
Graph Γ_f

Vertices: crit points $x \in \text{Crit}(f)$.
Edges: If $i_y = i_x - 1$ draw edge $x \rightarrow y \forall$ trajectory from x to y .



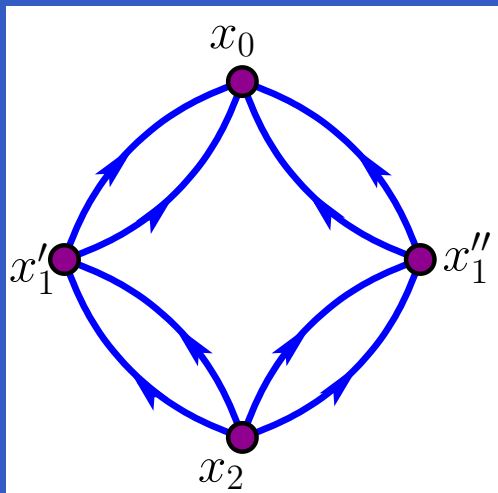
$$f : \mathbb{T}^2 = S^1 \times S^1 \rightarrow \mathbb{R}, \quad f(\theta_1, \theta_2) = \cos(\theta_1) + \sin(\theta_2).$$

Crit points: x_0 (ind = 0), x'_1, x''_1 (ind = 1), x_2 (ind = 2).



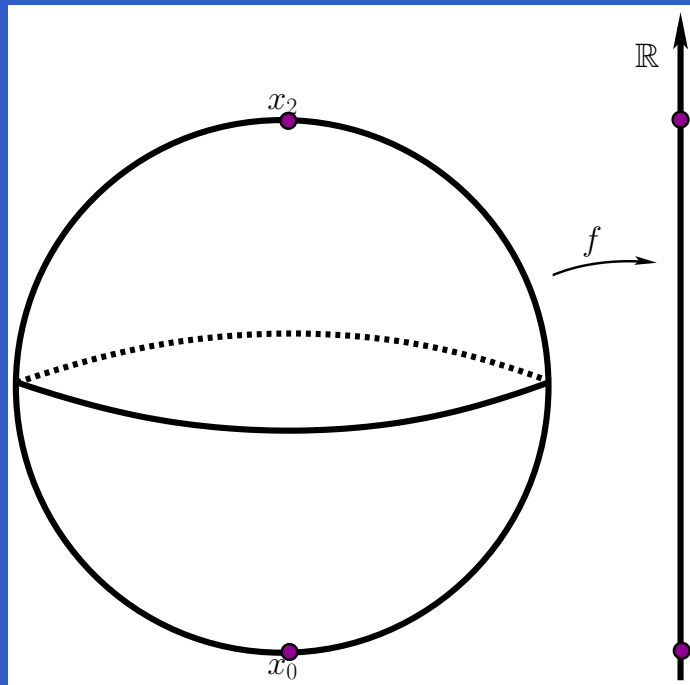
Graph Γ_f

Vertices: crit points $x \in \text{Crit}(f)$.
Edges: If $i_y = i_x - 1$ draw edge $x \rightarrow y \forall$ trajectory from x to y .

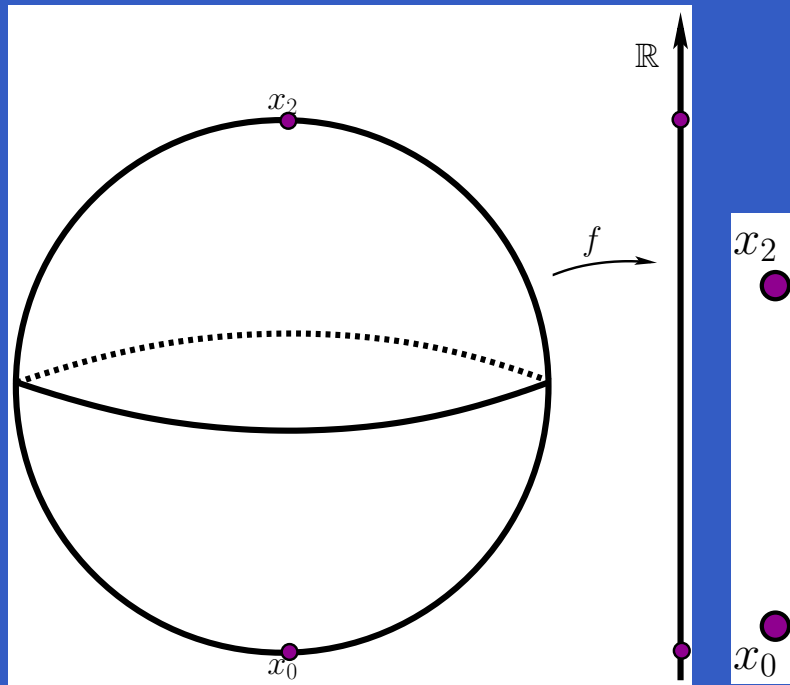


Thm: If $i_z = i_x - 2$ then
 $\#$ of length 2 paths from x to z
 is always even.

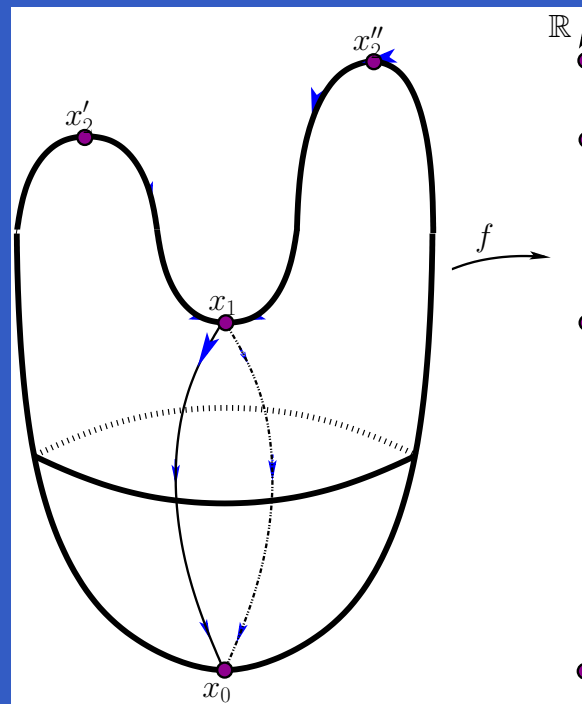
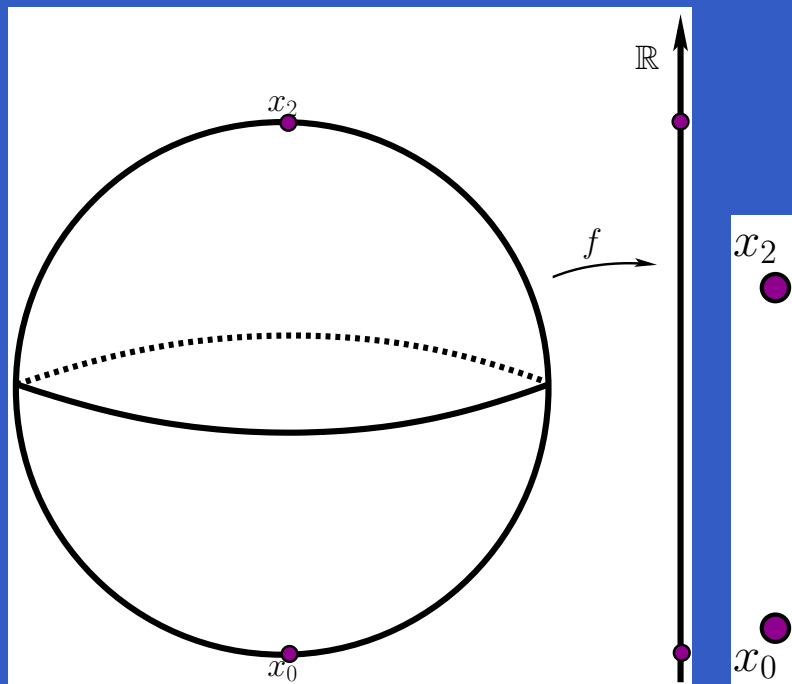
Another example: the sphere



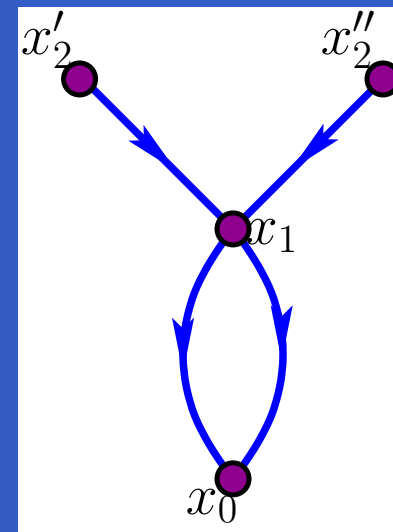
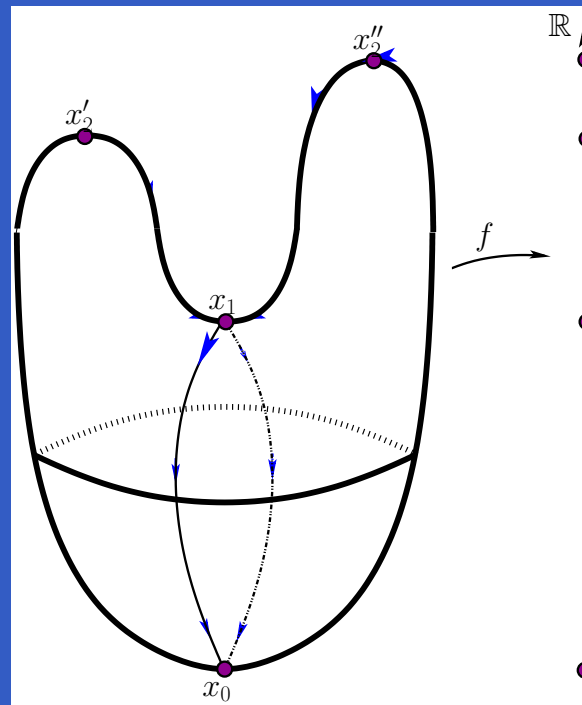
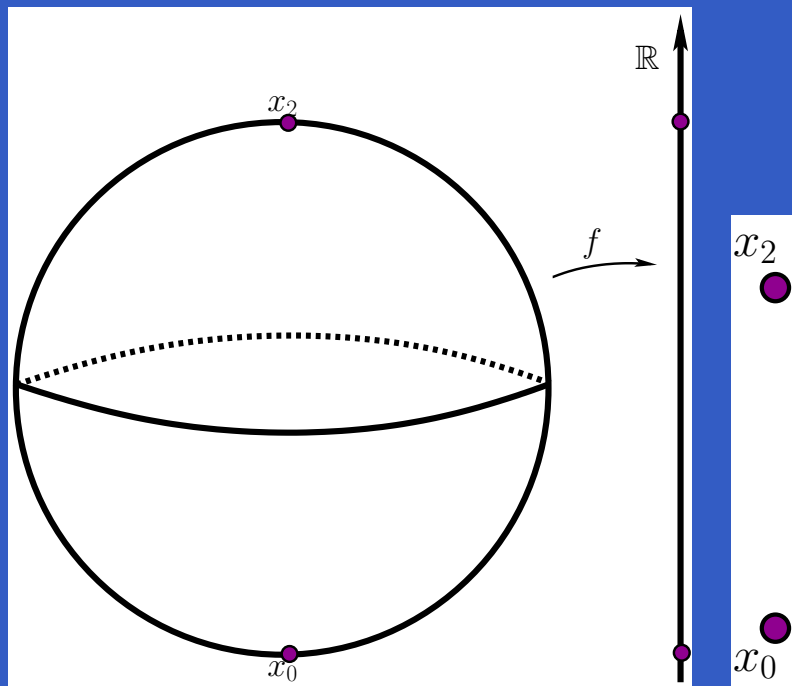
Another example: the sphere



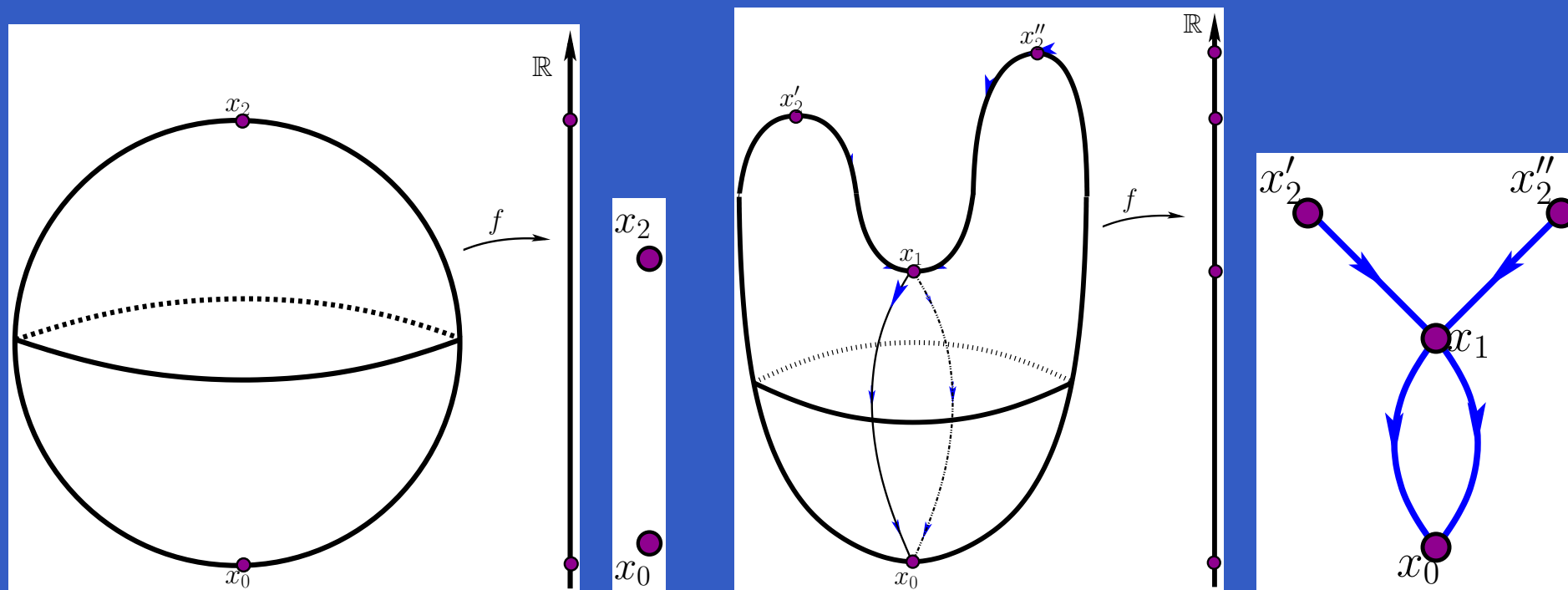
Another example: the sphere



Another example: the sphere

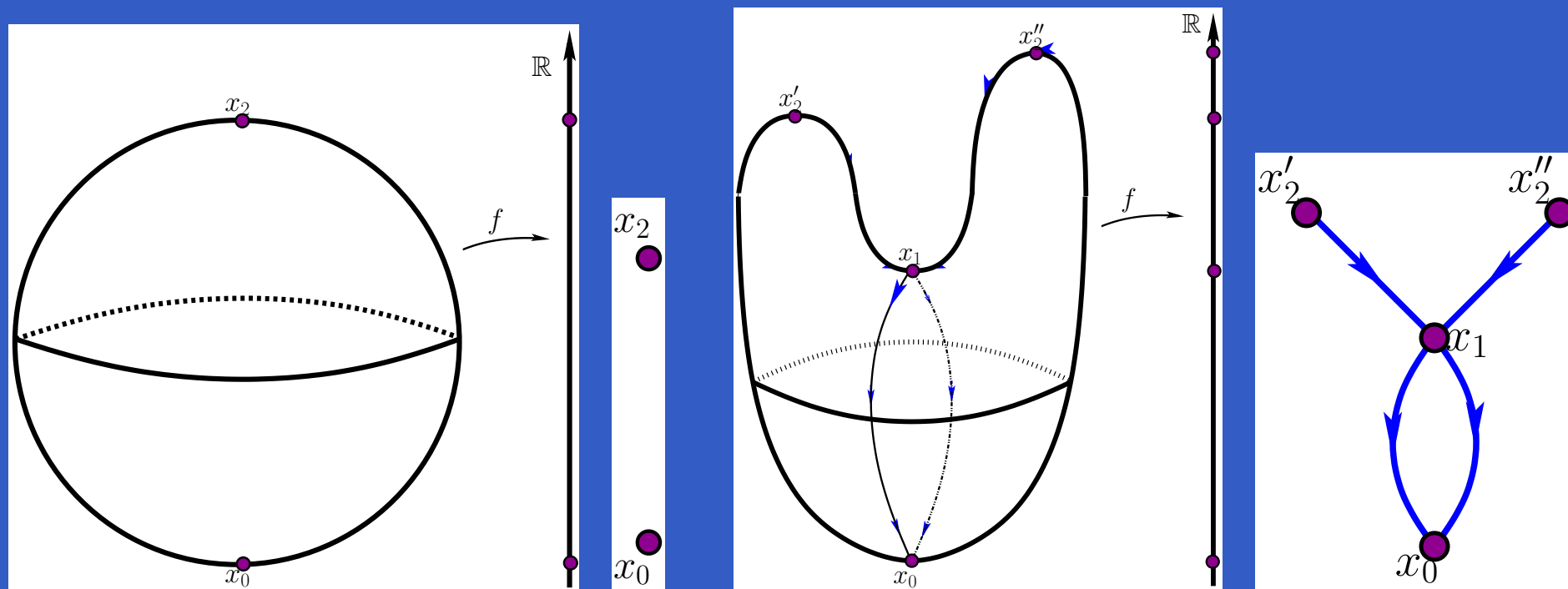


Another example: the sphere



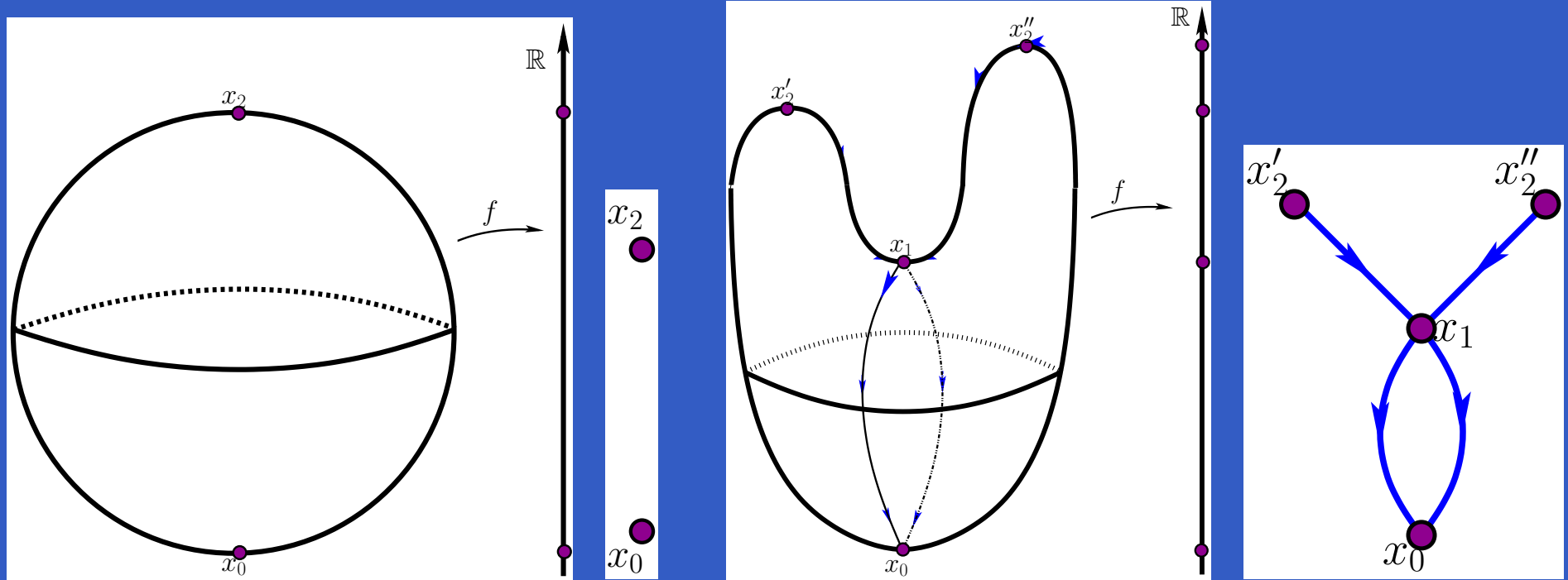
Proof: $i_z = i_x - 2 \implies$ space of trajectories $x \rightarrow z$ is parametrized by a *compact* 1-dimensional manifold with boundary.

Another example: the sphere

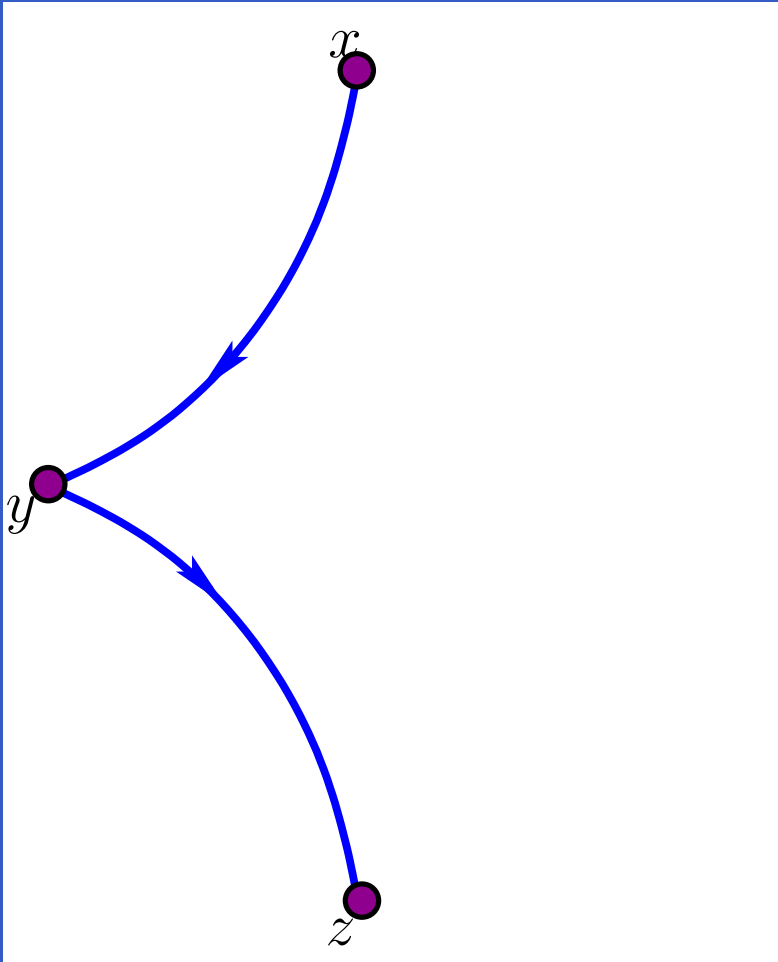


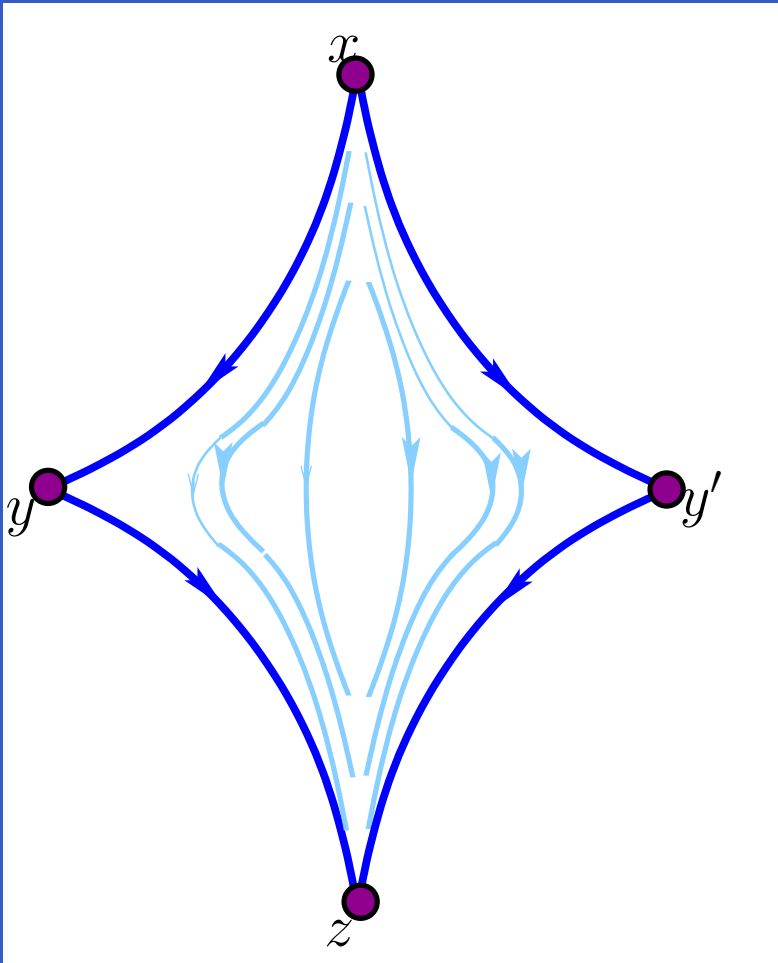
Proof: $i_z = i_x - 2 \implies$ space of trajectories $x \rightarrow z$ is parametrized by a *compact* 1-dimensional manifold with boundary. Boundary = broken trajectories.

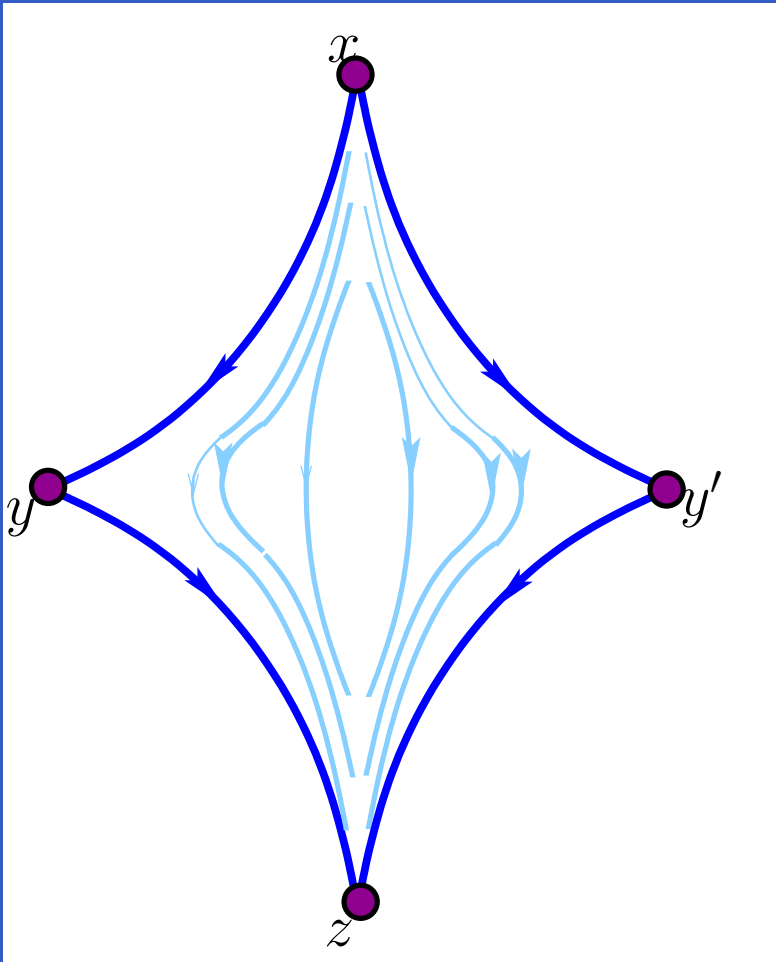
Another example: the sphere



Proof: $i_z = i_x - 2 \implies$ space of trajectories $x \rightarrow z$ is parametrized by a *compact* 1-dimensional manifold with boundary. Boundary = broken trajectories. The point is that a 1-dimensional compact manifold has an *even* number of boundary points.







Broken trajectories $x \rightarrow z$ come in pairs!

Morse homology

$f : M \rightarrow \mathbb{R}$ Morse function.

$C_k := \bigoplus_{i_x=k} \mathbb{Z}_2 \cdot x,$ vector space spanned by $\text{ind} = k$ crit points

Morse homology

$f : M \rightarrow \mathbb{R}$ Morse function.

$C_k := \bigoplus_{i_x=k} \mathbb{Z}_2 \cdot x$, vector space spanned by $\text{ind} = k$ crit points

$d : C_k \rightarrow C_{k-1}$, $d(x) = \sum_{i_y=i_x-1} n_{\mathbb{Z}_2}(x \rightarrow y)y$

Morse homology

$f : M \rightarrow \mathbb{R}$ Morse function.

$C_k := \bigoplus_{i_x=k} \mathbb{Z}_2 \cdot x$, vector space spanned by $\text{ind} = k$ crit points

$d : C_k \rightarrow C_{k-1}$, $d(x) = \sum_{i_y=i_x-1} n_{\mathbb{Z}_2}(x \rightarrow y)y$

$0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} C_{k+1} \xrightarrow{d} C_k \xrightarrow{d} \cdots$
 $\cdots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \longrightarrow 0$

Morse homology

$f : M \rightarrow \mathbb{R}$ Morse function.

$C_k := \bigoplus_{i_x=k} \mathbb{Z}_2 \cdot x$, vector space spanned by $\text{ind} = k$ crit points

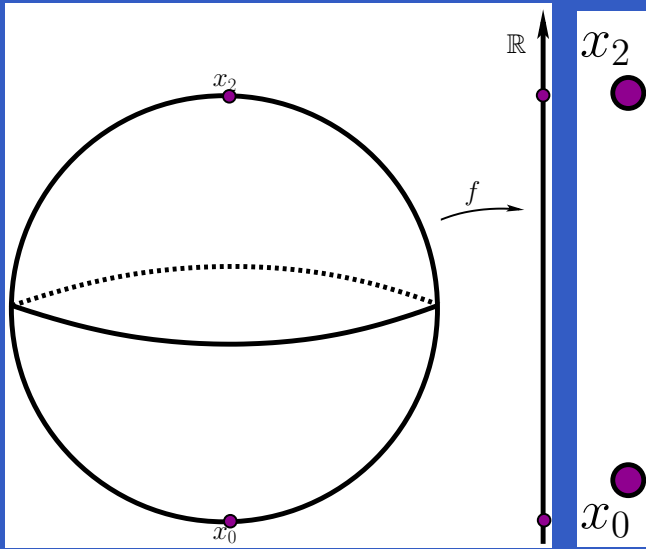
$d : C_k \rightarrow C_{k-1}$, $d(x) = \sum_{i_y=i_x-1} n_{\mathbb{Z}_2}(x \rightarrow y)y$

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & C_{k+1} & \xrightarrow{d} & C_k & \xrightarrow{d} & \cdots \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ \cdots & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_0 & \longrightarrow & 0 & & & & & & \end{array}$$

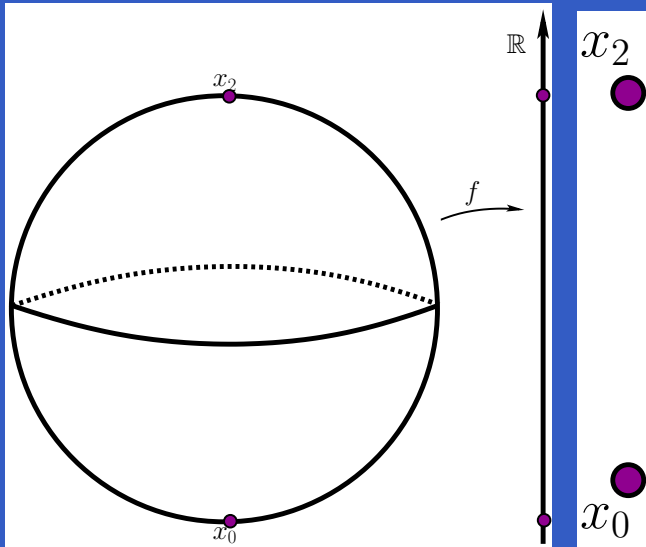
Prev statement on graphs $\Rightarrow d \circ d = 0$

$\Rightarrow d(C_{k+1}) \subset \ker d|_{C_k}$. **(Chain complex)**

Examples again

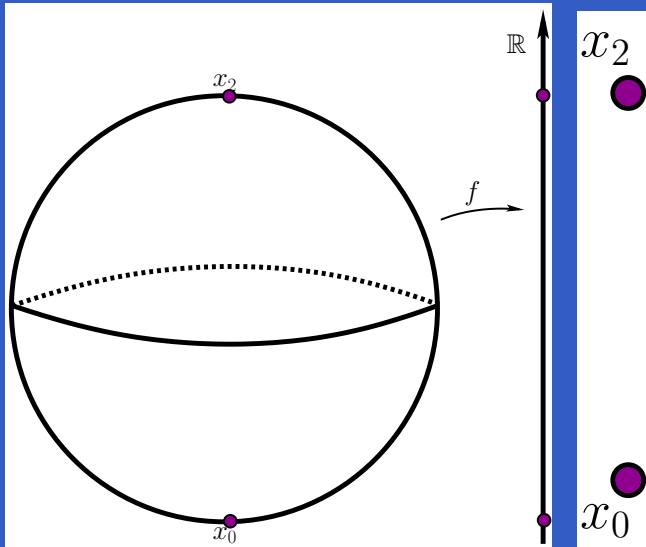


Examples again



$$d(x_2) = 0, d(x_0) = 0.$$

Examples again

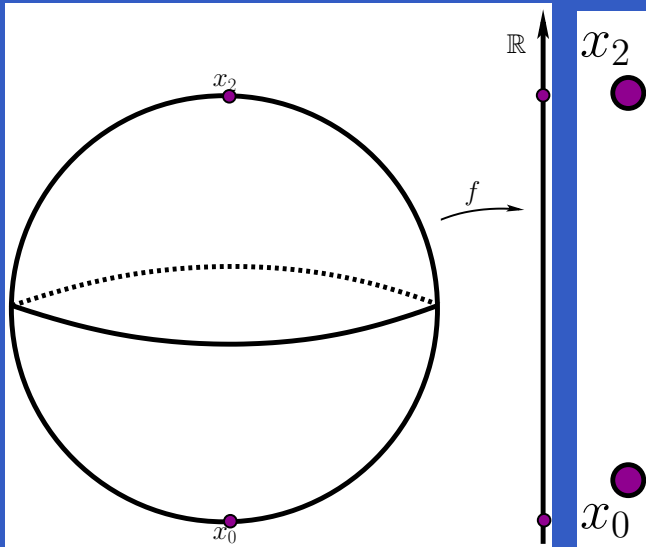


$$d(x_2) = 0, d(x_0) = 0.$$

$$H_2(S^2) = \mathbb{Z}_2, H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2.$$

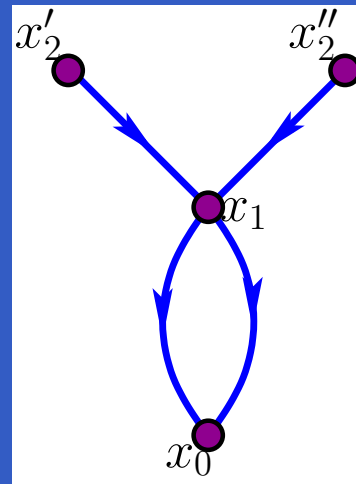
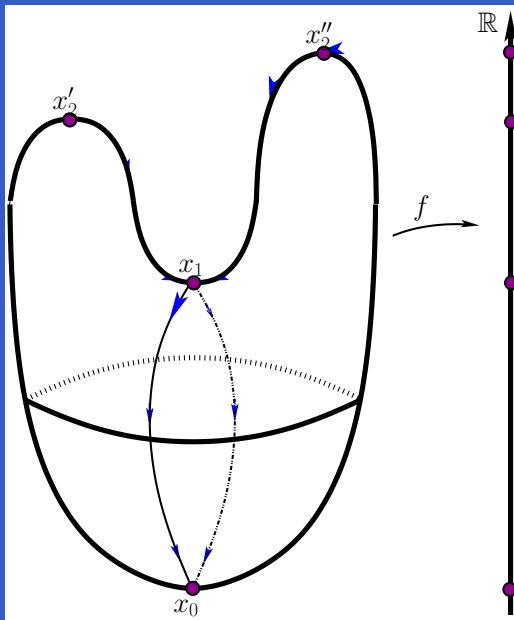
Examples again



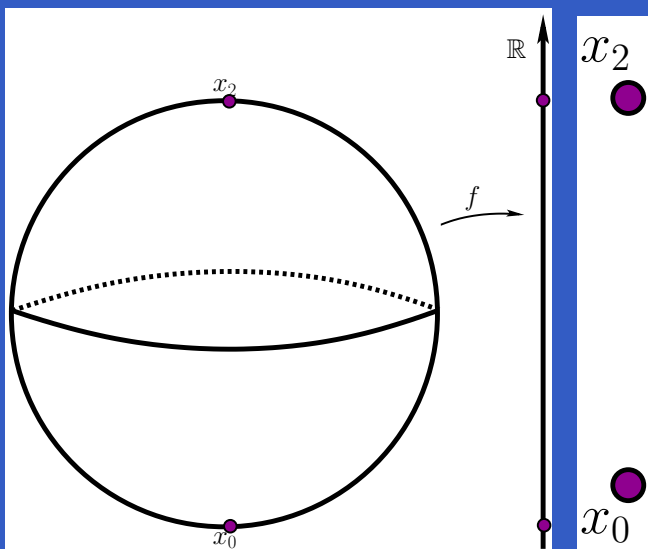
$$d(x_2) = 0, d(x_0) = 0.$$

$$H_2(S^2) = \mathbb{Z}_2, H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2.$$



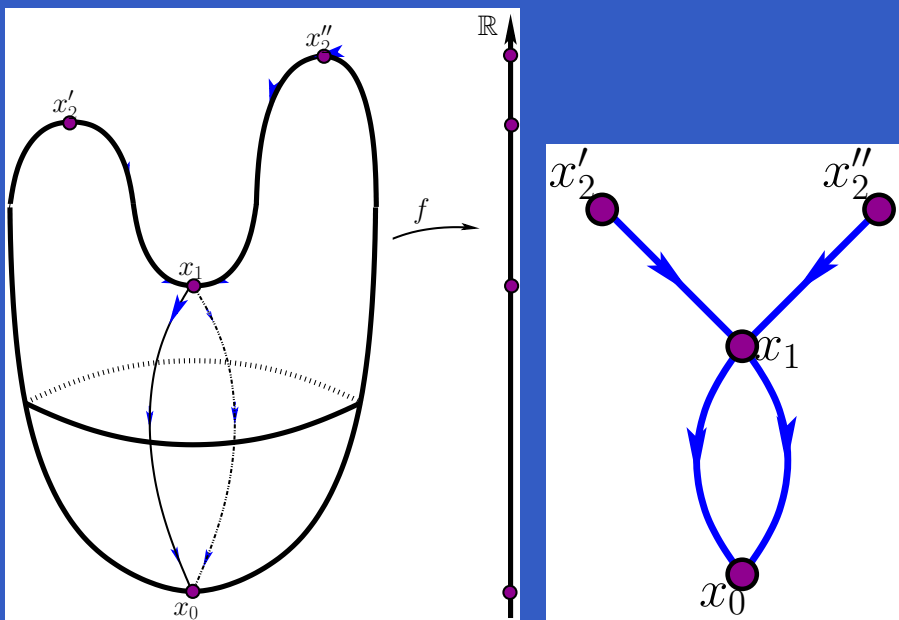
Examples again



$$d(x_2) = 0, d(x_0) = 0.$$

$$H_2(S^2) = \mathbb{Z}_2, H_1(S^2) = 0,$$

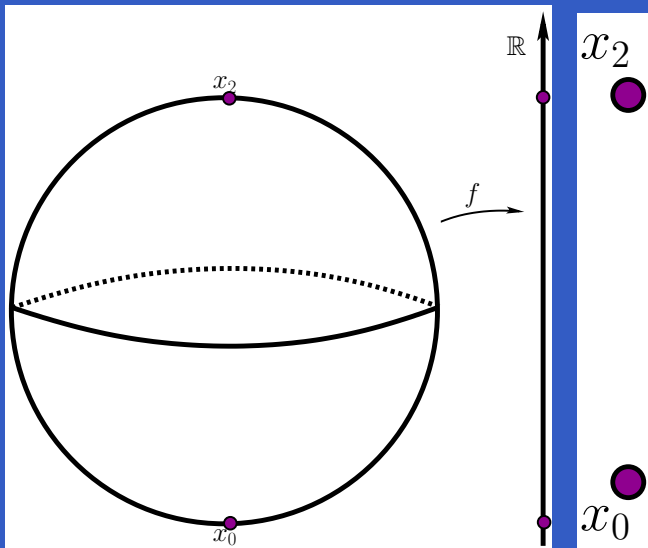
$$H_0(S^2) = \mathbb{Z}_2.$$



$$d(x'_2) = d(x''_2) = x_1,$$

$$d(x_1) = 0.$$

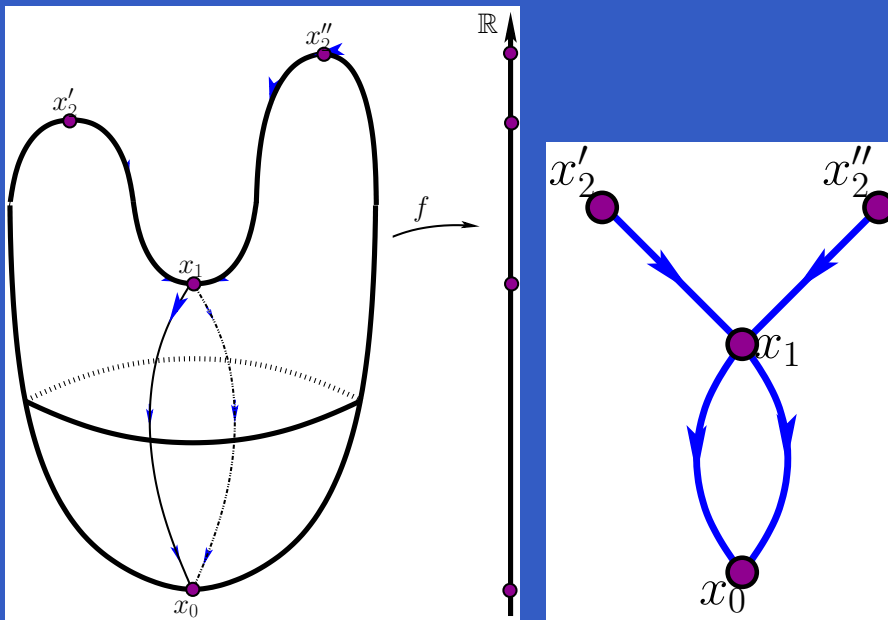
Examples again



$$d(x_2) = 0, d(x_0) = 0.$$

$$H_2(S^2) = \mathbb{Z}_2, H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2.$$



$$d(x'_2) = d(x''_2) = x_1,$$

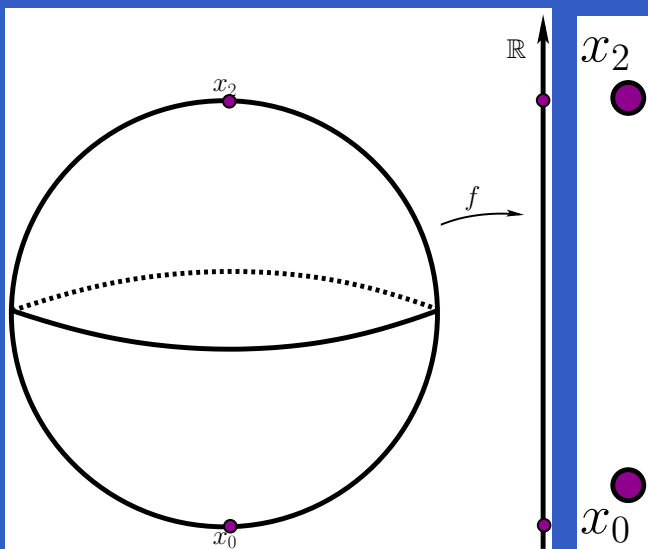
$$d(x_1) = 0.$$

$$H_2(S^2) \cong \mathbb{Z}_2(x'_2 + x''_2),$$

$$H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2 x_0.$$

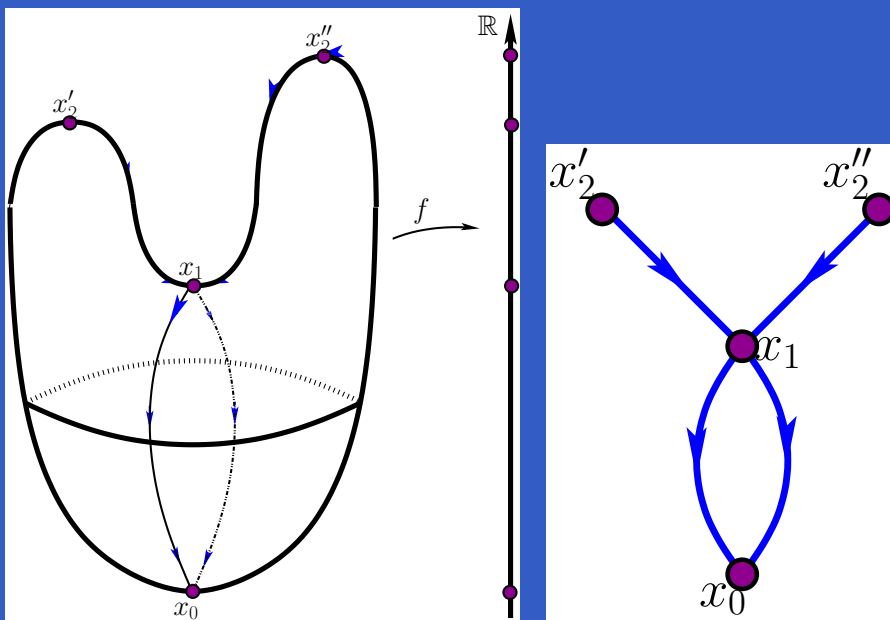
Examples again



$$d(x_2) = 0, d(x_0) = 0.$$

$$H_2(S^2) = \mathbb{Z}_2, H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2.$$



$$d(x'_2) = d(x''_2) = x_1,$$

$$d(x_1) = 0.$$

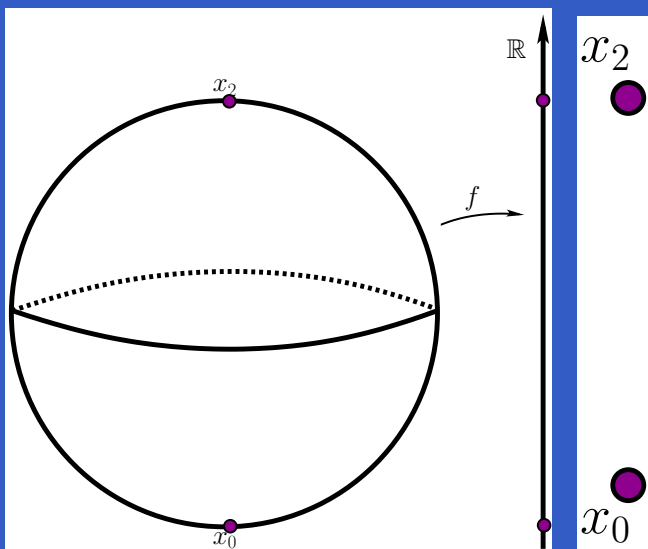
$$H_2(S^2) \cong \mathbb{Z}_2(x'_2 + x''_2),$$

$$H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2 x_0.$$

Isomorphic result

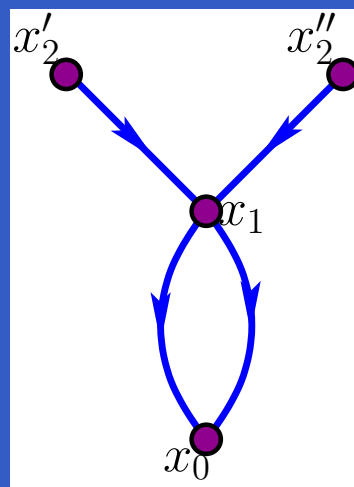
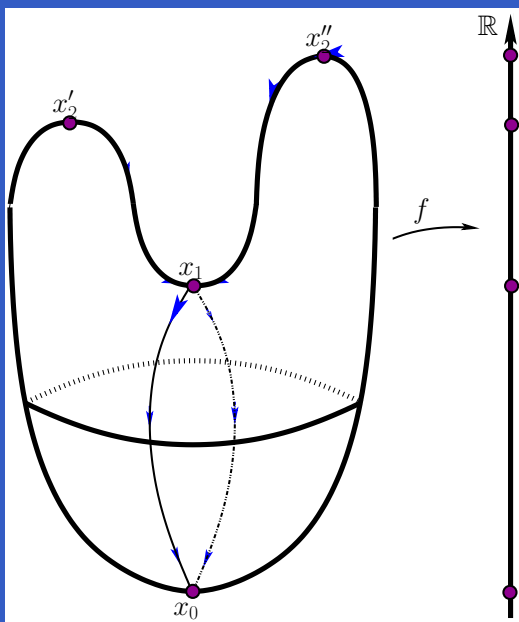
Examples again



$$d(x_2) = 0, d(x_0) = 0.$$

$$H_2(S^2) = \mathbb{Z}_2, H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2.$$



$$d(x'_2) = d(x''_2) = x_1,$$

$$d(x_1) = 0.$$

$$H_2(S^2) \cong \mathbb{Z}_2(x'_2 + x''_2),$$

$$H_1(S^2) = 0,$$

$$H_0(S^2) = \mathbb{Z}_2 x_0.$$

Isomorphic result

For \mathbb{T}^2 : $H_2 = \mathbb{Z}_2, H_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, H_0 = \mathbb{Z}_2.$

Algebraic operations (Betz-Cohen, Fukaya et al.)

$f, g, h : M \rightarrow \mathbb{R}$ (generic) Morse functions.

$x \in \text{Crit}(f), y \in \text{Crit}(g), z \in \text{Crit}(h)$.

Algebraic operations (Betz-Cohen, Fukaya et al.)

$f, g, h : M \rightarrow \mathbb{R}$ (generic) Morse functions.

$x \in \text{Crit}(f), y \in \text{Crit}(g), z \in \text{Crit}(h)$.

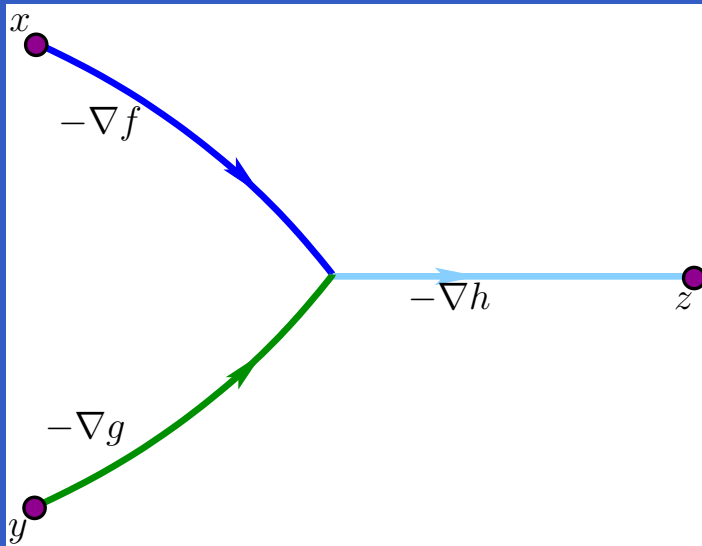
If $i_z = i_x + i_y - n$:

Algebraic operations (Betz-Cohen, Fukaya et al.)

$f, g, h : M \rightarrow \mathbb{R}$ (generic) Morse functions.

$x \in \text{Crit}(f)$, $y \in \text{Crit}(g)$, $z \in \text{Crit}(h)$.

If $i_z = i_x + i_y - n$:

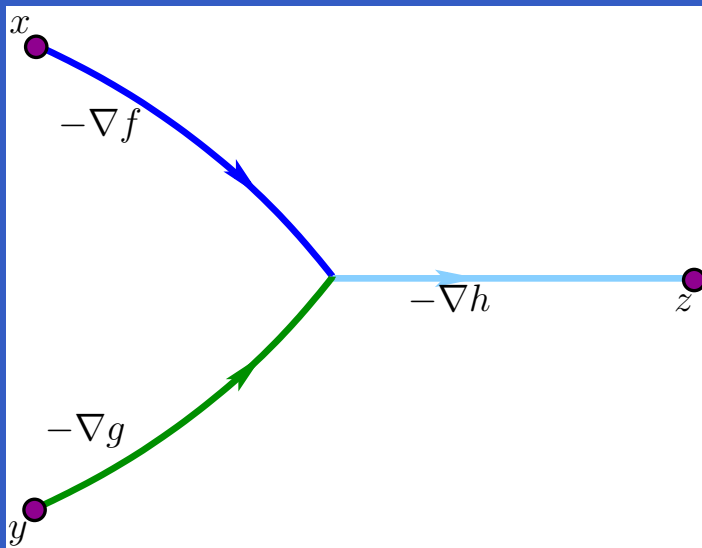


Algebraic operations (Betz-Cohen, Fukaya et al.)

$f, g, h : M \rightarrow \mathbb{R}$ (generic) Morse functions.

$x \in \text{Crit}(f)$, $y \in \text{Crit}(g)$, $z \in \text{Crit}(h)$.

If $i_z = i_x + i_y - n$:



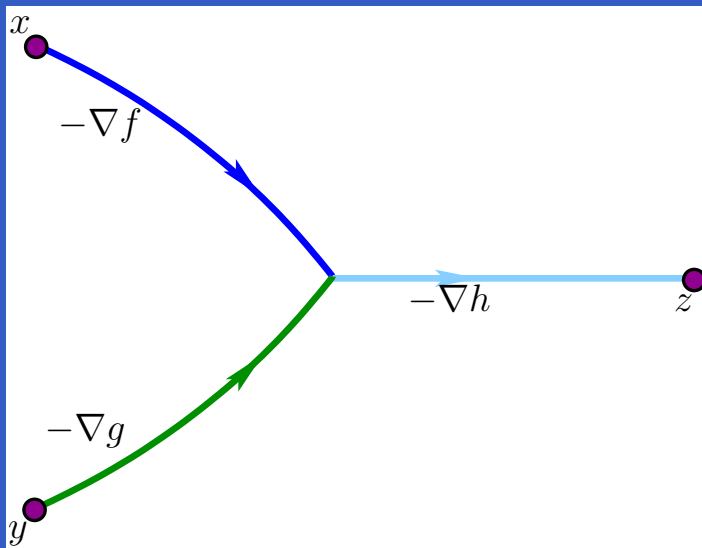
$$(x, y) \mapsto \sum_{i_z = i_x + i_y - n} n(x, y, z)_{\mathbb{Z}_2} z$$

Algebraic operations (Betz-Cohen, Fukaya et al.)

$f, g, h : M \rightarrow \mathbb{R}$ (generic) Morse functions.

$x \in \text{Crit}(f)$, $y \in \text{Crit}(g)$, $z \in \text{Crit}(h)$.

If $i_z = i_x + i_y - n$:



$$(x, y) \mapsto \sum_{i_z = i_x + i_y - n} n(x, y, z)_{\mathbb{Z}_2} z$$

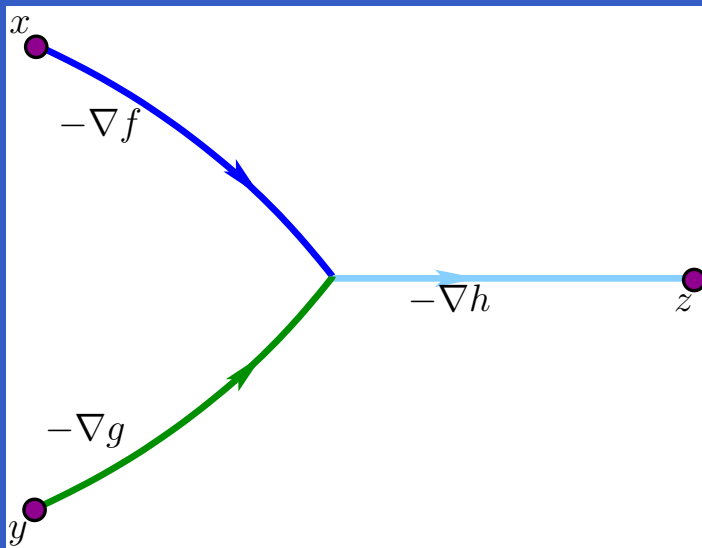
$$H_k(M) \otimes H_l(M) \longrightarrow H_{k+l-n}(M).$$

Algebraic operations (Betz-Cohen, Fukaya et al.)

$f, g, h : M \rightarrow \mathbb{R}$ (generic) Morse functions.

$x \in \text{Crit}(f)$, $y \in \text{Crit}(g)$, $z \in \text{Crit}(h)$.

If $i_z = i_x + i_y - n$:



$$(x, y) \mapsto \sum_{i_z = i_x + i_y - n} n(x, y, z)_{\mathbb{Z}_2} z$$

$$H_k(M) \otimes H_l(M) \longrightarrow H_{k+l-n}(M).$$

Associative ring.

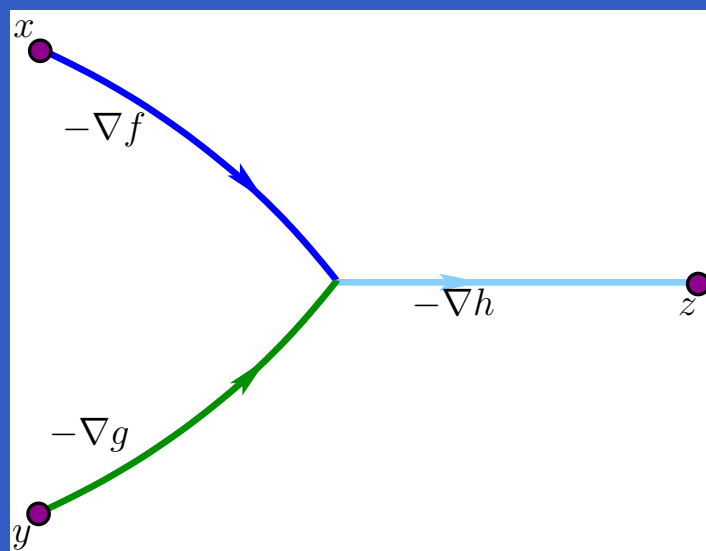
Unity = $[max] \in H_n(M)$.

Algebraic operations (Betz-Cohen, Fukaya et al.)

$f, g, h : M \rightarrow \mathbb{R}$ (generic) Morse functions.

$x \in \text{Crit}(f)$, $y \in \text{Crit}(g)$, $z \in \text{Crit}(h)$.

If $i_z = i_x + i_y - n$:



$$(x, y) \mapsto \sum_{i_z = i_x + i_y - n} n(x, y, z)_{\mathbb{Z}_2} z$$

$$H_k(M) \otimes H_l(M) \longrightarrow H_{k+l-n}(M).$$

Associative ring.

Unity = $[max] \in H_n(M)$.

Many more algebraic operations $\rightsquigarrow A_\infty$ -category.
(Fukaya et al.)

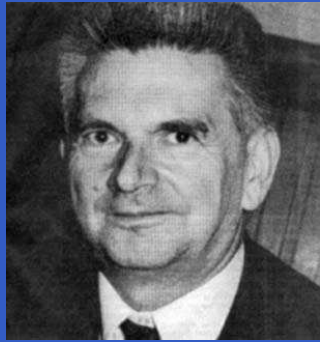


René Thom

1923 - 2002

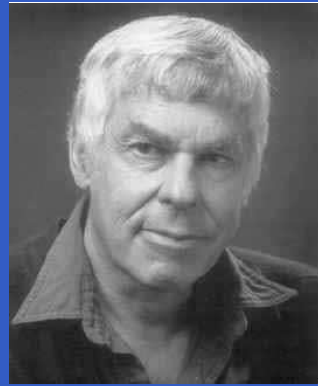
- Thom (1949): “Sur une partition en cellules associée à une fonction sur une variété.”

Credits



René Thom

1923 - 2002



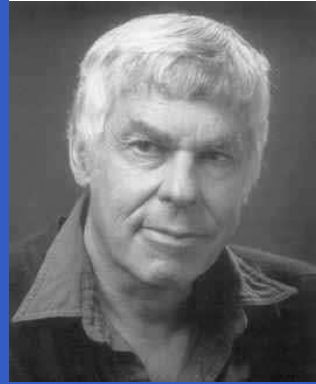
Stephen Smale

- Thom (1949): “Sur une partition en cellules associée à une fonction sur une variété.”
- Smale (1961): “Generalized Poincaré’s conjecture in dimensions greater than four.”

Credits



René Thom
1923 - 2002



Stephen Smale



Edward Witten

- Thom (1949): “Sur une partition en cellules associée à une fonction sur une variété.”
- Smale (1961): “Generalized Poincaré’s conjecture in dimensions greater than four.”
- Witten (1982): “Supersymmetry and Morse theory.”

Symplectic Geometry

(M^{2n}, ω) . $\omega =$ diff. 2-form. $\forall u, v \in T_x(M), \omega(u, v) \in \mathbb{R}$

Symplectic Geometry

(M^{2n}, ω) . $\omega =$ **diff. 2-form**. $\forall u, v \in T_x(M), \omega(u, v) \in \mathbb{R}$

Locally $(\mathbb{R}^{2n}, \omega_{\text{std}} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n)$.

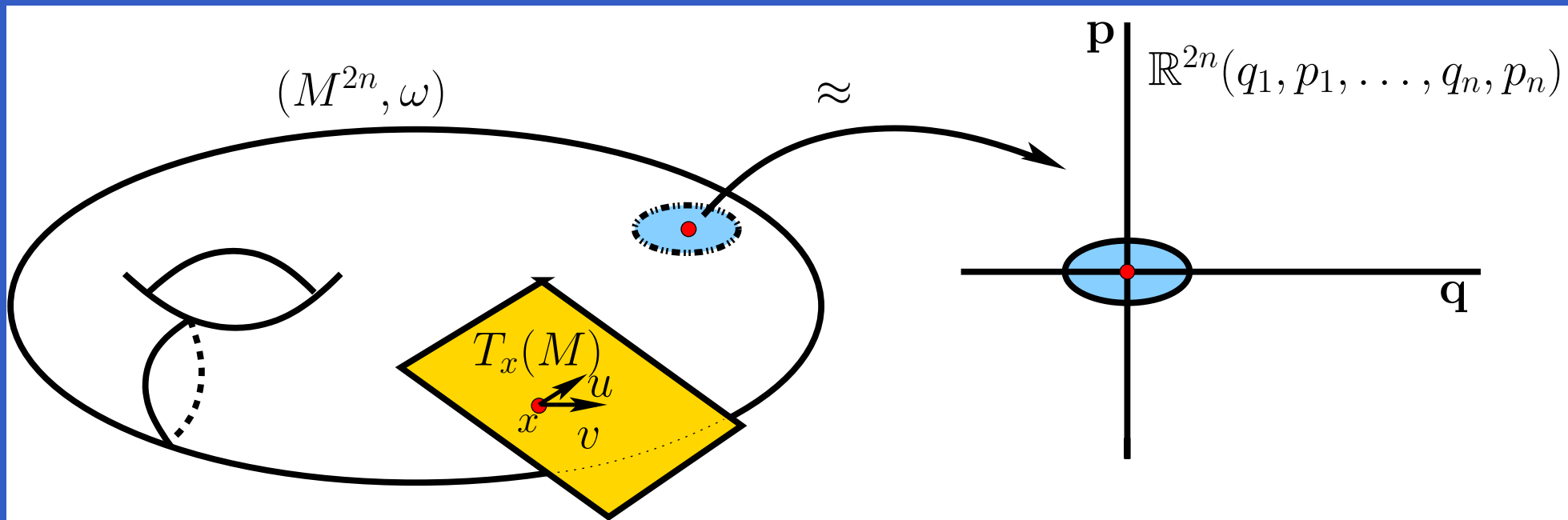
Or in “Complex coordinates”: $\omega_{\text{std}}(u, v) = -\text{Im}\langle u, v \rangle$.

Symplectic Geometry

(M^{2n}, ω) . $\omega = \text{diff. 2-form}$. $\forall u, v \in T_x(M), \omega(u, v) \in \mathbb{R}$

Locally $(\mathbb{R}^{2n}, \omega_{\text{std}} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n)$.

Or in “Complex coordinates”: $\omega_{\text{std}}(u, v) = -\text{Im}\langle u, v \rangle$.

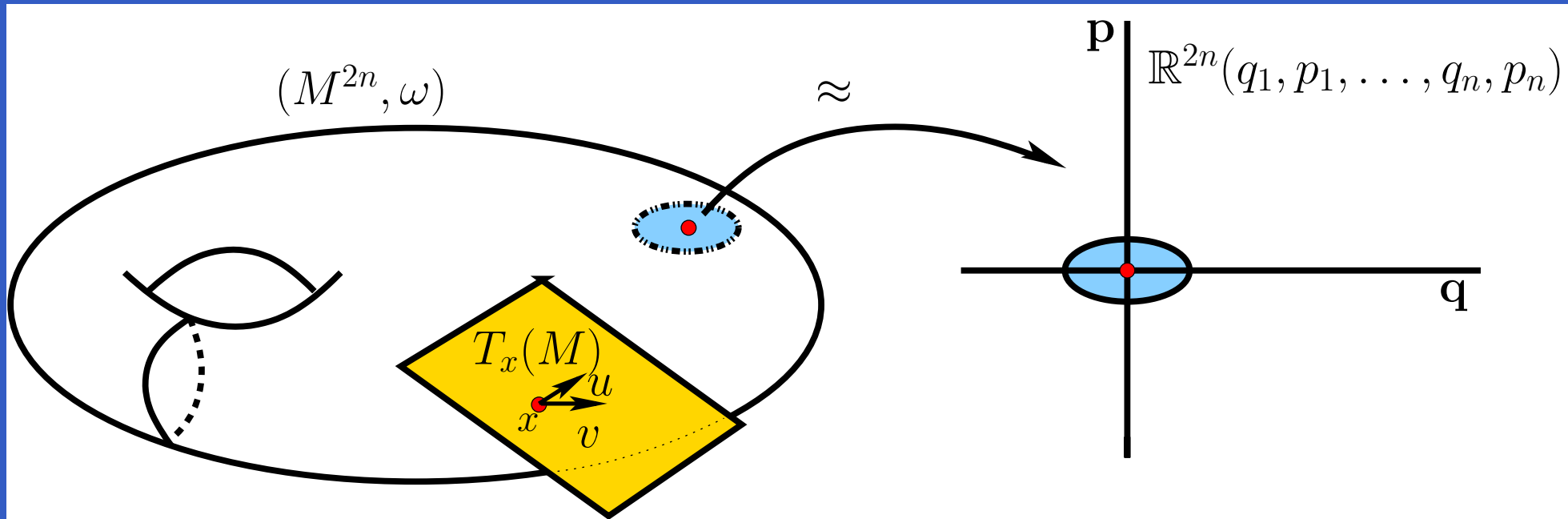


Symplectic Geometry

(M^{2n}, ω) . $\omega = \text{diff. 2-form}$. $\forall u, v \in T_x(M), \omega(u, v) \in \mathbb{R}$

Locally $(\mathbb{R}^{2n}, \omega_{\text{std}} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n)$.

Or in “Complex coordinates”: $\omega_{\text{std}}(u, v) = -\text{Im}\langle u, v \rangle$.



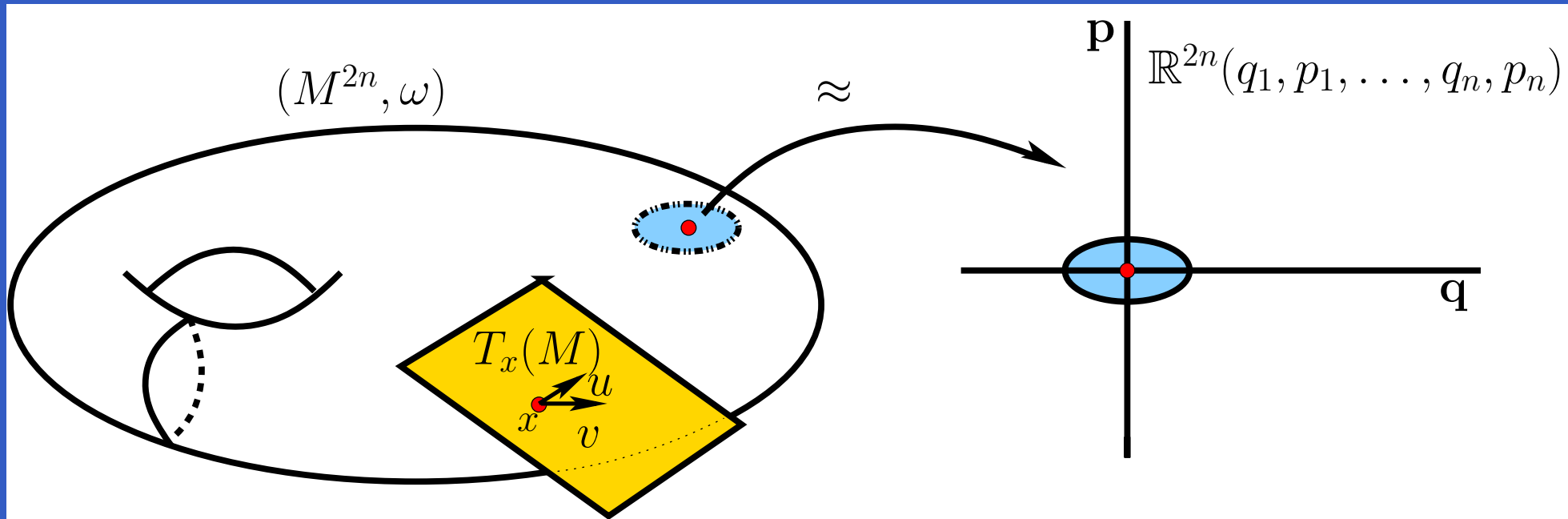
“Flat structure”, no local invariants !

Symplectic Geometry

(M^{2n}, ω) . $\omega = \text{diff. 2-form}$. $\forall u, v \in T_x(M), \omega(u, v) \in \mathbb{R}$

Locally $(\mathbb{R}^{2n}, \omega_{\text{std}} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n)$.

Or in “Complex coordinates”: $\omega_{\text{std}}(u, v) = -\text{Im}\langle u, v \rangle$.



“Flat structure”, no local invariants !

Origins: classical mechanics, calculus of variations

Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

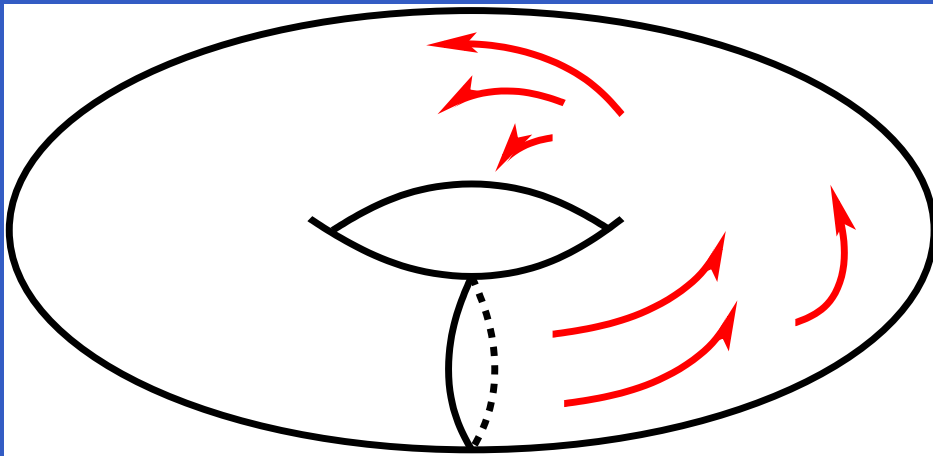
$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t), t) \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t), t) \end{cases}$$

Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t), t) \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t), t) \end{cases}$$

$$\text{Flow } (q, p) \xrightarrow{f_t} (q(t), p(t))$$

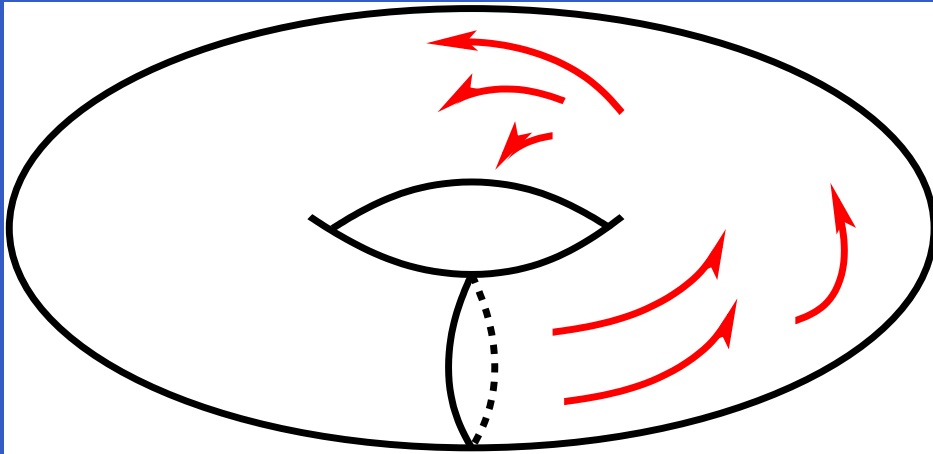


Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t), t) \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t), t) \end{cases}$$

$$\begin{aligned} \text{Flow } (q, p) &\xrightarrow{f_t} (q(t), p(t)) \\ f_t^* \omega &= \omega \end{aligned}$$

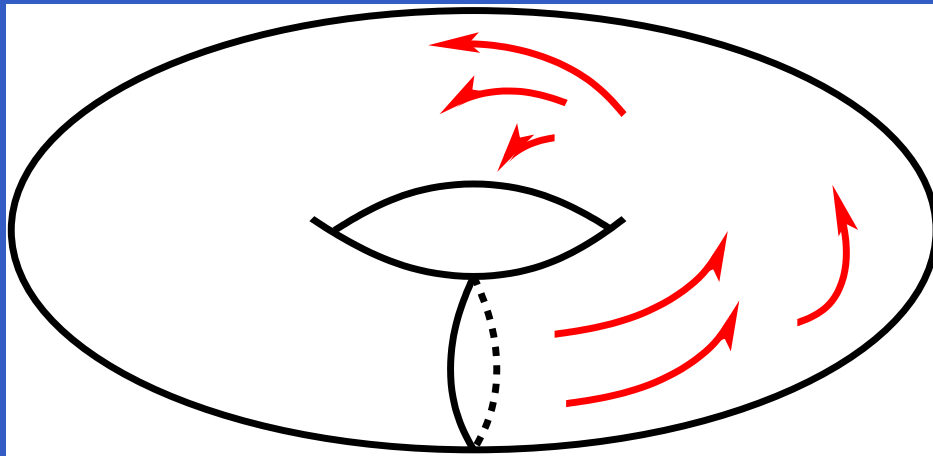


Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t), t) \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t), t) \end{cases}$$

$$\text{Flow } (q, p) \xrightarrow{f_t} (q(t), p(t)) \\ f_t^* \omega = \omega$$



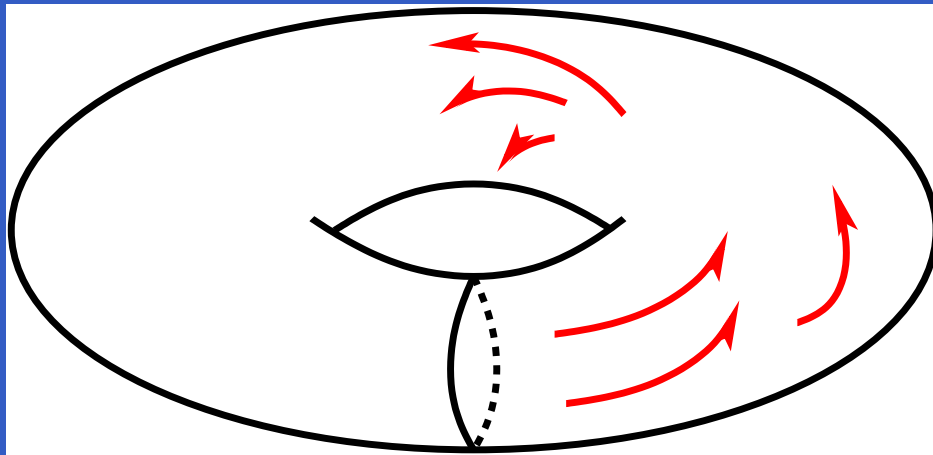
Group $\text{Ham}(M, \omega)$
= { all such maps f_t 's }
HUGE group

Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t), t) \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t), t) \end{cases}$$

$$\text{Flow } (q, p) \xrightarrow{f_t} (q(t), p(t)) \\ f_t^* \omega = \omega$$



Group $\text{Ham}(M, \omega)$
= { all such maps f_t 's }
HUGE group

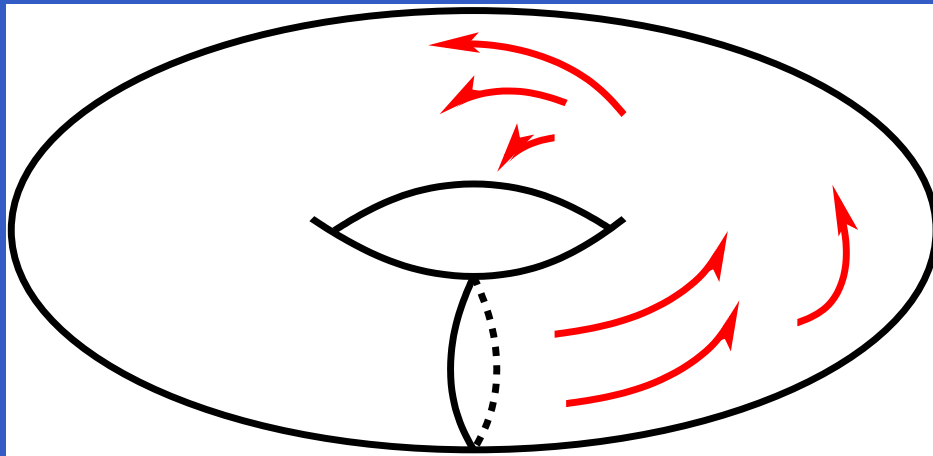
$\text{Ham}(M, \omega)$ looks very flexible: e.g. every 10^6 can be mapped to any other 10^6 points.

Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t), t) \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t), t) \end{cases}$$

$$\text{Flow } (q, p) \xrightarrow{f_t} (q(t), p(t)) \\ f_t^* \omega = \omega$$



Group $\text{Ham}(M, \omega)$
= { all such maps f_t 's }
HUGE group

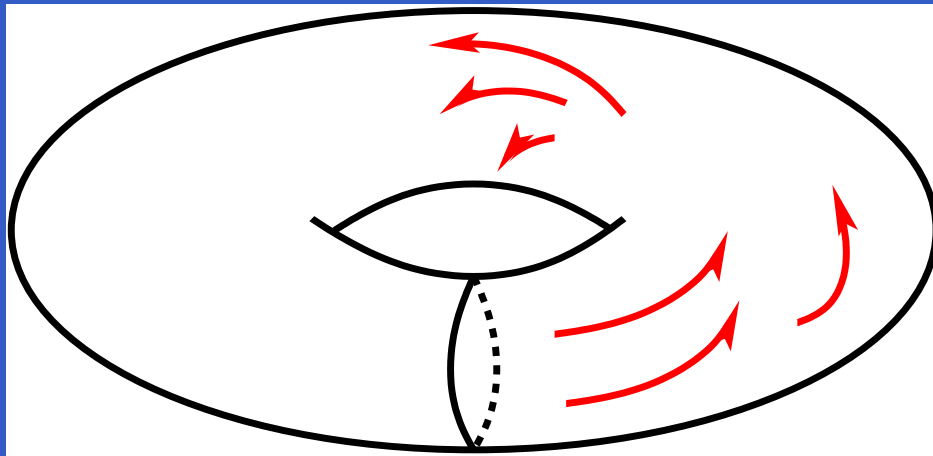
$\text{Ham}(M, \omega)$ looks very flexible: e.g. every 10^6 can be mapped to any other 10^6 points... **but it's an illusion...**

Hamiltonian diffeomorphisms (mechanical motions)

$$H : M \times [0, 1] \rightarrow \mathbb{R}. \quad H(x, t), \quad x = (q, p) \in M.$$

$$\begin{cases} \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t), t) \\ \dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t), t) \end{cases}$$

$$\text{Flow } (q, p) \xrightarrow{f_t} (q(t), p(t)) \\ f_t^* \omega = \omega$$



Group $\text{Ham}(M, \omega)$
= { all such maps f_t 's }
HUGE group

$\text{Ham}(M, \omega)$ looks very flexible: e.g. every 10^6 can be mapped to any other 10^6 points... **but it's an illusion...**

Qualitative study of Ham and its flows is central problem in symplectic topology.

Lagrangian submanifolds

$L^n \subset (M^{2n}, \omega)$ is called **Lagrangian** if
 $\dim L = \frac{1}{2} \dim M$, & $\omega(u, v) = 0, \forall u, v \in T_p(L)$.

Examples

Lagrangian submanifolds

$L^n \subset (M^{2n}, \omega)$ is called **Lagrangian** if
 $\dim L = \frac{1}{2} \dim M$, & $\omega(u, v) = 0, \forall u, v \in T_p(L)$.

Examples

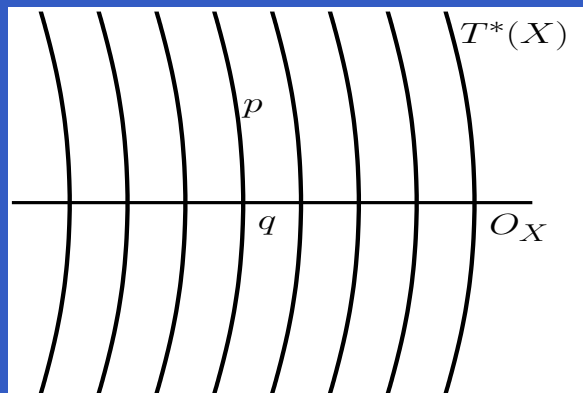
- $L = \mathbb{T}^n = S^1 \times \cdots \times S^1 \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2n}$.

Lagrangian submanifolds

$L^n \subset (M^{2n}, \omega)$ is called **Lagrangian** if
 $\dim L = \frac{1}{2} \dim M$, & $\omega(u, v) = 0, \forall u, v \in T_p(L)$.

Examples

- $L = \mathbb{T}^n = S^1 \times \cdots \times S^1 \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2n}$.
- Cotangent bundle: $M = T^*(X), \omega = dp \wedge dq, L = \mathbf{0}_X$.

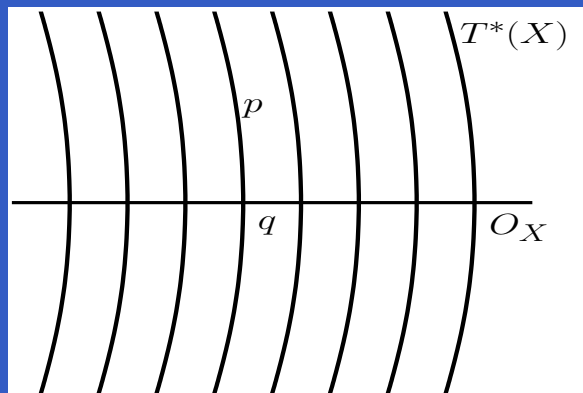


Lagrangian submanifolds

$L^n \subset (M^{2n}, \omega)$ is called **Lagrangian** if
 $\dim L = \frac{1}{2} \dim M$, & $\omega(u, v) = 0, \forall u, v \in T_p(L)$.

Examples

- $L = \mathbb{T}^n = S^1 \times \cdots \times S^1 \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2n}$.
- Cotangent bundle: $M = T^*(X), \omega = dp \wedge dq, L = \mathbf{0}_X$.
- Complex projective space: $M = \mathbb{C}P^n, L = \mathbb{R}P^n$.

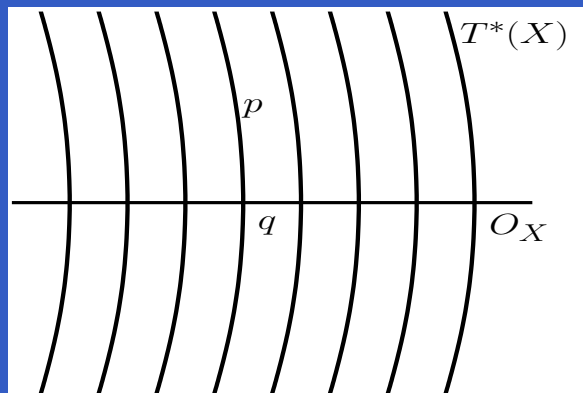


Lagrangian submanifolds

$L^n \subset (M^{2n}, \omega)$ is called **Lagrangian** if
 $\dim L = \frac{1}{2} \dim M$, & $\omega(u, v) = 0, \forall u, v \in T_p(L)$.

Examples

- $L = \mathbb{T}^n = S^1 \times \cdots \times S^1 \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2n}$.
- Cotangent bundle: $M = T^*(X), \omega = dp \wedge dq, L = \mathbf{0}_X$.
- Complex projective space: $M = \mathbb{C}P^n, L = \mathbb{R}P^n$.
- $\mathbb{T}_{\text{clif}}^n = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid |z_0| = \cdots = |z_n|\} \subset \mathbb{C}P^n$



Lagrangians don't obey the usual laws of topology!

Lagrangians don't obey the usual laws of topology!

Thm(Gromov 1985): \nexists simply connected compact
Lagrangians $L \subset \mathbb{R}^{2n}$.

Lagrangians don't obey the usual laws of topology!

Thm(Gromov 1985): \nexists simply connected compact Lagrangians $L \subset \mathbb{R}^{2n}$. (e.g. \nexists Lagrangian $S^3 \rightarrow \mathbb{R}^6$, but \nexists purely topological obstruction.)

Lagrangians don't obey the usual laws of topology!

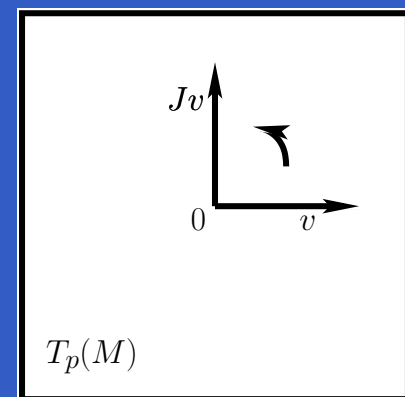
Thm(Gromov 1985): \nexists simply connected compact Lagrangians $L \subset \mathbb{R}^{2n}$. (e.g. \nexists Lagrangian $S^3 \rightarrow \mathbb{R}^6$, but \nexists purely topological obstruction.)

Gromov's idea: (pseudo)-holomorphic curves.

■ *Almost complex structure:* $J = \{J_p\}_{p \in M}$

$$J_p : T_p(M) \rightarrow T_p(M), \quad J_p^2 = -\mathbb{1}.$$

$\Rightarrow T_p(M)$ complex vector space.



Lagrangians don't obey the usual laws of topology!

Thm(Gromov 1985): \nexists simply connected compact Lagrangians $L \subset \mathbb{R}^{2n}$. (e.g. \nexists Lagrangian $S^3 \rightarrow \mathbb{R}^6$, but \nexists purely topological obstruction.)

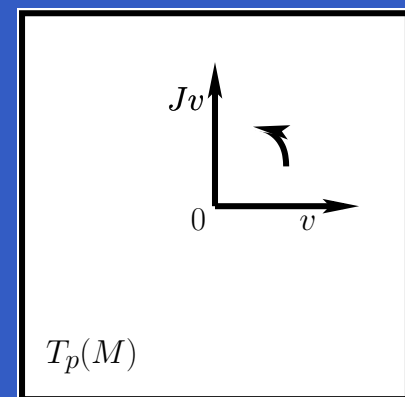
Gromov's idea: (pseudo)-holomorphic curves.

■ *Almost complex structure:* $J = \{J_p\}_{p \in M}$

$$J_p : T_p(M) \rightarrow T_p(M), J_p^2 = -\mathbb{1}.$$

$\Rightarrow T_p(M)$ complex vector space.

Positivity: $\omega(v, Jv) > 0, \forall 0 \neq v \in T_p(M)$.



Lagrangians don't obey the usual laws of topology!

Thm(Gromov 1985): \nexists simply connected compact Lagrangians $L \subset \mathbb{R}^{2n}$. (e.g. \nexists Lagrangian $S^3 \rightarrow \mathbb{R}^6$, but \nexists purely topological obstruction.)

Gromov's idea: (pseudo)-holomorphic curves.

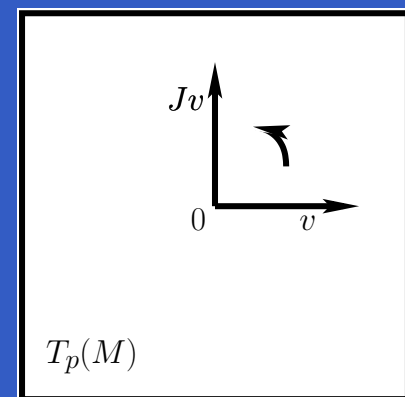
■ *Almost complex structure:* $J = \{J_p\}_{p \in M}$

$$J_p : T_p(M) \rightarrow T_p(M), \quad J_p^2 = -\mathbb{1}.$$

$\Rightarrow T_p(M)$ complex vector space.

Positivity: $\omega(v, Jv) > 0, \forall 0 \neq v \in T_p(M)$.

\exists plenty of such J 's on every (M, ω) .



Lagrangians don't obey the usual laws of topology!

Thm(Gromov 1985): \nexists simply connected compact Lagrangians $L \subset \mathbb{R}^{2n}$. (e.g. \nexists Lagrangian $S^3 \rightarrow \mathbb{R}^6$, but \nexists purely topological obstruction.)

Gromov's idea: (pseudo)-holomorphic curves.

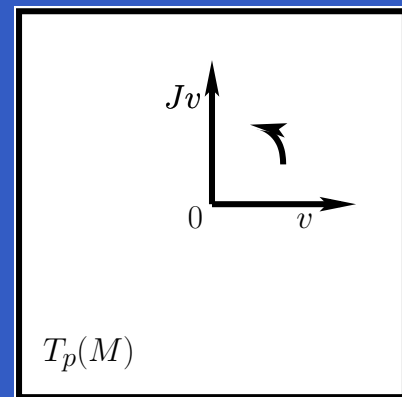
■ **Almost complex structure:** $J = \{J_p\}_{p \in M}$

$$J_p : T_p(M) \rightarrow T_p(M), J_p^2 = -\mathbb{1}.$$

$\Rightarrow T_p(M)$ complex vector space.

Positivity: $\omega(v, Jv) > 0, \forall 0 \neq v \in T_p(M)$.

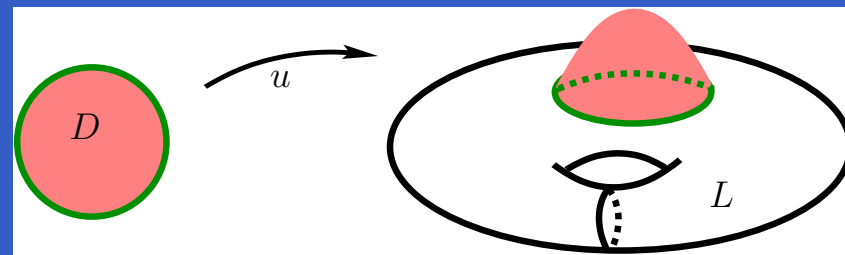
\exists plenty of such J 's on every (M, ω) .



■ **J -holomorphic disks:** $D \subset \mathbb{C}$ unit disk.

$u : (D, \partial D) \rightarrow (M, L)$ that satisfies Cauchy-Riemann

eq. Nonlinear elliptic PDE
$$\begin{cases} \frac{\partial u}{\partial \bar{z}} + J_{u(x,y)} \frac{\partial u}{\partial z} = 0 \\ u(z) \in L, \forall z \in \partial D \end{cases}$$



Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk
 $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Previous theorem follows:

Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Previous theorem follows: In \mathbb{R}^{2n} , $\omega_{\text{std}} = d\lambda$. $\lambda|_L = \text{closed}$.

Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Previous theorem follows: In \mathbb{R}^{2n} , $\omega_{\text{std}} = d\lambda$. $\lambda|_L = \text{closed}$. L simply connected $\Rightarrow u(\partial D)$ is contractible in L .

Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Previous theorem follows: In \mathbb{R}^{2n} , $\omega_{\text{std}} = d\lambda$. $\lambda|_L = \text{closed}$. L simply connected $\Rightarrow u(\partial D)$ is contractible in L . $\Rightarrow 0 = \int_{\partial D} u^* \lambda$

Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Previous theorem follows: In \mathbb{R}^{2n} , $\omega_{\text{std}} = d\lambda$. $\lambda|_L = \text{closed}$. L simply connected $\Rightarrow u(\partial D)$ is contractible in

$$L. \Rightarrow 0 = \int_{\partial D} u^* \lambda = \int_D u^* \omega_{\text{std}}$$

(Stokes)



Existence of holomorphic disks

Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Previous theorem follows: In \mathbb{R}^{2n} , $\omega_{\text{std}} = d\lambda$. $\lambda|_L = \text{closed}$. L simply connected $\Rightarrow u(\partial D)$ is contractible in

$$L. \Rightarrow 0 = \int_{\partial D} u^* \lambda = \int_D u^* \omega_{\text{std}} > 0. \quad (J \text{ is positive})$$

(Stokes)



□

Existence of holomorphic disks

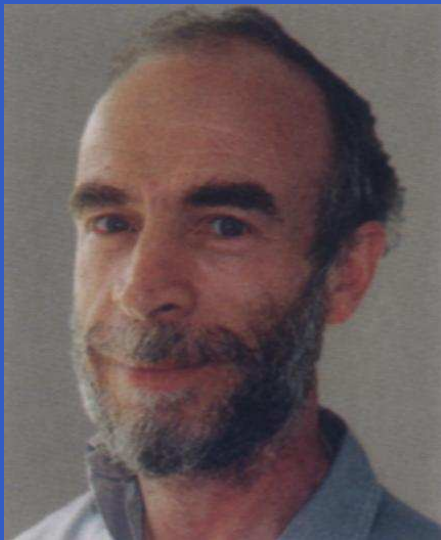
Gromov proved: $\forall J, \exists$ a non-constant J -holomorphic disk $u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$.

Proof is very indirect. Involves a hard compactness theorem for solutions of elliptic PDE's

Previous theorem follows: In \mathbb{R}^{2n} , $\omega_{\text{std}} = d\lambda$. $\lambda|_L = \text{closed}$. L simply connected $\Rightarrow u(\partial D)$ is contractible in

$$L. \Rightarrow 0 = \int_{\partial D} u^* \lambda = \int_D u^* \omega_{\text{std}}$$

□



Mikhail Gromov

- 1985: “Pseudo holomorphic curves in symplectic manifolds.”

Infinite dimensional Morse theory

Infinite dimensional Morse theory



Andreas Floer (1956-91)

- 1988: “Morse theory for Lagrangian intersections.”
- 1989: “Witten’s complex and infinite-dimensional Morse theory.”

Floer *Morsified* Gromov’s theory

Infinite dimensional Morse theory



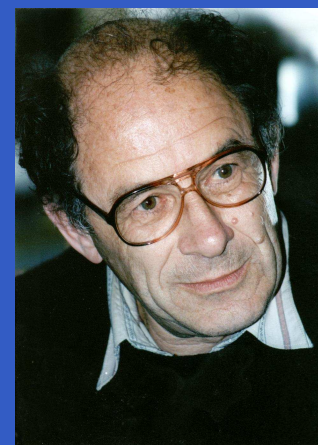
Andreas Floer (1956-91)

- 1988: “Morse theory for Lagrangian intersections.”
- 1989: “Witten’s complex and infinite-dimensional Morse theory.”

Floer *Morsified* Gromov’s theory

Arnold’s conjecture. $T^*(X)$, $\phi \in \text{Ham}$.

$$\Rightarrow \#(0_X \cap \phi(0_X)) \geq \sum_{i=0}^n \dim H_i(X).$$



Vladimir Arnold

Infinite dimensional Morse theory

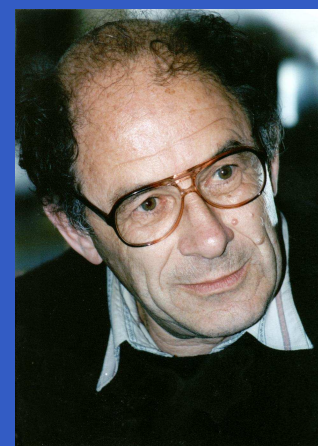
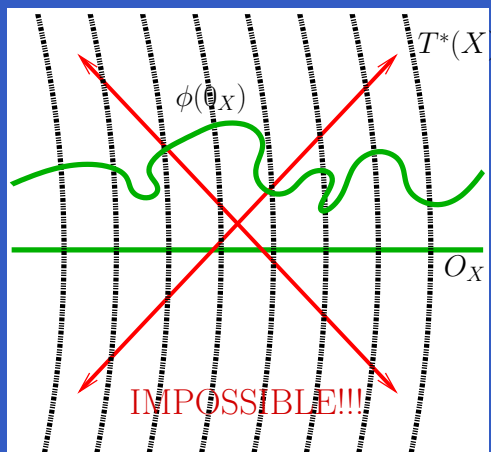
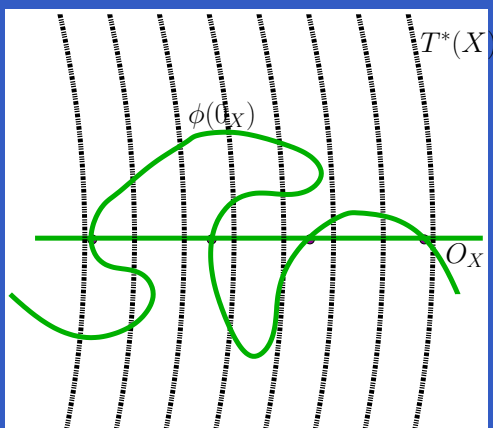


- 1988: “Morse theory for Lagrangian intersections.”
- 1989: “Witten’s complex and infinite-dimensional Morse theory.”

Andreas Floer (1956-91) Floer *Morsified* Gromov’s theory

Arnold’s conjecture. $T^*(X)$, $\phi \in \text{Ham}$.

$$\Rightarrow \#(0_X \cap \phi(0_X)) \geq \sum_{i=0}^n \dim H_i(X).$$



Vladimir Arnold

Floer theory

$L \subset M$ Lagrangian, $L' = \phi(L)$, $\phi \in \text{Ham}$, $L \pitchfork L'$.

Floer theory

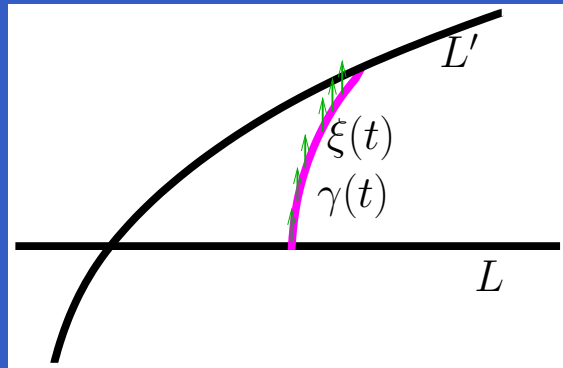
$L \subset M$ Lagrangian, $L' = \phi(L)$, $\phi \in \text{Ham}$, $L \pitchfork L'$.

■ Path space: $\mathcal{P} = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L, \gamma(1) \in L'\}$

Floer theory

$L \subset M$ Lagrangian, $L' = \phi(L)$, $\phi \in \text{Ham}$, $L \pitchfork L'$.

- Path space: $\mathcal{P} = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L, \gamma(1) \in L'\}$
- Tangent space: $T_\gamma \mathcal{P} =$ vector fields $\xi(t)$ along $\gamma(t)$.



Floer theory

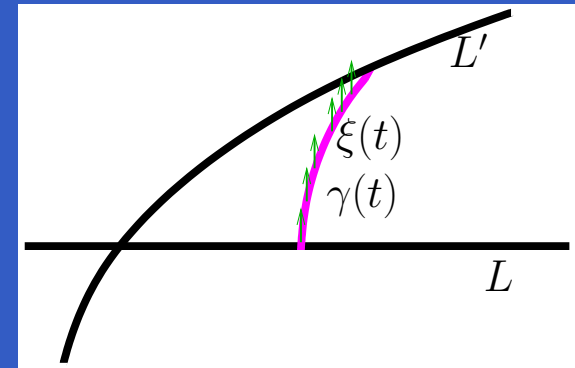
$L \subset M$ Lagrangian, $L' = \phi(L)$, $\phi \in \text{Ham}$, $L \pitchfork L'$.

- Path space: $\mathcal{P} = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L, \gamma(1) \in L'\}$
- Tangent space: $T_\gamma \mathcal{P} =$ vector fields $\xi(t)$ along $\gamma(t)$.

Action 1-form:

$$\alpha_\gamma(\xi) = \int_0^1 \omega(\xi(t), \dot{\gamma}(t)) dt.$$

Locally: $\alpha = d\mathcal{A}$, $\mathcal{A} : \mathcal{P} \rightarrow \mathbb{R}$



Floer theory

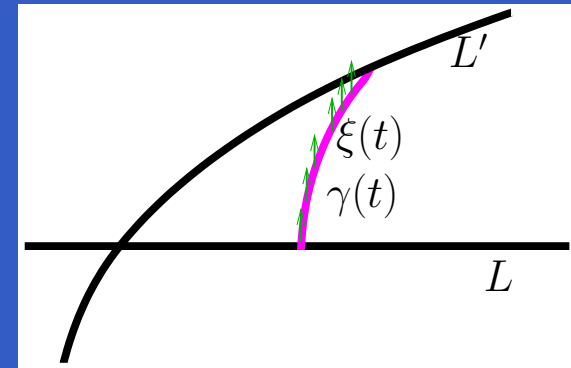
$L \subset M$ Lagrangian, $L' = \phi(L)$, $\phi \in \text{Ham}$, $L \pitchfork L'$.

- Path space: $\mathcal{P} = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L, \gamma(1) \in L'\}$
- Tangent space: $T_\gamma \mathcal{P} =$ vector fields $\xi(t)$ along $\gamma(t)$.

Action 1-form:

$$\alpha_\gamma(\xi) = \int_0^1 \omega(\xi(t), \dot{\gamma}(t)) dt.$$

Locally: $\alpha = d\mathcal{A}$, $\mathcal{A} : \mathcal{P} \rightarrow \mathbb{R}$



- Critical points = constant paths = $L \cap L'$.

Floer theory

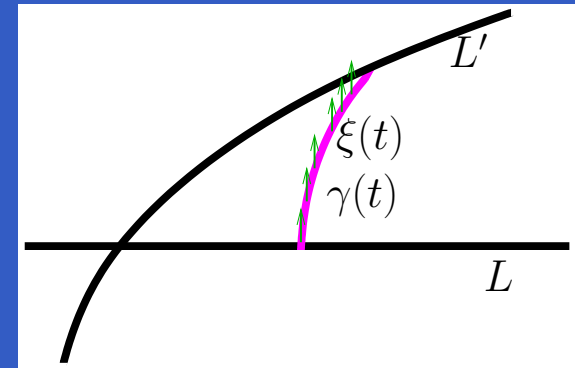
$L \subset M$ Lagrangian, $L' = \phi(L)$, $\phi \in \text{Ham}$, $L \pitchfork L'$.

- Path space: $\mathcal{P} = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L, \gamma(1) \in L'\}$
- Tangent space: $T_\gamma \mathcal{P} =$ vector fields $\xi(t)$ along $\gamma(t)$.

Action 1-form:

$$\alpha_\gamma(\xi) = \int_0^1 \omega(\xi(t), \dot{\gamma}(t)) dt.$$

Locally: $\alpha = d\mathcal{A}$, $\mathcal{A} : \mathcal{P} \rightarrow \mathbb{R}$



- Critical points = constant paths = $L \cap L'$.
- Gradient flow: $u : \mathbb{R} \rightarrow \mathcal{P}$, path of paths $\Rightarrow u(\tau, t)$

Floer theory

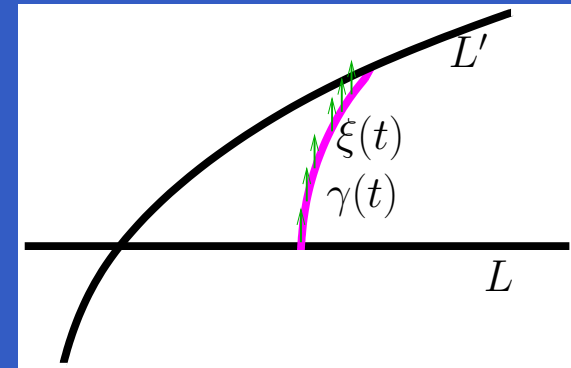
$L \subset M$ Lagrangian, $L' = \phi(L)$, $\phi \in \text{Ham}$, $L \pitchfork L'$.

- Path space: $\mathcal{P} = \{\gamma : [0, 1] \rightarrow M \mid \gamma(0) \in L, \gamma(1) \in L'\}$
- Tangent space: $T_\gamma \mathcal{P} =$ vector fields $\xi(t)$ along $\gamma(t)$.

Action 1-from:

$$\alpha_\gamma(\xi) = \int_0^1 \omega(\xi(t), \dot{\gamma}(t)) dt.$$

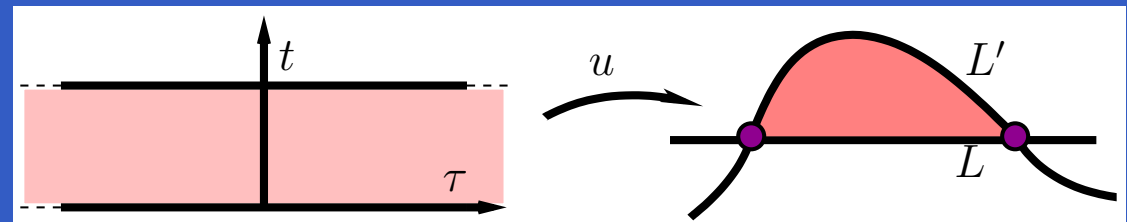
Locally: $\alpha = d\mathcal{A}$, $\mathcal{A} : \mathcal{P} \rightarrow \mathbb{R}$



- Critical points = constant paths = $L \cap L'$.

- Gradient flow: $u : \mathbb{R} \rightarrow \mathcal{P}$, path of paths $\Rightarrow u(\tau, t)$

$$\begin{cases} u : \mathbb{R} \times [0, 1] \rightarrow M \\ \frac{\partial u}{\partial \tau} + J_{u(\tau, t)} \frac{\partial u}{\partial t} = 0 \\ u(\tau, 0) \in L, u(\tau, 1) \in L' \end{cases}$$



Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

■ $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

- $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$
- **Invariance:** $L'' = \psi(L) \Rightarrow HF(L, L') \cong HF(L, L'').$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

- $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$
- Invariance: $L'' = \psi(L) \Rightarrow HF(L, L') \cong HF(L, L'').$
- Computation of $HF(L)...$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

- $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$
- Invariance: $L'' = \psi(L) \Rightarrow HF(L, L') \cong HF(L, L'').$
- Computation of $HF(L) \dots$

Note that: $\dim HF(L, L') \leq \dim CF(L, L') = \# L \cap L'.$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

- $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$
- Invariance: $L'' = \psi(L) \Rightarrow HF(L, L') \cong HF(L, L'').$
- Computation of $HF(L)...$

Note that: $\dim HF(L, L') \leq \dim CF(L, L') = \# L \cap L'.$

Floer proved: If $\langle \omega, \pi_2(M, L) \rangle = 0 \implies$ it works &
 $HF(L, L') \cong H_*(L) \forall \phi \in \text{Ham}.$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

- $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$
- Invariance: $L'' = \psi(L) \Rightarrow HF(L, L') \cong HF(L, L'').$
- Computation of $HF(L)...$

Note that: $\dim HF(L, L') \leq \dim CF(L, L') = \# L \cap L'.$

Floer proved: If $\langle \omega, \pi_2(M, L) \rangle = 0 \implies$ it works & $HF(L, L') \cong H_*(L) \forall \phi \in \text{Ham}.$ In particular \implies Arnold's

conjecture for $T^*(X)$: $\#(0_X \cap \phi(0_X)) \geq \sum_{i=1}^n \dim H_i(X).$

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

- $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$
- Invariance: $L'' = \psi(L) \Rightarrow HF(L, L') \cong HF(L, L'').$
- Computation of $HF(L)...$

Note that: $\dim HF(L, L') \leq \dim CF(L, L') = \# L \cap L'.$

Floer proved: If $\langle \omega, \pi_2(M, L) \rangle = 0 \implies$ it works & $HF(L, L') \cong H_*(L) \forall \phi \in \text{Ham}.$ In particular \implies Arnold's

conjecture for $T^*(X)$: $\#(0_X \cap \phi(0_X)) \geq \sum_{i=1}^n \dim H_i(X).$

Main point: \nexists holomorphic disks !

Floer homology

Floer complex: $CF = CF(L, L') = \bigoplus_{x \in L \cap L'} \mathbb{Z}_2 x$

$$d : CF \rightarrow CF, \quad d(x) = \sum_{y \in L \cap L'} \nu_{\mathbb{Z}_2}(x \rightarrow y) y.$$

Desired properties:

- $d \circ d = 0 \Rightarrow HF(L, L') = \ker d / \text{image } d.$
- Invariance: $L'' = \psi(L) \Rightarrow HF(L, L') \cong HF(L, L'').$
- Computation of $HF(L) \dots$

Note that: $\dim HF(L, L') \leq \dim CF(L, L') = \# L \cap L'.$

Floer proved: If $\langle \omega, \pi_2(M, L) \rangle = 0 \implies$ it works & $HF(L, L') \cong H_*(L) \forall \phi \in \text{Ham}.$ In particular \implies Arnold's

conjecture for $T^*(X)$: $\#(0_X \cap \phi(0_X)) \geq \sum_{i=1}^n \dim H_i(X).$

Main point: \nexists holomorphic disks !

Prev results: Chaperon, Hofer, Laudenbach-Sikorav, Chekanov.

Beyond Lagrangian intersections?

Topology of Lagrangians

Beyond Lagrangian intersections?

Topology of Lagrangians

- $L \subset \mathbb{C}P^n$ can't be simply connected.

Beyond Lagrangian intersections?

Topology of Lagrangians

- $L \subset \mathbb{C}P^n$ can't be simply connected.
But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Beyond Lagrangian intersections?

Topology of Lagrangians

■ $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$.

Beyond Lagrangian intersections?


Topology of Lagrangians

■ $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$. If $n = \text{odd}$ isomorphism of rings.



Beyond Lagrangian intersections?

Topology of Lagrangians

- $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$. If $n = \text{odd}$ isomorphism of rings.

- $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$?

Beyond Lagrangian intersections?

Topology of Lagrangians

- $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$. If $n = \text{odd}$ isomorphism of rings.

- $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$? Ex. $\mathbb{C}P^n \approx \text{anti-diag} \subset \mathbb{C}P^n \times \mathbb{C}P^n$.

Beyond Lagrangian intersections?

Topology of Lagrangians

■ $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$. If $n = \text{odd}$ isomorphism of rings.

■ $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$? Ex. $\mathbb{C}P^n \approx \text{anti-diag} \subset \mathbb{C}P^n \times \mathbb{C}P^n$.

Thm(B. 2004): $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$, $\pi_1(L) = 1$

$\Rightarrow H_*(L) \cong H_*(\mathbb{C}P^n)$ as rings.

Beyond Lagrangian intersections?

Topology of Lagrangians

■ $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$. If $n = \text{odd}$ isomorphism of rings.

■ $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$? Ex. $\mathbb{C}P^n \approx \text{anti-diag} \subset \mathbb{C}P^n \times \mathbb{C}P^n$.

Thm(B. 2004): $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$, $\pi_1(L) = 1$

$\Rightarrow H_*(L) \cong H_*(\mathbb{C}P^n)$ as rings.

Thm(Buhovski 2004): $L \subset T^*(S^n)$, $n = \text{odd}$, $\pi_1(L) = 1$

$\Rightarrow H_*(L) \cong H_*(S^n)$.

Beyond Lagrangian intersections?

Topology of Lagrangians

■ $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$. If $n = \text{odd}$ isomorphism of rings.

■ $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$? Ex. $\mathbb{C}P^n \approx \text{anti-diag} \subset \mathbb{C}P^n \times \mathbb{C}P^n$.

Thm(B. 2004): $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$, $\pi_1(L) = 1$

$\Rightarrow H_*(L) \cong H_*(\mathbb{C}P^n)$ as rings.

Thm(Buhovski 2004): $L \subset T^*(S^n)$, $n = \text{odd}$, $\pi_1(L) = 1$

$\Rightarrow H_*(L) \cong H_*(S^n)$. (later generalized by Seidel for $n = \text{even}$)

Beyond Lagrangian intersections?

Topology of Lagrangians

- $L \subset \mathbb{C}P^n$ can't be simply connected.

But $\exists L$ with $\pi_1(L) = \mathbb{Z}_2$. E.g. $L = \mathbb{R}P^n \subset \mathbb{C}P^n$.

Thm(Seidel, B.): $L \subset \mathbb{C}P^n$, $\pi_1(L) = \mathbb{Z}_2$

$\Rightarrow H_*(L) \cong H_*(\mathbb{R}P^n)$. If $n = \text{odd}$ isomorphism of rings.

- $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$? Ex. $\mathbb{C}P^n \approx \text{anti-diag} \subset \mathbb{C}P^n \times \mathbb{C}P^n$.

Thm(B. 2004): $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$, $\pi_1(L) = 1$

$\Rightarrow H_*(L) \cong H_*(\mathbb{C}P^n)$ as rings.

Thm(Buhovski 2004): $L \subset T^*(S^n)$, $n = \text{odd}$, $\pi_1(L) = 1$

$\Rightarrow H_*(L) \cong H_*(S^n)$. (later generalized by Seidel for $n = \text{even}$)

- Phenomenon: Homological uniqueness. Assumption on “low”-dim invariants \Rightarrow determine *all* $H_*(L)$.

Proofs come from Floer theory

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

But how to compute $HF(L)$?

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

But how to compute $HF(L)$?

- Geometric computation.

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

But how to compute $HF(L)$?

- Geometric computation. Sometimes (e.g in $\mathbb{C}P^n$) possible to transform L to a new manifold $\tilde{L} \hookrightarrow V$ and find $\phi \in \text{Ham}(V)$ s.t. $\phi(\tilde{L}) \cap \tilde{L} = \emptyset$.

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

But how to compute $HF(L)$?

- Geometric computation. Sometimes (e.g in $\mathbb{C}P^n$) possible to transform L to a new manifold $\tilde{L} \hookrightarrow V$ and find $\phi \in \text{Ham}(V)$ s.t. $\phi(\tilde{L}) \cap \tilde{L} = \emptyset. \implies HF(\tilde{L}) = 0$.

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

But how to compute $HF(L)$?

- Geometric computation. Sometimes (e.g in $\mathbb{C}P^n$) possible to transform L to a new manifold $\tilde{L} \hookrightarrow V$ and find $\phi \in \text{Ham}(V)$ s.t. $\phi(\tilde{L}) \cap \tilde{L} = \emptyset \implies HF(\tilde{L}) = 0$.
- Apply the spectral sequence:
Step₁ = $H_*(\tilde{L}) \implies$ Step _{∞} = $HF(\tilde{L}) = 0$.

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

But how to compute $HF(L)$?

- Geometric computation. Sometimes (e.g in $\mathbb{C}P^n$) possible to transform L to a new manifold $\tilde{L} \hookrightarrow V$ and find $\phi \in \text{Ham}(V)$ s.t. $\phi(\tilde{L}) \cap \tilde{L} = \emptyset. \implies HF(\tilde{L}) = 0$.
- Apply the spectral sequence:
Step₁ = $H_*(\tilde{L}) \implies$ Step _{∞} = $HF(\tilde{L}) = 0$.
 \rightsquigarrow info on $H_*(\tilde{L})$

Proofs come from Floer theory

- When $\pi_2(M, L) \neq 0$, $HF(L)$ is hard to compute!
- Y.-G. Oh: \exists inductive algorithm
Step₁ = $H_*(L) \implies$ Step _{∞} = $HF(L)$. (spectral sequence)

But how to compute $HF(L)$?

- Geometric computation. Sometimes (e.g in $\mathbb{C}P^n$) possible to transform L to a new manifold $\tilde{L} \hookrightarrow V$ and find $\phi \in \text{Ham}(V)$ s.t. $\phi(\tilde{L}) \cap \tilde{L} = \emptyset \implies HF(\tilde{L}) = 0$.
- Apply the spectral sequence:
Step₁ = $H_*(\tilde{L}) \implies$ Step _{∞} = $HF(\tilde{L}) = 0$.
 \rightsquigarrow info on $H_*(\tilde{L}) \implies$ info on $H_*(L)$.

Applications outside of symplectic geometry

Applications outside of symplectic geometry

Info on topology of Lagrangians has applications to...

Applications outside of symplectic geometry

Info on topology of Lagrangians has applications to... ..
algebraic geometry.

Applications outside of symplectic geometry

Info on topology of Lagrangians has applications to... .. algebraic geometry.

Many algebro geometric situations (e.g. degenerations)
↔ Lagrangian spheres.

Applications outside of symplectic geometry

Info on topology of Lagrangians has applications to... .. algebraic geometry.

Many algebro geometric situations (e.g. degenerations)
↔ Lagrangian spheres.

Using Floer theory we can study when such situations occur/can't occur.

Applications outside of symplectic geometry

Info on topology of Lagrangians has applications to... .. algebraic geometry.

Many algebro geometric situations (e.g. degenerations)
↔ Lagrangian spheres.

Using Floer theory we can study when such situations occur/can't occur.

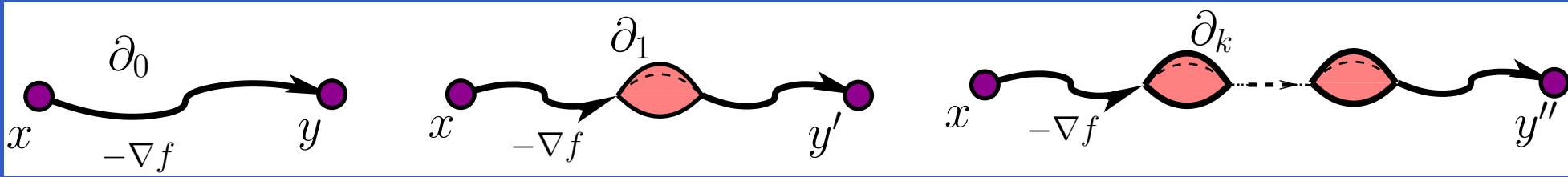
First steps in this direction: B., Jerby.

Strings of pearls

$f : L \rightarrow \mathbb{R}$ Morse function. $CP = \bigoplus_{x \in \text{Crit}(f)} \mathbb{Z}_2 x.$

Strings of pearls

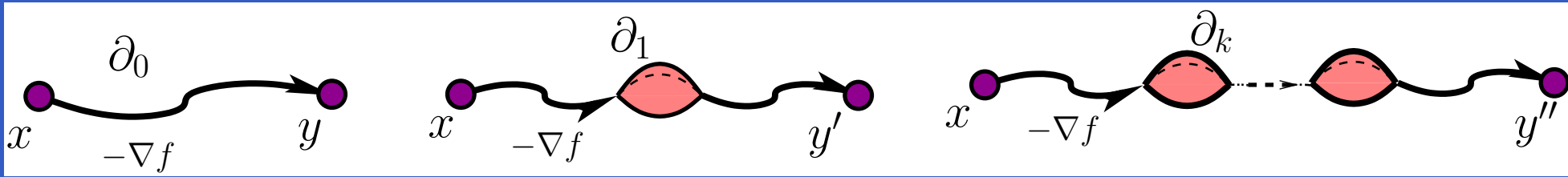
$f : L \rightarrow \mathbb{R}$ Morse function. $CP = \bigoplus_{x \in \text{Crit}(f)} \mathbb{Z}_2 x$.



$d : CP \rightarrow CP$, $d = \partial_0 + \partial_1 + \cdots + \partial_r$. $HP_* = \ker d / \text{image } d$.

Strings of pearls

$f : L \rightarrow \mathbb{R}$ Morse function. $CP = \bigoplus_{x \in \text{Crit}(f)} \mathbb{Z}_2 x$.

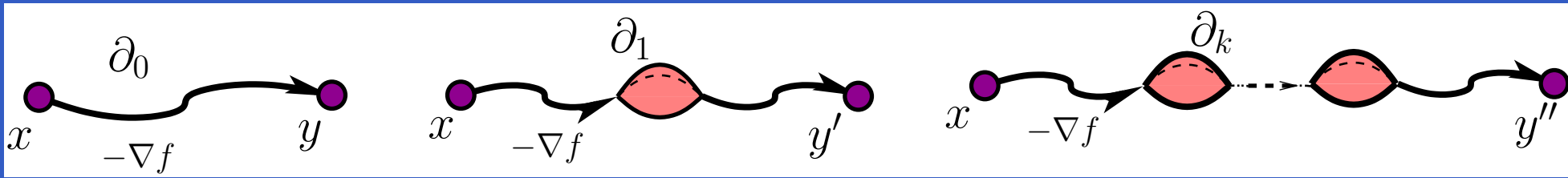


$d : CP \rightarrow CP$, $d = \partial_0 + \partial_1 + \cdots + \partial_r$. $HP_* = \ker d / \text{image } d$.

Thm: $HP_* \cong HF_*(L)$.

Strings of pearls

$f : L \rightarrow \mathbb{R}$ Morse function. $CP = \bigoplus_{x \in \text{Crit}(f)} \mathbb{Z}_2 x$.



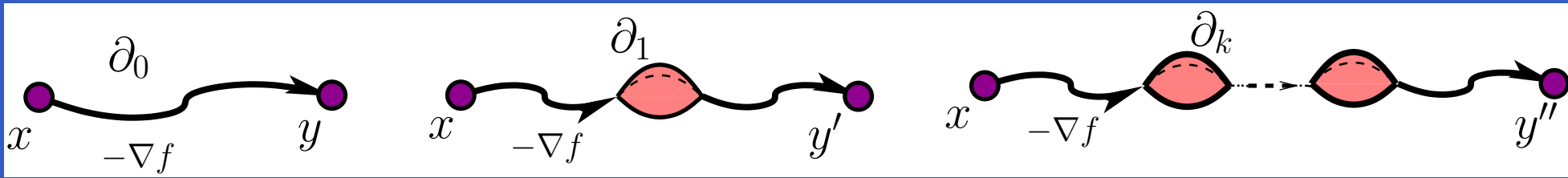
$d : CP \rightarrow CP$, $d = \partial_0 + \partial_1 + \cdots + \partial_r$. $HP_* = \ker d / \text{image } d$.

Thm: $HP_* \cong HF_*(L)$.

(Model suggested by Fukaya, Oh. Implemented by B.-Cornea)

Strings of pearls

$f : L \rightarrow \mathbb{R}$ Morse function. $CP = \bigoplus_{x \in \text{Crit}(f)} \mathbb{Z}_2 x$.



$d : CP \rightarrow CP$, $d = \partial_0 + \partial_1 + \cdots + \partial_r$. $HP_* = \ker / \text{image } d$.

Thm: $HP_* \cong HF_*(L)$.

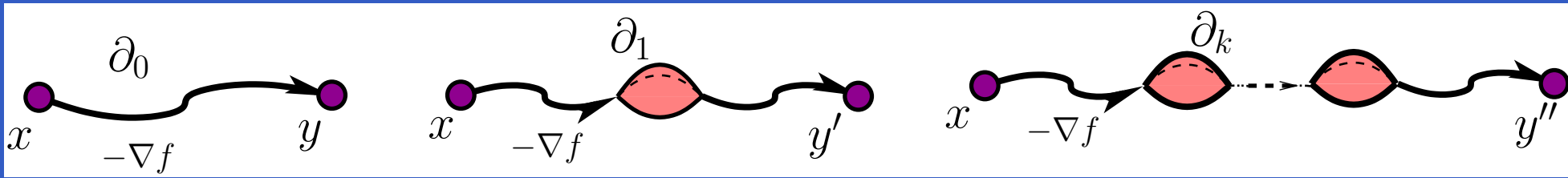
(Model suggested by Fukaya, Oh. Implemented by B.-Cornea)

Quantum products.

$f, f', f'' : L \rightarrow \mathbb{R}$ generic triple of Morse functions.

Strings of pearls

$f : L \rightarrow \mathbb{R}$ Morse function. $CP = \bigoplus_{x \in \text{Crit}(f)} \mathbb{Z}_2 x$.



$d : CP \rightarrow CP$, $d = \partial_0 + \partial_1 + \cdots + \partial_r$. $HP_* = \ker d / \text{image } d$.

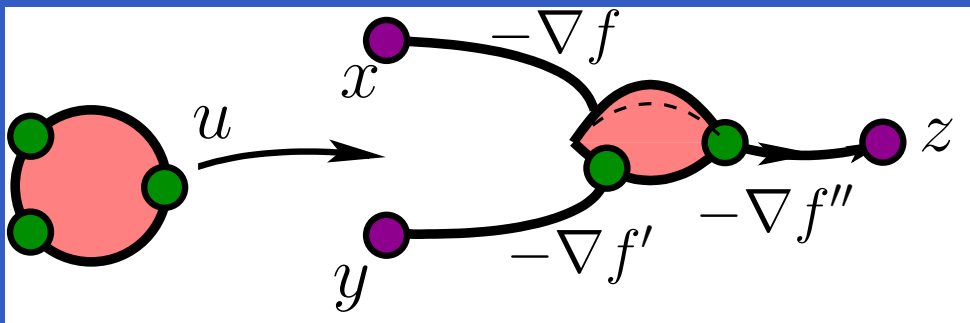
Thm: $HP_* \cong HF_*(L)$.

(Model suggested by Fukaya, Oh. Implemented by B.-Cornea)

Quantum products.

$f, f', f'' : L \rightarrow \mathbb{R}$ generic triple of Morse functions.

$\circ : HF(L) \otimes HF(L) \rightarrow HF(L)$ associative ring.



More structure

(M^{2n}, ω) . $QH(M) := H_*(M) \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{Z}_2[T, T^{-1}]$.

More structure

(M^{2n}, ω) . $QH(M) := H_*(M) \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{Z}_2[T, T^{-1}]$.

$h, h', h'' : M \rightarrow \mathbb{R}$.

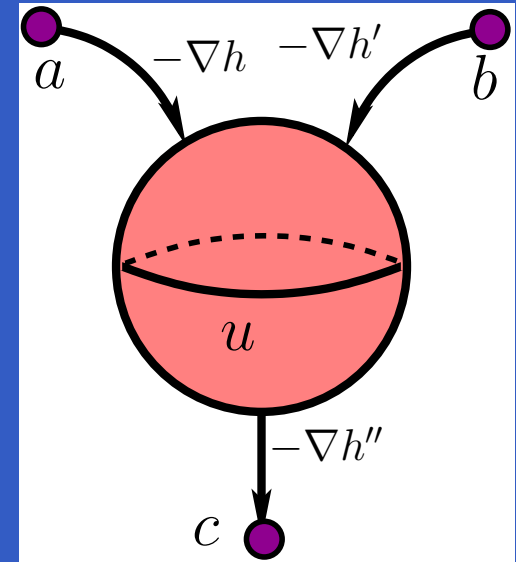
More structure

(M^{2n}, ω) . $QH(M) := H_*(M) \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{Z}_2[T, T^{-1}]$.

$h, h', h'' : M \rightarrow \mathbb{R}$.

$*$: $QH_k \otimes QH_l \rightarrow QH_{k+l-2n}$

$a * b = \sum_{c,u} n(a, b, c) c \otimes T^{c_1(u)}$.



More structure

(M^{2n}, ω) . $QH(M) := H_*(M) \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{Z}_2[T, T^{-1}]$.

$h, h', h'' : M \rightarrow \mathbb{R}$.

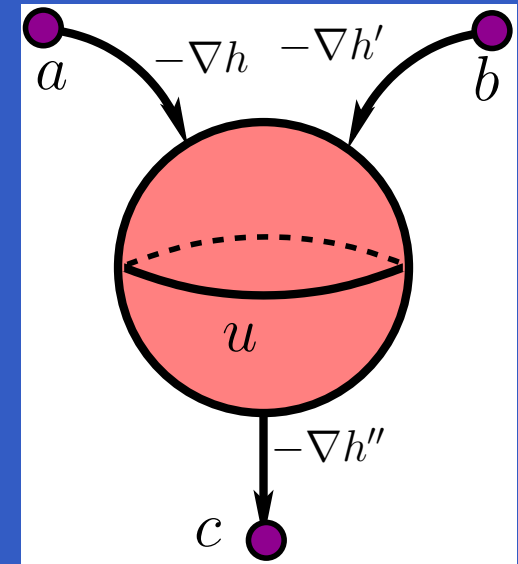
$* : QH_k \otimes QH_l \rightarrow QH_{k+l-2n}$

$a * b = \sum_{c, u} n(a, b, c) c \otimes T^{c_1(u)}$.

Quantum homology

Enumerative algebraic geometry.

(Witten, Ruan, Kontsevich-Manin et al.)



More structure

(M^{2n}, ω) . $QH(M) := H_*(M) \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{Z}_2[T, T^{-1}]$.

$h, h', h'' : M \rightarrow \mathbb{R}$.

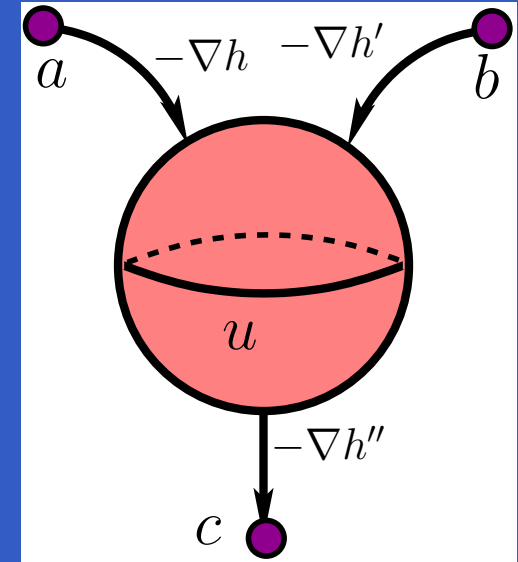
$*$: $QH_k \otimes QH_l \rightarrow QH_{k+l-2n}$

$a * b = \sum_{c, u} n(a, b, c) c \otimes T^{c_1(u)}$.

Quantum homology

Enumerative algebraic geometry.

(Witten, Ruan, Kontsevich-Manin et al.)



Module structure

More structure

(M^{2n}, ω) . $QH(M) := H_*(M) \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{Z}_2[T, T^{-1}]$.

$h, h', h'' : M \rightarrow \mathbb{R}$.

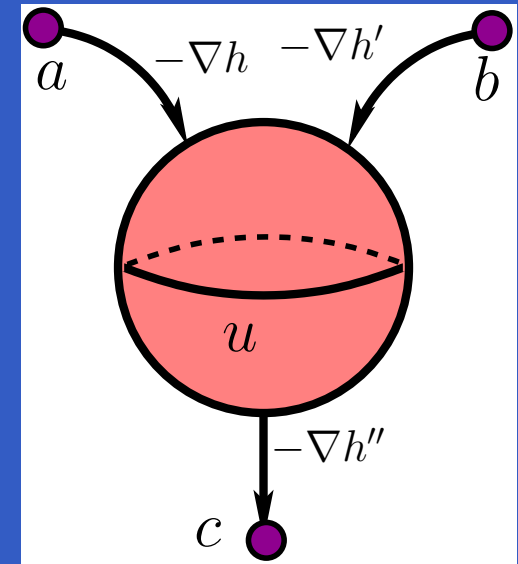
$* : QH_k \otimes QH_l \rightarrow QH_{k+l-2n}$

$a * b = \sum_{c, u} n(a, b, c) c \otimes T^{c_1(u)}$.

Quantum homology

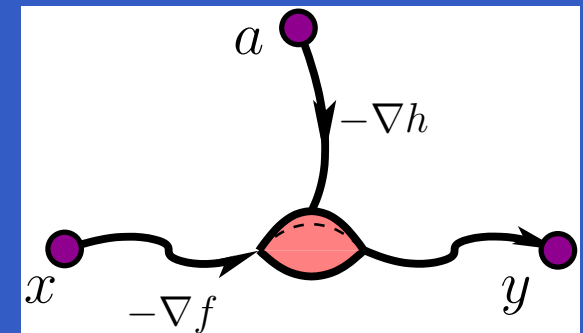
Enumerative algebraic geometry.

(Witten, Ruan, Kontsevich-Manin et al.)



Module structure

$QH_k(M) \otimes HF_l(L) \rightarrow HF_{k+l-2n}(L)$.



More structure

(M^{2n}, ω) . $QH(M) := H_*(M) \otimes \mathcal{A}$, where $\mathcal{A} = \mathbb{Z}_2[T, T^{-1}]$.

$h, h', h'' : M \rightarrow \mathbb{R}$.

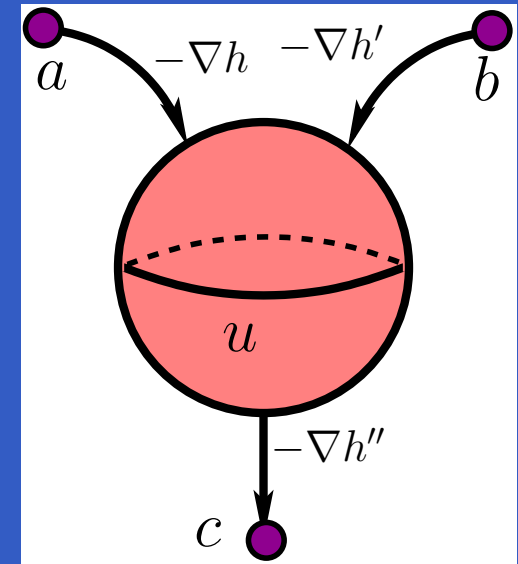
$*$: $QH_k \otimes QH_l \rightarrow QH_{k+l-2n}$

$a * b = \sum_{c, u} n(a, b, c) c \otimes T^{c_1(u)}$.

Quantum homology

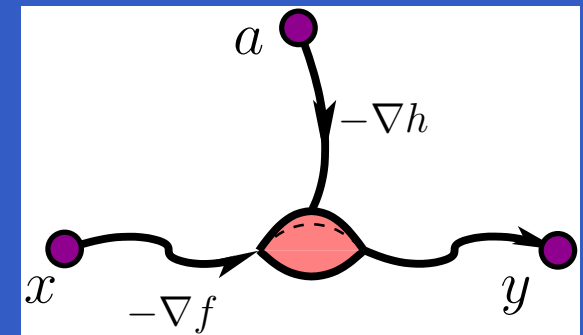
Enumerative algebraic geometry.

(Witten, Ruan, Kontsevich-Manin et al.)



Module structure

$QH_k(M) \otimes HF_l(L) \rightarrow HF_{k+l-2n}(L)$.



Thm(B.-Cornea 2006) $HF_*(L)$ is a module over $QH_*(M)$.

In fact even an algebra over QH .

What to do with all these structures ?

Topology of Lagrangians again.

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

$\Rightarrow HF_*(L) \cong HF_{*+k-2n}(L)$ (periodicity).

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

$\Rightarrow HF_*(L) \cong HF_{*+k-2n}(L)$ (periodicity).

(exp. $QH_{2n-2}(\mathbb{C}P^n)$ has such an element.)

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

$\Rightarrow HF_*(L) \cong HF_{*+k-2n}(L)$ (periodicity).

(exp. $QH_{2n-2}(\mathbb{C}P^n)$ has such an element.)

Using this possible to compute $HF \rightsquigarrow$ info on $H_*(L)$.

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

$\Rightarrow HF_*(L) \cong HF_{*+k-2n}(L)$ (periodicity).

(exp. $QH_{2n-2}(\mathbb{C}P^n)$ has such an element.)

Using this possible to compute $HF \rightsquigarrow$ info on $H_*(L)$.

Thm(B.-Cornea 2006): $Q^n =$ complex quadric, $n =$ even,
 $L \subset Q^n$, $\pi_1(L) = 1$. $\Rightarrow H_*(L) = H_*(S^n)$.

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

$\Rightarrow HF_*(L) \cong HF_{*+k-2n}(L)$ (periodicity).

(exp. $QH_{2n-2}(\mathbb{C}P^n)$ has such an element.)

Using this possible to compute $HF \rightsquigarrow$ info on $H_*(L)$.

Thm(B.-Cornea 2006): $Q^n =$ complex quadric, $n = \text{even}$,
 $L \subset Q^n$, $\pi_1(L) = 1$. $\Rightarrow H_*(L) = H_*(S^n)$.

Relative enumerative geometry.

Related to Welschinger's \mathbb{R} Real enumerative invariants

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

$\Rightarrow HF_*(L) \cong HF_{*+k-2n}(L)$ (periodicity).

(exp. $QH_{2n-2}(\mathbb{C}P^n)$ has such an element.)

Using this possible to compute $HF \rightsquigarrow$ info on $H_*(L)$.

Thm(B.-Cornea 2006): $Q^n =$ complex quadric, $n = \text{even}$,
 $L \subset Q^n$, $\pi_1(L) = 1$. $\Rightarrow H_*(L) = H_*(S^n)$.

Relative enumerative geometry.

Related to Welschinger's \mathbb{R} real enumerative invariants

$L \subset M$, generic J . $\exists J$ -hol disk through every $p \in L$?

What to do with all these structures ?

Topology of Lagrangians again.

If \exists invertible element $a \in QH_k(M)$

$\Rightarrow HF_*(L) \cong HF_{*+k-2n}(L)$ (periodicity).

(exp. $QH_{2n-2}(\mathbb{C}P^n)$ has such an element.)

Using this possible to compute $HF \rightsquigarrow$ info on $H_*(L)$.

Thm(B.-Cornea 2006): $Q^n =$ complex quadric, $n = \text{even}$,
 $L \subset Q^n$, $\pi_1(L) = 1$. $\Rightarrow H_*(L) = H_*(S^n)$.

Relative enumerative geometry.

Related to Welschinger's \mathbb{R} real enumerative invariants

$L \subset M$, generic J . $\exists J$ -hol disk through every $p \in L$?

Through every 2 points in L ? ... 3, 4 ... points?

Results in this direction

Results in this direction

Thm(Buhovski 2006): $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$ monotone.

$\Rightarrow \forall J, \forall p \in L, \exists$ a J -holomorphic disk

$u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$ with $u(\partial D) \ni p$ & $\mu(u) = 2$.

Results in this direction

Thm(Buhovski 2006): $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$ monotone.

$\Rightarrow \forall J, \forall p \in L, \exists$ a J -holomorphic disk

$u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$ with $u(\partial D) \ni p$ & $\mu(u) = 2$.

$\mu(u)$ = Maslov index of u , measures complexity of u .

Results in this direction

Thm(Buhovski 2006): $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$ monotone.

$\Rightarrow \forall J, \forall p \in L, \exists$ a J -holomorphic disk

$u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$ with $u(\partial D) \ni p$ & $\mu(u) = 2$.

$\mu(u)$ = Maslov index of u , measures complexity of u .

Monotone = Maslov is proportional to area.

$\mu(u) = 2 \Rightarrow$ small complexity \Rightarrow small area.

Results in this direction

Thm(Buhovski 2006): $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$ monotone.

$\Rightarrow \forall J, \forall p \in L, \exists$ a J -holomorphic disk

$u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$ with $u(\partial D) \ni p$ & $\mu(u) = 2$.

$\mu(u)$ = Maslov index of u , measures complexity of u .

Monotone = Maslov is proportional to area.

$\mu(u) = 2 \Rightarrow$ small complexity \Rightarrow small area.

■ Thm solves partial case of Audin's conjecture (1988)

Results in this direction

Thm(Buhovski 2006): $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$ monotone.

$\Rightarrow \forall J, \forall p \in L, \exists$ a J -holomorphic disk

$u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$ with $u(\partial D) \ni p$ & $\mu(u) = 2$.

$\mu(u)$ = Maslov index of u , measures complexity of u .

Monotone = Maslov is proportional to area.

$\mu(u) = 2 \Rightarrow$ small complexity \Rightarrow small area.

■ Thm solves partial case of Audin's conjecture (1988)

Prev results: Polterovich, Viterbo, Oh, B.-Cieliebak

Results in this direction

Thm(Buhovski 2006): $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$ monotone.

$\Rightarrow \forall J, \forall p \in L, \exists$ a J -holomorphic disk

$u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$ with $u(\partial D) \ni p$ & $\mu(u) = 2$.

$\mu(u)$ = Maslov index of u , measures complexity of u .

Monotone = Maslov is proportional to area.

$\mu(u) = 2 \Rightarrow$ small complexity \Rightarrow small area.

■ Thm solves partial case of Audin's conjecture (1988)

Prev results: Polterovich, Viterbo, Oh, B.-Cieliebak

Thm(B.-Cornea 2006): $L = \mathbb{T}^2 \subset M$ Lag. torus. Pick *any* J .

Explicit and computable conditions on existence of

J -holomorphic disk with $\mu = 4$ through 3 points

$p, p', p'' \in L$.

Results in this direction

Thm(Buhovski 2006): $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$ monotone.

$\Rightarrow \forall J, \forall p \in L, \exists$ a J -holomorphic disk

$u : (D, \partial D) \rightarrow (\mathbb{R}^{2n}, L)$ with $u(\partial D) \ni p$ & $\mu(u) = 2$.

$\mu(u)$ = Maslov index of u , measures complexity of u .

Monotone = Maslov is proportional to area.

$\mu(u) = 2 \Rightarrow$ small complexity \Rightarrow small area.

■ Thm solves partial case of Audin's conjecture (1988)

Prev results: Polterovich, Viterbo, Oh, B.-Cieliebak

Thm(B.-Cornea 2006): $L = \mathbb{T}^2 \subset M$ Lag. torus. Pick *any* J .

Explicit and computable conditions on existence of

J -holomorphic disk with $\mu = 4$ through 3 points

$p, p', p'' \in L$.

Possible to consider also mixed problems: \exists hol disk

through $p_1, \dots, p_k \in L$ and $q_1, \dots, q_r \in M$?

Main ideas

- $T^n \approx L \subset \mathbb{R}^{2n}$:

Main ideas

■ $T^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.

Main ideas

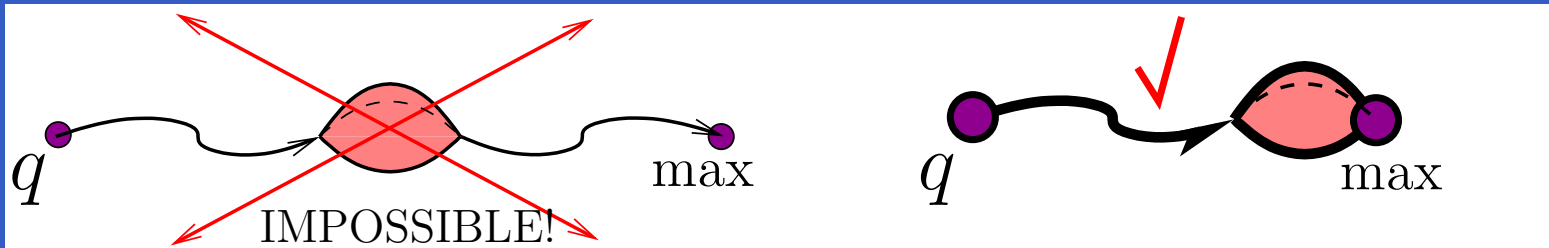
- $T^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [\max] = 0 \in HF(L)$

Main ideas

- $T^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [\max] = 0 \in HF(L) \Rightarrow \max = d(q)$.

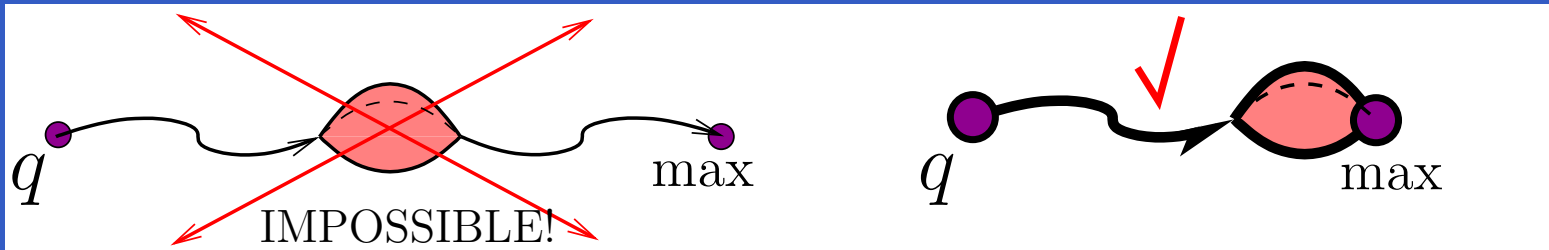
Main ideas

- $T^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [\max] = 0 \in HF(L) \Rightarrow \max = d(q)$.



Main ideas

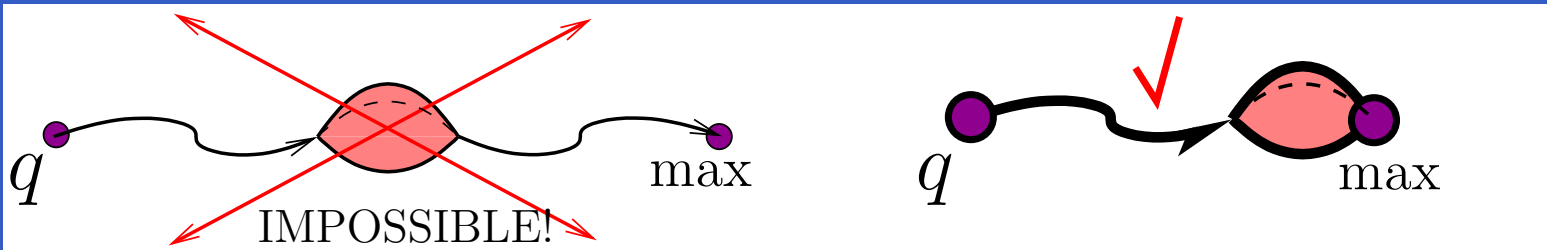
- $T^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [\max] = 0 \in HF(L) \Rightarrow \max = d(q)$.



- $T^2 \subset M$:

Main ideas

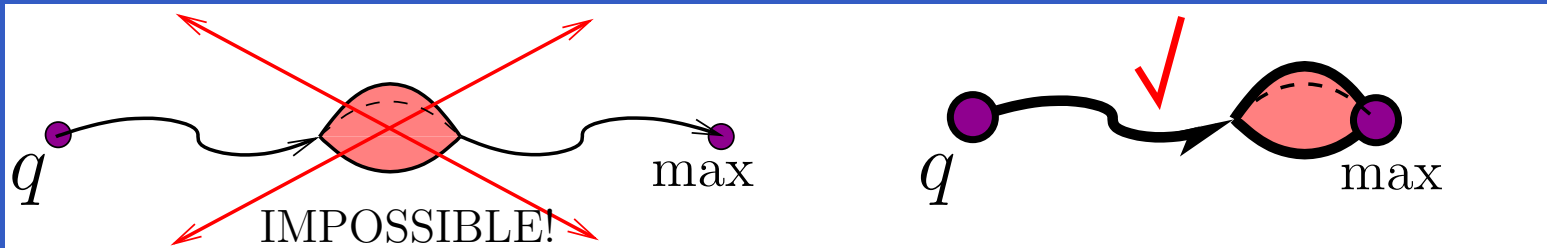
- $T^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [\max] = 0 \in HF(L) \Rightarrow \max = d(q)$.



- $T^2 \subset M$: Pick $f, f', f'' \rightarrow \mathbb{R}$ with $\min(f)$ at p , $\min(f')$ at p' , $\max(f'')$ at p'' .

Main ideas

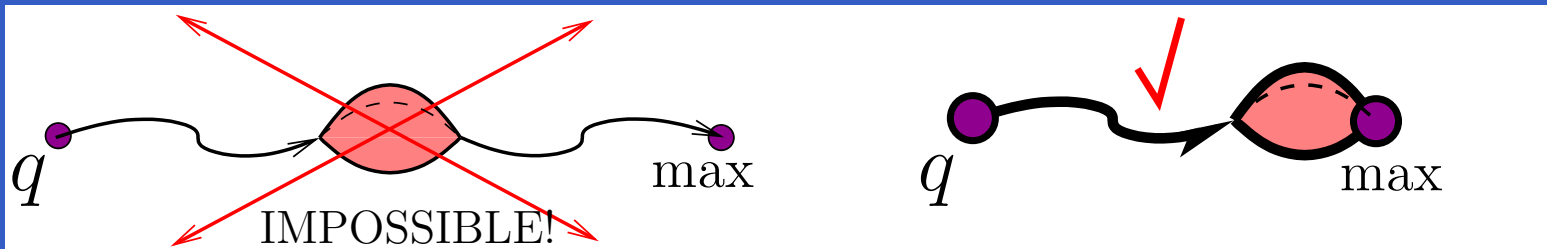
- $T^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [p] = 0 \in HF(L) \Rightarrow \max = d(q)$.



- $T^2 \subset M$: Pick $f, f', f'' \rightarrow \mathbb{R}$ with $\min(f)$ at p , $\min(f')$ at p' , $\max(f'')$ at p'' .
Computation: if $[p] \circ [p'] = \dots + [p''] + \dots$ then:

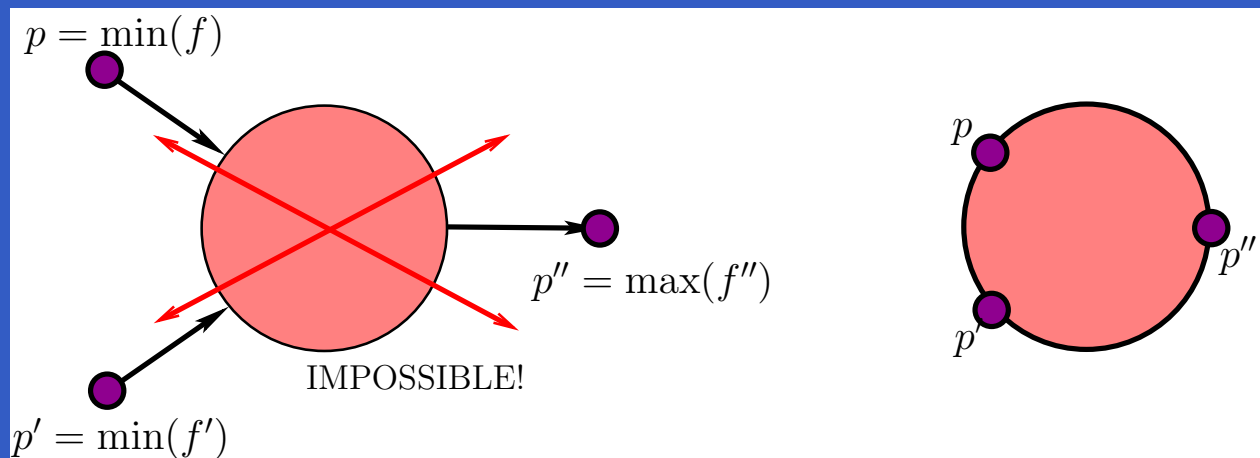
Main ideas

- $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [p] = 0 \in HF(L) \Rightarrow \max = d(q)$.



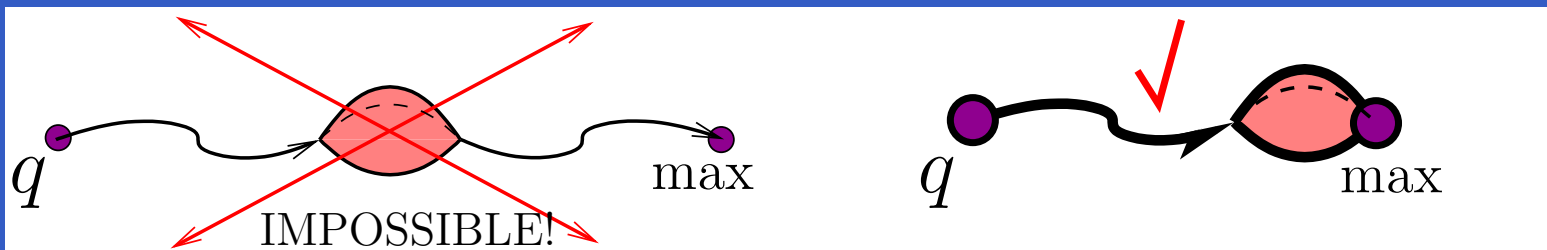
- $\mathbb{T}^2 \subset M$: Pick $f, f', f'' \rightarrow \mathbb{R}$ with $\min(f)$ at p , $\min(f')$ at p' , $\max(f'')$ at p'' .

Computation: if $[p] \circ [p'] = \dots + [p''] + \dots$ then:



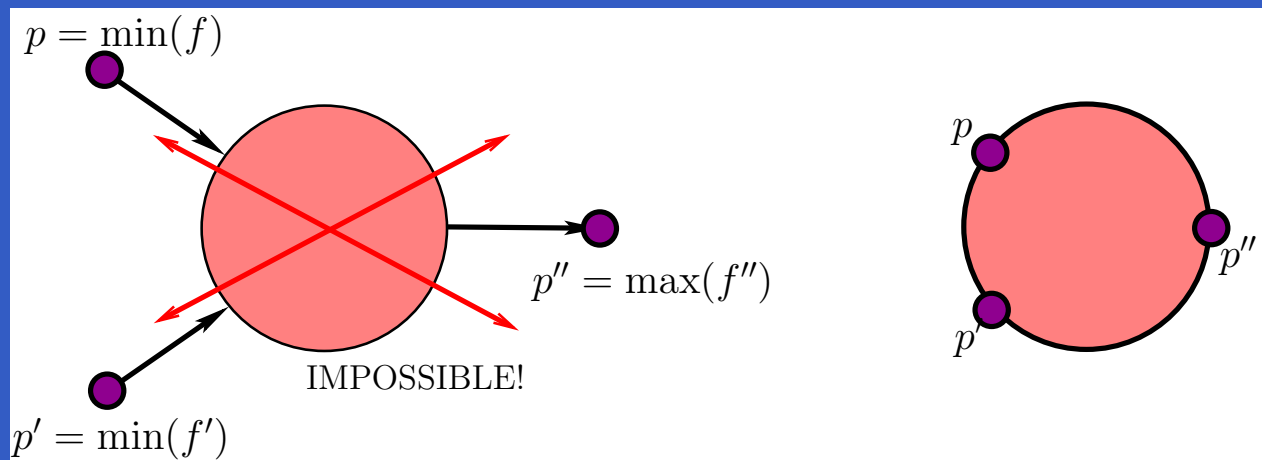
Main ideas

- $\mathbb{T}^n \approx L \subset \mathbb{R}^{2n}$: Pick $f : L \rightarrow \mathbb{R}$ with $\max = p$.
 $HF(L) = 0 \Rightarrow [p] = 0 \in HF(L) \Rightarrow \max = d(q)$.



- $\mathbb{T}^2 \subset M$: Pick $f, f', f'' \rightarrow \mathbb{R}$ with $\min(f)$ at p , $\min(f')$ at p' , $\max(f'')$ at p'' .

Computation: if $[p] \circ [p'] = \dots + [p''] + \dots$ then:



- \rightsquigarrow applications to (relative) symplectic packing.

Final remarks

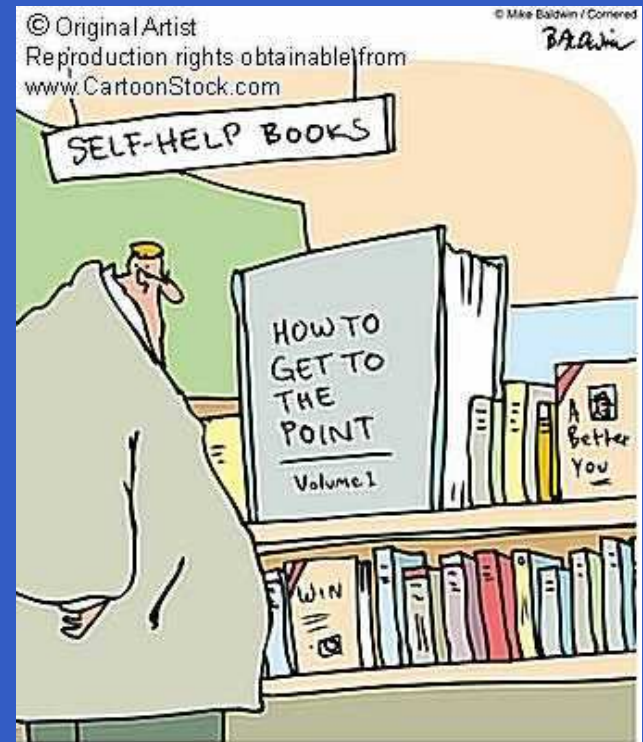
- Rich (homological) algebraic structure: A_∞ -category.
FO3=Fukaya-Oh-Ohta-Ono, Seidel.

Final remarks

- Rich (homological) algebraic structure: A_∞ -category.
FO3=Fukaya-Oh-Ohta-Ono, Seidel.
- Currently theory does not work for all Lagrangians.
Hopefully works of FO3 and Cornea-Lalonde will...

Final remarks

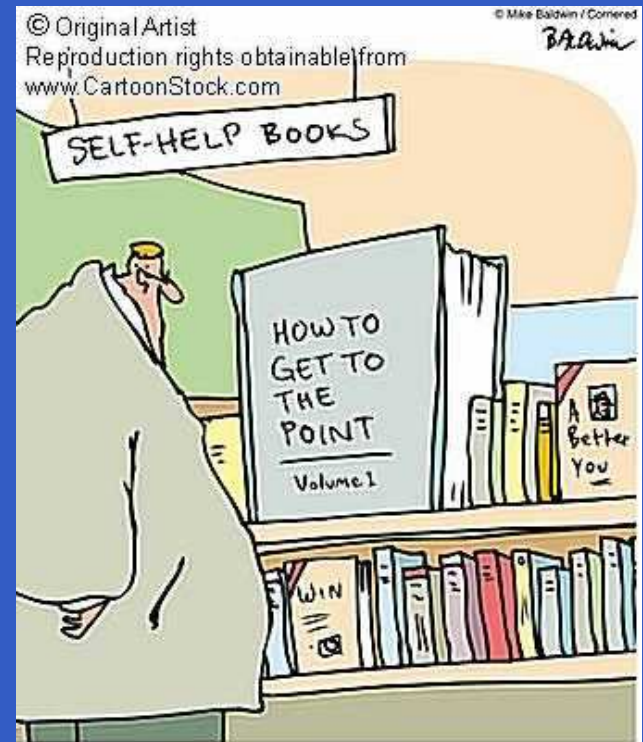
- Rich (homological) algebraic structure: A_∞ -category.
FO3=Fukaya-Oh-Ohta-Ono, Seidel.
- Currently theory does not work for all Lagrangians.
Hopefully works of FO3 and Cornea-Lalonde will...
- Another feature: # pages $\longrightarrow \infty$



Final remarks

- Rich (homological) algebraic structure: A_∞ -category.
FO3=Fukaya-Oh-Ohta-Ono, Seidel.
- Currently theory does not work for all Lagrangians.
Hopefully works of FO3 and Cornea-Lalonde will...
- Another feature: # pages $\longrightarrow \infty$

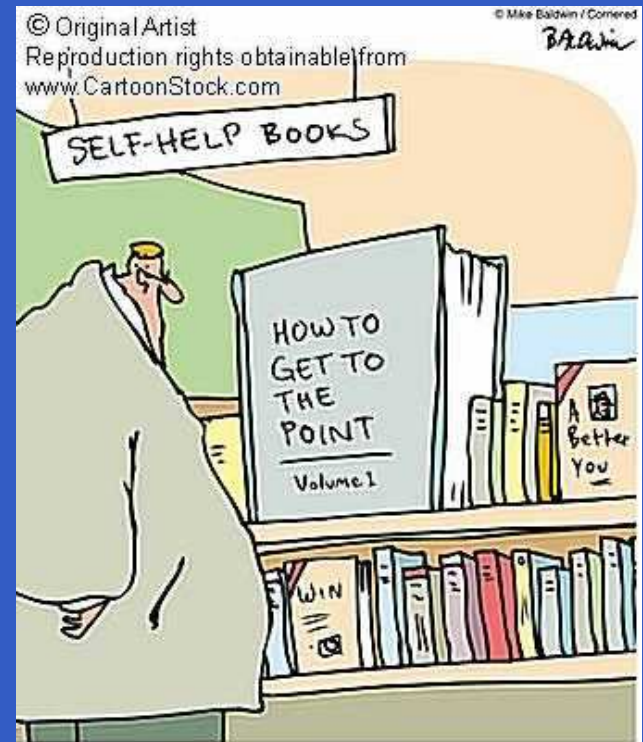
Gromov's paper: 41 pp



Final remarks

- Rich (homological) algebraic structure: A_∞ -category.
FO3=Fukaya-Oh-Ohta-Ono, Seidel.
- Currently theory does not work for all Lagrangians.
Hopefully works of FO3 and Cornea-Lalonde will...
- Another feature: # pages $\longrightarrow \infty$

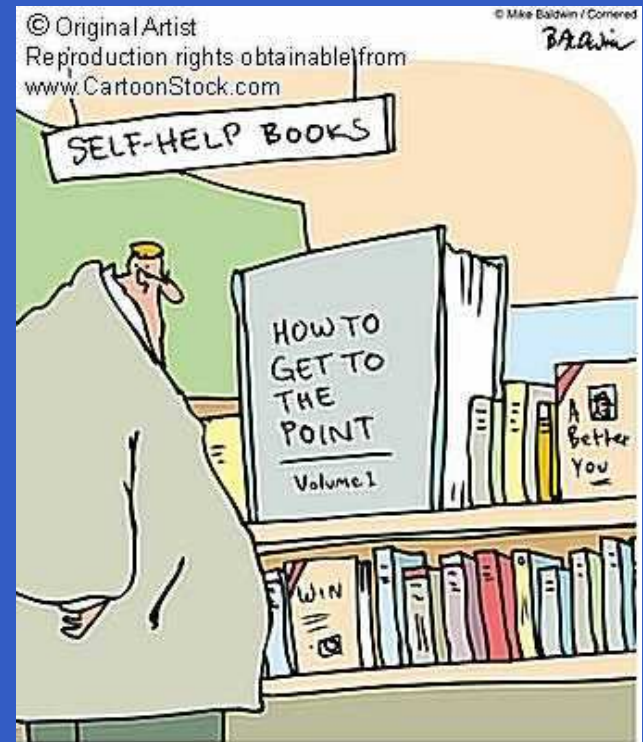
Gromov's paper: 41 pp
Cornea-Lalonde: 58 pp



Final remarks

- Rich (homological) algebraic structure: A_∞ -category.
FO3=Fukaya-Oh-Ohta-Ono, Seidel.
- Currently theory does not work for all Lagrangians.
Hopefully works of FO3 and Cornea-Lalonde will...
- Another feature: # pages $\longrightarrow \infty$

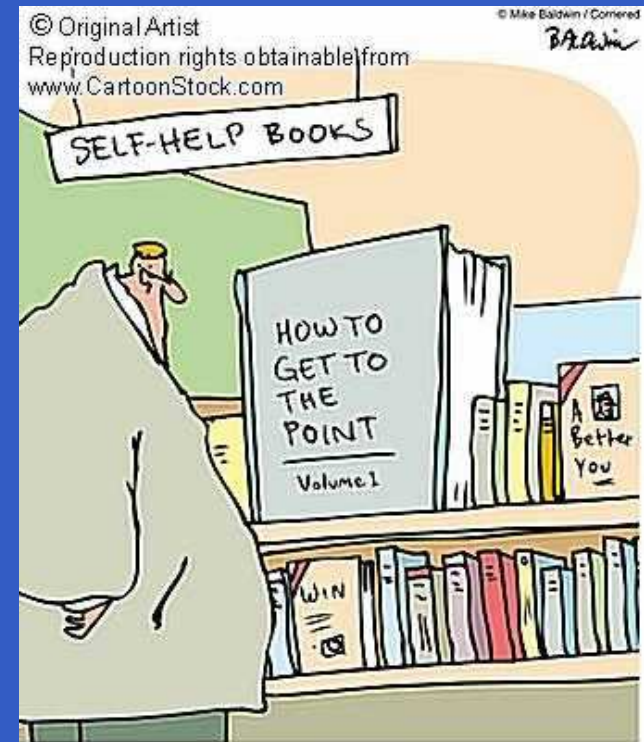
Gromov's paper: 41 pp
Cornea-Lalonde: 58 pp
B.-Cornea: \approx 200 pp



Final remarks

- Rich (homological) algebraic structure: A_∞ -category.
FO3=Fukaya-Oh-Ohta-Ono, Seidel.
- Currently theory does not work for all Lagrangians.
Hopefully works of FO3 and Cornea-Lalonde will...
- Another feature: # pages $\longrightarrow \infty$

Gromov's paper: 41 pp
Cornea-Lalonde: 58 pp
B.-Cornea: ≈ 200 pp
FO3: > 930 pp



On the other hand ...

... there is no free lunch in math !

On the other hand ...

... there is no free lunch in math !

