Invent. math. 136, 123–155 (1999)
DOI 10.1007/s002229900900

Inventiones mathematicae

A stability property of symplectic packing

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Oblatum 9-I-1998 & 1-VII-1998 / Published online: 14 January 1999

Abstract. We prove that for any closed symplectic 4-manifold (M, Ω) with $[\Omega] \in H^2(M, \mathbb{Q})$ there exists a number N_0 such that for every $N \geq N_0$, (M, Ω) admits full symplectic packing by N equal balls. We also indicate how to compute this N_0. Our approach is based on Donaldson's symplectic submanifold theorem and on tools from the framework of Taubes theory of Gromov invariants.

1. Introduction and main results

Let (M, Ω) be a closed symplectic 4-manifold and consider the following question:

Given an integer N, how much of the volume of (M, Ω) can be filled by symplectic packing with N equal balls?

By a *symplectic packing* with N equal balls we mean a symplectic embedding

$$\varphi : B(\lambda) \coprod \cdots \coprod B(\lambda) \to (M, \Omega)$$

of a disjoint union of N equal standard 4-dimensional balls of any radius λ. We say that (M, Ω) admits *full symplectic packing* by N equal balls if the volume that can be filled via such embeddings is arbitrarily close to the volume of (M, Ω).

The present paper is devoted to proving the following stability property of symplectic packing in dimension 4:

Theorem 1.A. *Let* (M, Ω) *be a closed symplectic* 4*-manifold with* $[\Omega] \in H^2(M, \mathbb{Q})$. *Then, there exists* N_0 *such that for every* $N \geq N_0$, (M, Ω) *admits full symplectic packing by* N *equal balls. In fact, if for some* $k_0 \in \mathbb{Q}$ *the Poincaré dual to* $k_0[\Omega]$ *can be represented by a symplectic submanifold of genus at least* 1, *then one can assume that* $N_0 = 2k_0^2 \operatorname{Vol}(M, \Omega)$, *where* $\operatorname{Vol}(M, \Omega) = \frac{1}{2}\int_M \Omega \wedge \Omega$.

We remark that the existence of k_0 with the property mentioned in the theorem is assured by a theorem due to Donaldson [Do].

Example. Consider $(\mathbb{C}P^2, \sigma_{\mathrm{std}})$, where σ_{std} is the standard Kähler form of $\mathbb{C}P^2$, normalized so that its integral over projective lines is 1. Since the Poincaré dual to $3[\sigma_{\mathrm{std}}]$ can be represented by a smooth algebraic cubic it follows that $(\mathbb{C}P^2, \sigma_{\mathrm{std}})$ admits full symplectic packing by N equal balls for every $N \geq 9$ (compare with [M-P] and [Bi 1]).

More interesting examples appear in the following corollary of our main theorem:

Corollary 1.B. *In each of the following cases* N_0 *is a number* (*not necessarily minimal*) *for which the relevant symplectic manifold admits full symplectic packing by* N *equal balls for every* $N \geq N_0$:

1. *Let* $S \subset \mathbb{C}P^n$ *be an irrational smooth complex projective surface of degree* d, *and let* Ω *be the restriction of the standard Kähler form of* $\mathbb{C}P^n$ *to* S. *Then for* (S, Ω) *we have* $N_0 = d$.
2. *For* $(\mathbb{T}^2 \times \mathbb{T}^2, \sigma \oplus \sigma)$ *the* 4*-dimensional symplectic split-torus, where* σ *is an area form on* $\mathbb{T}^2$, *we have* $N_0 = 2$.
3. *Let* (C_1, σ_1), (C_2, σ_2) *be* (*real*) *symplectic surfaces with* $\int_{C_1} \sigma_1 = \int_{C_2} \sigma_2 = 1$, *and let* $a, b \in \mathbb{N}$. *Then, for* $(C_1 \times C_2, a\sigma_1 \oplus b\sigma_2)$ *we have* $N_0 = 8ab$.

The proof of this corollary, more examples and sharper estimates on N_0 appear in Section 5 below.

1.1. The symplectic packing problem

Recall that a *symplectic packing* of a $2n$-dimensional symplectic manifold (M^{2n}, Ω) is a symplectic embedding of a disjoint union of N standard $2n$-dimensional closed balls into (M, Ω)

$$\varphi : B(\lambda_1) \coprod \cdots \coprod B(\lambda_N) \to (M, \Omega) \ .$$

Here, $B(\lambda_q)$ denotes the standard $2n$-dimensional closed ball of radius λ_q in $\mathbb{R}^{2n}$, endowed with the standard symplectic structure of $\mathbb{R}^{2n}$, $\omega_{\mathrm{std}} = \sum_{i=1}^n dx_i \wedge dy_i$.

Symplectic packings were studied for the first time by Gromov in [Gr]. Gromov discovered that symplectic embeddings are much more rigid than volume preserving ones. For example, in [Gr] he proved that in contrast to volume preserving packings, it is impossible to fill more than half of the volume of $\mathbb{C}P^2$ via symplectic packing with two equal balls. These rigidity phenomena are called *packing obstructions*.

This however was only the beginning of the story of symplectic packing. In [M-P], McDuff and Polterovich extended Gromov's results for packing with more than two balls and gave a complete description of all the possible packing obstructions for $\mathbb{C}P^2$ with $N \leq 9$ balls and for $N = k^2$ equal balls. In the same paper they introduced the following quantities associated to any symplectic manifold of finite volume:

$$v_N(M, \Omega) = \sup_{\lambda} \frac{\text{Vol}(\text{Image } \varphi_{\lambda})}{\text{Vol}(M, \Omega)} \quad .$$

Here, λ passes over all the positive real numbers for which there exists a symplectic packing φ_{λ} of (M, Ω) with N equal balls of radius λ. Two phenomena of different nature can be distinguished in terms of the quantities v_N:

- $v_N(M, \Omega) = 1$ *full packing.*
- $v_N(M, \Omega) < 1$ *packing obstruction.*

In [Bi 1] we detected several symplectic 4-manifolds which have a *packing stability* property in the sense that all packing obstructions disappear for large enough number of balls, that is, $v_N(M, \Omega) = 1$ for large enough N's. Our list of manifolds having this property consisted of $\mathbb{C}P^2$, ruled surfaces as well as several other Kähler surfaces (see [Bi 1] for more details). The basic problem of whether or not this phenomenon occurs for general symplectic manifolds remained open. The main goal of this paper is to prove that essentially all symplectic 4-manifolds have the preceeding stability property.

1.2. Strategy of the proof

The geometric idea behind the proof is the following. By a theorem due to Donaldson [Do] there exists k_0 such that the Poincaré dual to $k_0[\Omega]$ can be represented by a connected 2-dimensional symplectic submanifold $\Sigma \subset (M, \Omega)$. Taking k_0 to be large enough we can also assume that genus$(\Sigma) \geq 1$. Consider the following ruled surface over Σ:

$$S = \mathbb{P}(N_{\Sigma/M} \oplus \mathbb{C}) \to \Sigma \ .$$

Here, $\mathbb{P}$ stands for complex projectivization, $N_{\Sigma/M}$ is the (symplectic) normal bundle of Σ in M, viewed as a complex line bundle, and $\mathbb{C}$ denotes the trivial line bundle over Σ. Note that $S \to \Sigma$ has two distinguished sections, namely the "zero section" $Z_0 = \mathbb{P}(0 \oplus \mathbb{C})$ and the "section at infinity" $Z_\infty = \mathbb{P}(N_{\Sigma/M} \oplus 0)$.

Local considerations show that it is possible to endow S with a symplectic form for which $S \setminus Z_\infty$ can be symplectically identified with a (small) tubular neighborhood of Σ in M. The next step is to "inflate" this tubular neighborhood, that is to enlarge it, until it essentially fills the entire volume of the manifold. The main idea is that it is possible to do so in such a way that this tubular neighborhood still symplectically compactifies into a ruled surface. Having done this, the problem of symplectic packing of (M, Ω) is reduced to symplectic packing of ruled surfaces. The problem of packing irrational-ruled surfaces is much more tractable and can be solved by similar methods to [Bi 1]. Indeed, it turns out that for such ruled surfaces all packing obstructions disappear for large enough number of balls. Putting all together we conclude that the same must also hold for (M, Ω).

Here is a more elaborated, though still heuristic, argument which explains how it is possible to enlarge the tubular neighborhood of Σ in M and still compactify it into a symplectic ruled surface. Put $\Omega' = k_0 \Omega$ and endow S with a smooth family of symplectic forms $\{\omega_t\}_{0<t<1}$, which have the following properties:

- $\int_{Z_\infty} \omega_t = t(\Sigma \cdot \Sigma)$.
- $\int_{Z_0} \omega_t = \Sigma \cdot \Sigma$.
- $\int_F \omega_t = 1 - t$, where F denotes the homology class of a fiber of $S \to \Sigma$.

Next, consider the following family of Gompf fiber sums (see [Go] and [Mc-Wo] for more details on fiber sums):

$$(M_t, \Omega_t) = (M, t\Omega') \underset{\Sigma = Z_\infty}{\#} (S, \omega_t), \quad 0 < t < 1 \ . \qquad (*)$$

It is not hard to see that topologically all the manifolds M_t are diffeomorphic to M. Moreover, it is possible to find a smooth family of diffeomorphisms $f_t : M \to M_t (0 < t < 1)$ which identify Σ with Z_0 for every t.

The first important observation is that the family $\Omega'_t = f_t^* \Omega_t (0 < t < 1)$ forms an isotopy of symplectic forms on M, all lying in the cohomology class $[\Omega']$. Next, note that it is possible to choose the f_t's in such a way that for $t = t_1$ close enough to 1, (M, Ω'_{t_1}) is symplectomorphic to (M, Ω'). The reason is, roughly speaking, that when t is very close to 1, the contribution to (M_t, Ω_t) coming from (S, ω_t) becomes neglectable because it essentially consists of a disc bundle over Σ with a section (Z_0) having constant area and fibers of area which tends to 0 as $t \to 1$. On the other hand, on the rest of M_t, the form Ω_t equals to $t\Omega'$ which becomes arbitrarily close to Ω' as $t \to 1$.

The final and crucial point is that when $t \to 0$ most of the contribution to the volume of (M_t, Ω_t) comes from (S, ω_t). Thus, in order to prove that (M, Ω) admits full symplectic packing by N equal balls it is enough to show that the "(S, ω_t) part of (M, Ω_t)" admits such a packing for t's which are arbitrarily close to 0. In other words, the problem of symplectic packing of (M, Ω) can be reduced to symplectic packing of $(S \backslash Z_\infty, \omega_t)$. This can be done by essentially similar methods to those of [Bi 1] and indeed it turns out that (S, ω_t) admits full symplectic packing by N equal balls for every $N \geq N_0(t)$, where

$$N_0(t) = \frac{2\mathrm{Vol}(S, \omega_t)}{\left(\int_F \omega_t\right)^2} \quad .$$

A simple calculation shows that when $t \to 0$, $N_0(t) \to Z_0 \cdot Z_0 = \Sigma \cdot \Sigma = 2k_0^2 \mathrm{Vol}(M, \Omega)$.

The present case is somewhat more complicated, because we have to prove that $(S \backslash Z_\infty, \omega_t)$ admits such packing rather than just (S, ω_t). This is a subtle point, mainly because $Z_\infty \subset S$ usually does not occur as a pseudo-holomorphic curve for generic almost complex structures. Nevertheless, there is a way to get around this difficulty and still produce the needed symplectic packing of $(S \backslash Z_\infty)$.

This is a rough description of the geometric idea of the proof. In practice, we shall not perform explicitly a Gompf fiber sum construction, mainly because it seems hard to find a canonical identification between the M_t's and M which will be suitable for our needs. Instead, we shall always work on the same manifold M, and produce the symplectic forms Ω_t using the *inflation procedure* which was introduced by Lalonde and McDuff in [McD 5], [L-M], [La], [McD 3]. As suggested above, the idea is first to embedd into M a "small" copy of $S \backslash Z_\infty$ and then using inflation to increase the area of the fibers until its volume becomes arbitrarily close to that of (M, Ω). It turns out that this procedure and Gompf fiber sum of M with a ruled surface in fact give the same result.

The other steps of the proof will be carried out along the general lines we have just described.

1.3. Organization of the paper

The rest of the paper is organized as follows. Section 2 is devoted to various preparations towards proving our main theorem. We first extend the inflation procedure of Lalonde-McDuff to work in our situation, and then turn to developing some tools from the framework of pseudo-holomorphic curves which will be necessary in the sequel. In particular we shall prove a criterion for existence of pseudo-holomorphic curves for non-generic almost complex structures. Section 3 is central and is devoted to proving the existence of full packings of a ruled surface which do not intersect the section at infinity. Finally, in Section 4 we give the proof of the main theorem and present some examples in Section 5.

2. The inflation procedure and pseudo-holomorphic curves

This section is divided into two parts. In the first we generalize the inflation procedure of Lalonde-McDuff in order to deform a symplectic form through a family of symplectic forms in a suitable direction while keeping a given submanifold symplectic during the deformation. In the second part we develop a criterion for existence of pseudo-holomorphic curves representing several homology classes simultaneously for the same almost complex structure.

2.1. The inflation procedure

The following lemma is an extension of the inflation procedure of Lalonde and McDuff (see [McD 5, McD 3], [La], [L-M]). Here and in what follows we shall write PD for Poincaré duality.

Lemma 2.1.A (Inflation Lemma)**.** *Let $Z, C \subset (M^4, \Omega)$ be two distinct* 2-*dimensional symplectic submanifolds where Z is possibly disconnected but then assumed to consists of pairwise disjoint components. Suppose that $C \cdot C \geq 0$ and that C intersects Z transversally and positively at a finite number of points. Then, there exists a closed* 2-form ρ, *supported in an arbitrarily small neighborhood of C, with the following properties*:

1. $[\rho] = PD[C]$.
2. $\Omega_{s,t} = t\Omega + s\rho$ *is symplectic for every* $t > 0, s \geq 0$.
3. Z *is symplectic with respect to* $\Omega_{s,t}$ *for every* $t > 0, s \geq 0$.

Remark. When $Z = \emptyset$ we obtain exactly the inflation lemma of Lalonde and McDuff.

The proof of the inflation lemma is given in Section 6.1 below. We turn now to developing some essential tools from the framework of the theory of pseudo-holomorphic curves which will be needed later.

2.2. A pseudo holomorphic lemma

In order to make the inflation procedure work we have to assure that the submanifolds Z and C intersect positively. The situation that we shall encounter is the following: $Z \subset M$ will be a given symplectic submanifold, and A will be a given 2-homology class which we shall need to represent by a symplectic submanifold, C, in such a way that C intersects Z transversally and positively so that the inflation procedure would be applicable. One way to do this is to try to find an almost complex structure J for which Z is J-holomorphic and for which there exists a J-holomorphic representative of the class A. The purpose of this subsection is to establishing a general criterion for the existence of such an almost complex structure J. This type of problem can be easily handled when the homology classes $[Z]$ and A are "generic" in the sense that they both admit J-holomorphic representatives for generic J. However, the situation becomes more delicate if Z is "non-generic" in that sense. A typical example is the case when $Z \cdot Z < 0$ and $\text{genus}(Z) \geq 1$. This is precisely the case which will appear in our applications.

We shall work in the following setting: (M^4, ω) will be a closed symplectic 4-manifold and $\mathcal{J} = \mathcal{J}(M, \omega)$ will be the space of almost complex structures, J, which are *tamed by* ω, that is, $\omega(X, JX) > 0$ for every non-zero $X \in TM$.

Definition 2.2.A. *Let* $A \in H_2(M, \mathbb{Z})$ *and* $J \in \mathcal{J}$. *We say that* A *is* ***J-effective and simple*** if for a generic choice of $k(A) = \frac{A \cdot A + c_1(A)}{2} \geq 0$ *distinct points in* M, *say* $\Omega_{k(A)} = \{p_1, \ldots, p_{k(A)}\} \subset M$, *there exists a smooth, connected and reduced* (*ie non-multiply covered*) J*-holomorphic curve* $C \subset M$ *which represents the class* A *and passes through all the points of* $\Omega_{k(A)}$.

We are ready now to state our criterion.

Lemma 2.2.B. *Let (M^4, ω) be a closed symplectic 4-manifold and $Z_1, \dots, Z_n \subset (M, \omega)$ be pairwise disjoint 2-dimensional symplectic submanifolds with $Z_i \cdot Z_i < 0$ for every $1 \le i \le n$. Suppose that $A \in H_2(M, \mathbb{Z})$ satisfies the following conditions*:

1. *$A \cdot Z_i \ge K \cdot Z_i$ for every $1 \le i \le n$, where K is the canonical class of (M, ω).*
2. *A is J-effective and simple for generic $J \in \mathscr{J}$.*
3. *In case $A^2 = 0$ and $K \cdot A = 0$ assume also that A is not divisible in $H_2(M, \mathbb{Z})$, that is, A cannot be written as $A = kA'$, with $k \ge 2$ and $A' \in H_2(M, \mathbb{Z})$.*

Then, there exists an almost complex structure $J' \in \mathscr{J}$ with the following properties:

1. *$Z_1, \dots, Z_n$ are all J'-holomorphic.*
2. *There exists a smooth, connected and reduced J'-holomorphic curve $C \in M$ which represents the class A. Moreover, the curve C can be assumed to intersect all the Z_i's transversally, unless $A = [Z_j]$ for some j in which case $C = Z_j$.*

Furthermore, one can assume A to be J' effective and simple.

Remarks. 1. If we assume in addition that $g(A) < \text{genus}(Z_i)$ for every i, where $g(A) = 1 + \frac{A \cdot A - c_1(A)}{2}$, then condition 1 of the lemma can be dropped. The reason for this will become clear throughout the proof.

2. There is no need to assume that $A \cdot Z_i \ge 0$, for if $K \cdot Z_i < 0$ then the condition $Z_i \cdot Z_i < 0$ implies that Z_i is a sphere with self intersection -1. It is well known that such spheres are J-holomorphic for generic J, hence $A \cdot Z_i \ge 0$ or $A = Z_i$.

The proof of Lemma 2.2.B is rather technical and therefore postponed to Section 6.2 below. Meanwhile, let us explain the main ideas and techniques we shall use to prove it.

The idea is to choose an almost complex structure J_0 for which all the Z_i's are J_0-holomorphic, and restrict to working with the subspace $\mathscr{J}' \subset \mathscr{J} = \mathscr{J}(M, \omega)$ consisting of the almost complex structures which coincide with J_0 near $Z = Z_1 \cup \cdots \cup Z_n$, rather than work with the entire space $\mathscr{J}$. It turns out that the space $\mathscr{J}'$ is still "large" enough for a smooth and connected J'-holomorphic curve in the class A to exist for some $J' \in \mathscr{J}'$. Note that $Z = Z_1 \cup \cdots \cup Z_n$ will be automatically J'-holomorphic.

In order to show the existence of such a curve, one has first to show that for generic $J \in \mathscr{J}'$ the moduli spaces of J-curves which represent some homology classes related to A are still smooth manifolds of the expected dimension. Next, we take a suitable sequence J_n

of generic almost complex structures in $\mathscr{J}$ which converge to some generic $J' \in \mathscr{J}'$ and consider a sequence of smooth and connected J_n-holomorphic curves C_n in the class A. The main point of the proof is that it is possible to choose J_n and C_n so that the sequence C_n will converge to a smooth and connected J'-holomorphic curve. The main tool here is Gromov's compactness theorem (the "arbitrary" genus version). Indeed, a careful computation of the dimensions of the spaces of all possible occurring cusp curves shows that these dimensions are lower than the dimension of the moduli space of pseudo-holomorphic curves in the class A, and so curves in the class A persist when we pass to the limit $J' \in \mathscr{J}'$. Condition 1 in the statement of the lemma is precisely what makes possible to "lower" the dimensions of the moduli spaces of the relevant cusp curves.

A criterion for effectiveness. Lemma 2.2.B requires the class A to be J-effective and simple. Usually it is not an easy matter to verify this condition, however as we shall now see, for manifolds of *SW non-simple type* the situation improves considerably. We refer the reader to [Tau 1, Tau 2] and to [McD 4, McD 3] for more details about this class of manifolds. For the time being let us just remark that ruled surfaces as well as their blow-ups have SW non-simple type.

Let $H_2(M)$ be the torsion free part of $H_2(M, \mathbb{Z})$. We denote by $\mathscr{E} = \mathscr{E}(M, \omega)$ the set of all 2-homology classes in M which can be represented by *exceptional spheres*, that is, by symplectically embedded spheres of self intersection -1. The following criterion for a class to be effective and simple is due to McDuff [McD 3].

Theorem 2.2.C (McDuff). *Let* (M^4, ω) *be a closed symplectic* 4-manifold of *SW non-simple type*, *and* $A \in H_2(M)$. *Suppose that*:

1. $A \cdot A > 0$ *and that* $\int_A \omega > 0$.
2. $A \cdot E \geq 0$ *for every* $E \in \mathscr{E}$.

Then for large enough n and generic $J \in \mathscr{J}$ *the class nA is J-effective and simple.*

3. Symplectic packing of ruled surfaces

Let $N \to \Sigma^2$ be a symplectic rank-2 vector bundle over an oriented (real) surface Σ. By endowing N with a complex structure J, compatible with its symplectic structure, we obtain a complex line bundle, still denoted by N, whose isomorphism class is independent of the choice of J. Consider the manifold $S = \mathbb{P}(N \oplus \mathbb{C})$, where $\mathbb{C}$ is the trivial complex line bundle over Σ. We have a fibration $S \to \Sigma$ with fibers diffeomorphic to $\mathbb{C}P^1$, and henceforth we shall call S by abuse of language a ruled surface over Σ. Note that S has a natural orientation inherited from Σ and N. The ruled surface $S \to \Sigma$ has two distinguished disjoint sections, $Z_0 = \mathbb{P}(0 \oplus \mathbb{C})$ and $Z_\infty = \mathbb{P}(N \oplus 0)$ which we shall refer to as the zero section and the section at infinity.

The main result we shall need about packing of ruled surfaces is the following:

Theorem 3.A. *Let* $S = \mathbb{P}(N \oplus \mathbb{C}) \to \Sigma$ *be a ruled surface over a* (real) oriented surface Σ of *genus* ≥ 1, *and assume that* $\deg(N) > 0$. *Let* ω *be a symplectic form on* S *for which* $Z_0, Z_\infty \subset S$ *are symplectic. Then for every* $\epsilon > 0$,

$$v_N(S \backslash Z_\infty, \omega) \geq 1 - \epsilon, \quad \textit{provided that } N > \frac{2\mathrm{Vol}(S, \omega)}{\left(\int_F \omega\right)^2}(1 - \epsilon) \ ,$$

where F *is the homology class of a fiber of* $S \to \Sigma$. *In particular,*

$$v_N(S \backslash Z_\infty, \omega) = 1 \quad \textit{for every } N \geq \frac{2\mathrm{Vol}(S, \omega)}{\left(\int_F \omega\right)^2} \ .$$

The following two subsections are devoted to preparations needed for the proof of this theorem. The proof itself appears in Subsection 3.3 below.

3.1. Pseudo-holomorphic curves on blow-ups of ruled surfaces

Let $S = \mathbb{P}(N \oplus \mathbb{C}) \to \Sigma$ be a ruled surface over Σ, where $N \to \Sigma$ is a complex line bundle, and fix a symplectic form ω on S for which Z_0, Z_∞ are symplectic. Let $x_1, \ldots, x_N \in S \backslash (Z_0 \cup Z_\infty)$ be N distinct points and choose an ω-compatible almost complex structure J_0 which is integrable near the x_q's. Having fixed this, we can consider the complex blow-up $\Theta : (\tilde{S}, \tilde{J}_0) \to (S, J_0)$ of S at $x_1, \ldots, x_N$. Denoting by F the homology class of a fiber of $S \to \Sigma$ and by $E_q = [\Theta^{-1}(x_q)]$ the homology classes of the exceptional divisors over the x_q's, we have

$$H_2(\tilde{S}, \mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}[Z_0] \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_N \ .$$

Note that $[Z_\infty] = [Z_0] - \deg(N)F$.

Consider now a symplectic embedding $\varphi_0 : B(\delta) \coprod \ldots \coprod B(\delta) \to (S, \omega)$ of a disjoint union of N balls of radius δ, which takes the center of the q'th ball to x_q. Clearly, for δ small enough such an embedding exists, hence we can consider the symplectic blowing-up $\tilde{\omega}_\delta$ of ω associated to φ_0. The symplectic form $\tilde{\omega}_\delta$ lies in the cohomology class

$$[\tilde{\omega}_\delta] = [\Theta^*\omega] - \pi\delta^2(e_1 + \cdots + e_N) \ ,$$

where e_q is the Poincaré dual to E_q.

Note that by taking δ to be small enough we can assume that $Z_0, Z_\infty \subset \tilde{S}$ remain $\tilde{\omega}_\delta$-symplectic. The main result of this subsection is the following lemma:

Lemma 3.1.A. *Assume that* $\deg(N) > 0$, $\mathrm{genus}(\Sigma) \geq 1$ *and let* $A \in H_2(\tilde{S}, \mathbb{Q})$ *be a homology class which satisfies the following conditions*:

1. $A \cdot A > 0$, $\int_A \tilde{\omega}_\delta > 0$, $A \cdot Z_\infty > 0$.
2. $A \cdot E_q > 0$, $A \cdot (F - E_q) > 0$ *for every* $1 \leq q \leq N$.

Then, there exists an $\tilde{\omega}_\delta$*-tamed almost complex structure* J *which has the following properties*:

1. $Z_\infty \subset \tilde{S}$ *is* J*-holomorphic.*
2. *The exceptional divisors* $\Theta^{-1}(x_1), \ldots, \Theta^{-1}(x_N)$ *are* J*-holomorphic.*
3. *For large enough* n, *there exists a connected, smooth an reduced* J*-holomorphic curve* $C \subset \tilde{S}$ *which represents the class* nA, *and intersects all of* $\Theta^{-1}(x_1), \ldots, \Theta^{-1}(x_N)$ *and* Z_∞ *transversally and positively.*

Proof. Denote by $\mathscr{E} \subset H_2(\tilde{S}, \mathbb{Z})$ the set of all classes which can be represented by symplectic exceptional spheres. Since $\mathrm{genus}(\Sigma) \geq 1$ it is not hard to see that

$$\mathscr{E} = \{E_1, \ldots, E_N, F - E_1, \ldots, F - E_N\} \quad \text{(see [Bi 1] for more details)} \ .$$

As ruled surfaces have SW non-simple type, it follows from Theorem 2.2.C that for large enough n and generic J the class nA is J-effective and simple in the sense of Definition 2.2.A. It follows from our assumptions that by taking n to be large enough we can assume that $nA \cdot Z_\infty > K \cdot Z_\infty$. Clearly we also have that $nA \cdot E_q > K \cdot E_q$ for every $1 \leq q \leq N$. The result now follows immediately from Lemma 2.2.B with the Z_i's being the symplectic submanifolds: $Z_\infty, \Theta^{-1}(x_1), \ldots, \Theta^{-1}(x_N)$. □

3.2. Blowing down

In order to obtain symplectic embeddings of balls one needs to perform *symplectic blowing-down*. We refer the reader to [M-P] for more information on this operation. We shall work in the following setting. Let (M^4, J) be a 4-dimensional almost complex manifold with J integrable near $x_1, \ldots, x_N \in M$. We denote by $\Theta : (\tilde{M}, \tilde{J}) \to (M, J)$ the complex blowing-up of (M, J) at $x_1, \ldots, x_N$, and by $\Sigma_q = \Theta^{-1}(x_q), q = 1, \ldots, N$ the exceptional divisors. Finally, we write E_q for the homology classes of the Σ_q's and e_q for their Poincaré duals. Recall that a symplectic form Ω which tames an almost complex structure J is said to be *J-standard near $x \in M$* if the pair (Ω, J) is diffeomorphic to the standard pair (ω_{std}, i) of $\mathbb{C}^2$ near x.

In what follows we shall need to prove existence of symplectic packing of an (open) symplectic manifold $M \backslash Z$, where $Z \subset M$ is a 2-dimensional symplectic submanifold. The technical tool for obtaining this is the following blowing down proposition which is an obvious generalization of Proposition 2.1.C from [M-P].

Proposition 3.2.A. *Let (M^4, Ω) be a closed symplectic 4-manifold, and $Z \subset M$ be a 2-dimensional symplectic submanifold. Let J be an almost complex structure which is tamed by Ω and suppose that Ω is J-standard near $x_1, \ldots, x_N \in M \backslash Z$. Let $\mu_1, \ldots, \mu_N$ be positive numbers and*

$$\varphi : \coprod_{q=1}^{N} B(\mu_q) \to (M \backslash Z, \Omega)$$

be a symplectic embedding which is also (i, J)-holomorphic. Denote by $(\tilde{M}, \tilde{\Omega}_0)$ the symplectic blow-up of (M, Ω) associated with φ and by $\tilde{Z} = \Theta^{-1}(Z)$ the proper transform of Z in $\tilde{M}$. Suppose that $\tilde{\Omega}_0$ can be included into a deformation of symplectic forms $\{\tilde{\Omega}_t\}_{0 \le t \le 1}$ with the following properties:

1. *$\tilde{\Omega}_t$ lies in the cohomology class*

$$[\tilde{\Omega}_t] = [\Theta^* \Omega] - \pi \sum_{q=1}^{N} \mu_q(t)^2 e_q \quad \textit{for every } 0 \le t \le 1 \ ,$$

where $\mu_1(t), \ldots, \mu_N(t)$ are smooth functions of t with $\mu_q(0) = \mu_q$ for every $1 \le q \le N$.

2. *$\tilde{\Omega}_t$ is non-degenerate on the exceptional divisors $\Sigma_1, \ldots, \Sigma_N$ and on $\tilde{Z}$, for every $0 \le t \le 1$.*

Then $(M\backslash Z, \Omega)$ *admits a symplectic packing by* N *balls of radii* $\mu_1(1), \ldots, \mu_N(1)$.

The proof goes exactly along the same lines as the one of Proposition 2.1.C of [M-P] except of the following modification of the last step: when blowing down the family $\tilde{\Omega}_t$ one obtains an isotopy of symplectic forms Ω_t on M with $\Omega_0 = \Omega$ and a smooth family of symplectic embeddings

$$\varphi_t : \coprod_{q=1}^{N} B\big(\mu_q(t)\big) \to (M, \Omega_t) \ .$$

Since $\tilde{Z} = \Theta^{-1}(Z)$ is disjoint from $\Sigma_1, \ldots, \Sigma_N$ and $\tilde{\Omega}_t$ is non-degenerate on $\Sigma_1, \ldots, \Sigma_N$ and on $\tilde{Z}$ it easily follows from the proof in [M-P] that the φ_t's can be chosen so that their images do not intersect $Z = \Theta(\tilde{Z})$ in M, and also that Ω_t is non-degenerate on Z for every t. The proof is concluded by Moser's stability theorem for pairs rather than the usual Moser argument, namely, it follows that there exists an isotopy $F_t : M \to M$ with

$$F_0 = \mathbb{1}, \quad F_t(Z) = Z, \quad F_t^*\Omega_t = \Omega_0 = \Omega \qquad \text{for every } 0 \leq t \leq 1 \ .$$

As $(M\backslash Z, \Omega_1)$ admits a symplectic packing by N balls of radii $\mu_1(1), \ldots, \mu_N(1)$ so does also $(M\backslash Z, \Omega)$. □

3.3. Proof of Theorem 3.A

Throughout the proof we shall assume that $[\omega] \in H^2(S, \mathbb{Q})$. The case of a non-rational cohomology class can be reduced to the rational one using the same method as in the proof of Theorem 4.1.A of [Bi 1].

Fix $\epsilon > 0$ and let N be an integer as in the statement of the theorem, that is,

$$N > \frac{2\mathrm{Vol}(S, \omega)}{\left(\int_F \omega\right)^2}(1 - \epsilon) \ . \tag{1}$$

Choose an ω-tamed almost complex structure J_0 which is integrable near some N distinct points $x_1, \ldots, x_N \in S\backslash Z_\infty$ and let $\Theta : (\tilde{S}, \tilde{J}_0) \to (S, J_0)$ be the complex blow-up of S at $x_1, \ldots, x_N$. Denote by $E_q = [\Theta^{-1}(x_q)], q = 1, \ldots, N$, the homology classes of the exceptional divisors and by $e_q = \mathrm{PD}(E_q)$ their Poincaré duals. Finally, write c_1 for the first Chern class of (TS, J_0) and by $\tilde{c}_1$ for the one of $(T\tilde{S}, \tilde{J}_0)$. Clearly we have $\tilde{c}_1 = c_1 - \sum_{q=1}^{N} e_q$ under the natural decomposition $H^2(\tilde{S}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_N$.

The next step is to endow $\tilde{S}$ with an auxiliary symplectic form. For this purpose note that we may assume without loss of generality that ω is J_0-standard near the points $x_1, \ldots, x_N$. Indeed, ω is isotopic to such a J_0-standard form via an isotopy which may be assumed to be supported in any prescribed neighborhood of the x_q's (see [M-P] or [Bi 2] for more details). Let $\tilde{\omega}_\delta$ be the symplectic blow-up of ω, associated to a symplectic and holomorphic embedding $\varphi_\delta : B(\delta) \coprod \cdots \coprod B(\delta) \to (S, \omega)$ of a disjoint union of N equal balls of radius δ which sends the center of the q'th balls to x_q. The form $\tilde{\omega}_\delta$ lies in the cohomology class $[\tilde{\omega}_\delta] = [\Theta^*\omega] - \pi\delta^2 \sum_{q=1}^{N} e_q$. By taking δ to be small enough we may assume that the following holds:

- Image φ_δ is disjoint from Z_∞.
- $\Theta^{-1}(Z_\infty)$ is a symplectic submanifold of $(\tilde{S}, \tilde{\omega}_\delta)$.

To simplify notations we shall write from now on Z_∞ instead of $\Theta^{-1}(Z_\infty)$ since we have chosen $x_1, \ldots, x_N$ not to lie on $Z_\infty \subset S$.

Let $\lambda_s > 0$ be the real positive number defined by the equality

$$\pi\lambda_s^2 = \sqrt{\frac{2(1-\epsilon)\mathrm{Vol}(S, \omega) + s}{N}} \ ,$$

where $s > 0$ is chosen so that:

- $\pi\lambda_s^2 < \sqrt{2\mathrm{Vol}(S, \omega)/N}$.
- $\pi\lambda_s^2 \in \mathbb{Q}$.

Note that there are arbitrarily small choices of $s > 0$ for which these two conditions are satisfied.

Define now $A_s \in H_2(\tilde{S}, \mathbb{Q})$ to be

$$A_s = \mathrm{PD}[\Theta^*\omega] - \pi\lambda_s^2 \sum_{q=1}^{N} E_q \ .$$

The main point is that the class A_s satisfies the conditions of Lemma 3.1.A provided that δ and s are small enough. Indeed, by our assumptions and the choice of λ_s we have $A_s \cdot Z_\infty = \int_{Z_\infty} \omega > 0$ and

$$\int_{A_s} \tilde{\omega}_\delta = 2\mathrm{Vol}(S, \omega) - \pi^2\delta^2 N\lambda_s^2 > 2\mathrm{Vol}(S, \omega) - \pi\delta^2 N\sqrt{\frac{2\mathrm{Vol}(S, \omega)}{N}} \ .$$

Hence $\int_{A_s} \tilde{\omega}_\delta > 0$ whenever $\delta^2 < \frac{\sqrt{2\mathrm{Vol}(S,\omega)}}{\pi\sqrt{N}}$. Finally, $A_s \cdot A_s = 2\mathrm{Vol}(S,\omega) - N(\pi\lambda_s^2)^2 > 0$, and $A_s \cdot E_q = \pi\lambda_s^2 > 0$. It remains to check that when s is small enough, $A_s \cdot (F - E_q) \geq 0$ for every q. Indeed, by inequality (1) we have:

$$A_s(F - E_q) = \int_F \omega - \pi\lambda_s^2 \underset{s\to 0^+}{\longrightarrow} \int_F \omega - \sqrt{\frac{2(1-\epsilon)\mathrm{Vol}(S,\omega)}{N}} > 0 \ .$$

This proves that for $\delta, s > 0$ small enough the conditions of Lemma 3.1.A are satisfied for the class A_s.

Fix s and δ as above and write $A = A_s, \lambda = \lambda_s$. By Lemma 3.2 there exists $n \in \mathbb{N}$ and an almost complex structure J, tamed by $\tilde{\omega}_\delta$, and a connected 2-dimensional submanifold $C \subset \tilde{S}$ with $[C] = nA$, such that C, Z_∞ and $\Theta^{-1}(x_1), \ldots, \Theta^{-1}(x_N)$ are all J-holomorphic and such that C intersects Z_∞ and the $\Theta^{-1}(x_q)$'s transversally and positively. It now follows from the inflation lemma (Lemma 2.1.A) that there exists a closed 2-form ρ with the following properties:

- $[\rho] = \mathrm{PD}([C]) = \mathrm{PD}(nA)$.
- $\tilde{\Omega}_t = (1-t)\tilde{\omega}_\delta + \frac{t}{n}\rho$ is symplectic for all $0 \leq t < 1$.
- Z_∞ and $\Theta^{-1}(x_1), \ldots, \Theta^{-1}(x_N)$ are symplectic with respect to $\tilde{\Omega}_t$ for every $0 \leq t < 1$.

Note that $\tilde{\Omega}_0 = \tilde{\omega}_\delta$ and that

$$[\tilde{\Omega}_t] = [\Theta^*\omega] - \pi\big(t\lambda^2 + (1-t)\delta^2\big)\sum_{q=1}^{N} e_q \ ,$$

hence,

$$\begin{aligned}\int_{\tilde{S}} \tilde{\Omega}_t \wedge \tilde{\Omega}_t \ &= \ 2\,\mathrm{Vol}(S,\omega) - N\pi^2\big(t\lambda^2 + (1-t)\delta^2\big)^2 \\ &\underset{t\to 1^-}{\longrightarrow} 2\,\mathrm{Vol}(S,\omega) - N(\pi\lambda^2)^2 < 2\epsilon\,\mathrm{Vol}(S,\omega) \ .\end{aligned}$$

Choose $t = t_1 < 1$ close enough to 1 so that $\int_{\tilde{S}} \tilde{\Omega}_{t_1} \wedge \tilde{\Omega}_{t_1} < 2\epsilon$ $\mathrm{Vol}(S,\omega)$. Applying Proposition 3.2.A to the family $\{\tilde{\Omega}_t\}_{0\leq t\leq t_1}$ we see that $(S\backslash Z_\infty, \omega)$ admits a symplectic packing by N equal balls, say φ, such that

$$\mathrm{Vol}(S\backslash \mathrm{Image}\ \varphi, \omega) = \frac{1}{2}\int_{\tilde{S}} \tilde{\Omega}_{t_1} \wedge \tilde{\Omega}_{t_1} \ .$$

This implies that

$$v_N(S \backslash Z_\infty, \omega) \geq 1 - \frac{\frac{1}{2}\int_{\tilde{S}} \tilde{\Omega}_{t_1} \wedge \tilde{\Omega}_{t_1}}{\mathrm{Vol}(S,\omega)} \geq 1 - \epsilon \ .$$

The first statement of the theorem is proved.

The second statement follows immediately from the first one. □

4. Proof of the main theorem

As explained in the introduction the first step towards the proof is to embed into (M,Ω) a (small) symplectic disc-bundle which can be symplectically compactified into a ruled surface.

Given a 2-dimensional symplectic submanifold $\Sigma \subset (M,\Omega)$, the normal bundle of Σ in M, $N_{\Sigma/M} \to \Sigma$, is a symplectic rank-2 vector bundle and we can consider the ruled surface $S = \mathbb{P}(N_{\Sigma/M} \oplus \mathbb{C})$ over Σ. Here and below we shall use the notations of Section 3.

Proposition 4.A. *Let* (M^4,Ω) *be a symplectic 4-manifold and* $\Sigma \subset M$ *be a* 2-dimensional symplectic submanifold with $\Sigma \cdot \Sigma \geq 0$. *Then* $S = \mathbb{P}(N_{\Sigma/M} \oplus \mathbb{C})$ *admits a symplectic form* ω_0 *with the following properties*:

1. $Z_0, Z_\infty \subset (S,\omega_0)$ *are symplectic submanifolds.*
2. *There exists a symplectic embedding*

$$\phi_0 : (S \backslash Z_\infty, \omega_0) \to (M,\Omega) \quad \textit{with } \phi_0(Z_0) = \Sigma \ .$$

We defer the proof of the proposition to Section 7, and turn to the proof of the main theorem.

Proof of the main theorem. By a theorem of Donaldson [Do] there exists $k_0 > 0$ such that the Poincaré dual to $[k_0\Omega]$ can be represented by a connected 2-dimensional symplectic submanifold, say $\Sigma \subset (M,\Omega)$, of genus at least 1. Put $\Omega' = k_0\Omega$ and $N_0 = \Sigma \cdot \Sigma = 2k_0^2\mathrm{Vol}(M,\Omega)$.

Consider the ruled surface $S = \mathbb{P}(N_{\Sigma/M} \oplus \mathbb{C}) \to \Sigma$ and let Z_0, Z_∞ be its zero section and the section at infinity. By Proposition 4.A there exists a symplectic form ω_0 on S for which Z_0, Z_∞ are symplectic submanifolds and for which there exists a symplectic embedding

$$\phi_0 : (S \backslash Z_\infty, \omega_0) \to (M,\Omega) \quad \text{with } \phi_0(Z_0) = \Sigma \ .$$

Since $Z_0 \cdot Z_0 > 0$, Lemma 2.1.A (this time with $Z = \emptyset$) implies that there exists a closed 2-form ρ_0' on S such that $[\rho_0'] = \mathrm{PD}([Z_0])$ and $t\omega_0 + s\rho_0'$ is symplectic for every $t > 0, s \geq 0$. Moreover, ρ_0' can be assumed to have support in an arbitrarily small neighborhood of Z_0 in S.

Put $\rho_0 = \frac{1}{k_0}\rho_0'$ and define a (closed) 2-form ρ on M by pushing forward ρ_0 via ϕ_0 and extending by zero to the rest of M. More precisely, put

$$\rho = \begin{cases} 0 & \text{on } M\backslash \text{Image } \phi_0 \\ (\phi_0^{-1})^* \rho_0 & \text{on Image } \phi_0 \end{cases}$$

Clearly $[\rho] = \mathrm{PD}(\frac{1}{k_0}[\Sigma]) = [\Omega]$.

By construction, $t\Omega + s\rho$ is symplectic for every $s \geq 0, t > 0$. Consider now the following family of symplectic forms on M:

$$\Omega_t = (1-t)\Omega + t\rho \qquad 0 \leq t < 1 \ .$$

Clearly $\{\Omega_t\}_{0\leq t<1}$ is an isotopy of symplectic forms, all lying in the cohomology class $[\Omega]$. As $\Omega_0 = \Omega$, it is enough to prove the existence of the wanted packing for one of the Ω_t's.

Writing $\omega_t = (1-t)\omega_0 + t\rho_0$ we obtain that the embedding $\phi_0 : S\backslash Z_\infty \to M$ satisfies $\phi_0^*\Omega_t = \omega_t$ for every $0 \leq t < 1$. Next, note that

$$\lim_{t\to 1^-} \mathrm{Vol}(\text{Image } \phi_0, \Omega_t) = \mathrm{Vol}(M, \Omega)$$

because ρ is supported inside Image ϕ_0.

Fix $\epsilon > 0$. By Theorem 3.A we have

$$v_N(S\backslash Z_\infty, \omega_t) \geq 1 - \epsilon \quad \text{for every } N > \frac{2\mathrm{Vol}(S, \omega_t)}{\left(\int_F \omega_t\right)^2}(1-\epsilon) \ . \qquad (1)$$

Denoting by v_t the right hand side of (1) we have

$$\lim_{t\to 1^-} v_t = \frac{\int_S \rho_0 \wedge \rho_0}{\left(\int_F \rho_0\right)^2}(1-\epsilon) = 2k_0^2 \ \mathrm{Vol}(M, \Omega)(1-\epsilon) = N_0(1-\epsilon) \ .$$

Choose $t_1 < 1$ close enough to 1 so that the following holds:

- $\mathrm{Vol}(\text{Image } \phi_0, \Omega_{t_1}) \geq (1-\epsilon)\mathrm{Vol}(M, \Omega)$.
- $N_0 > v_{t_1}$.

Now we have that for *every* $N \geq N_0$,

$$v_N(M, \Omega) = v_N(M, \Omega_{t_1}) \geq v_N(S\backslash Z_\infty, \omega_{t_1}) \frac{\mathrm{Vol}(S\backslash Z_\infty, \omega_{t_1})}{\mathrm{Vol}(M, \Omega_{t_1})} \geq (1-\epsilon)^2 \ .$$

This inequality holds for every $N \geq N_0$ and $\epsilon > 0$, hence $v_N(M, \Omega) = 1$ for every $N \geq N_0$. □

5. Examples

We begin with

Proof of Corollary 1.B. 1. This follows immediately from Theorem 1.A.

2. Consider a Riemann surface C of genus 2 and let $J(C) = \mathrm{Pic}^0(C)$ be its Jacobian. Let Ω be the principle polarizing Kähler form, induced by the cup product $H^1(C, \mathbb{Z}) \otimes H^1(C, \mathbb{Z}) \to \mathbb{Z}$.

Without loss of generality we may assume that $\int_{\mathbb{T}^2} \sigma = 1$ and so $\sigma \oplus \sigma$ also determines a principle polarization on $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ for some complex structure. As the moduli space of principle polarized tori of a given dimension is connected, it is not hard to see that $(\mathbb{T}^2 \times \mathbb{T}^2, \sigma \oplus \sigma)$ is symplectomorphic to $(J(C), \Omega)$. We shall show now that $(J(C), \Omega)$ admits full symplectic packing by N equal balls for every $N \geq 2$.

To see this, fix $p_0 \in C$ and consider the Abel map $C \ni p \mapsto [p - p_0] \in J(C)$. In our situation this map is a holomorphic embedding and its image is homologous to the theta divisor of $J(C)$ (see [G-H] or [Sh]). Thus, the Poincaré dual to $[\Omega] \in H^2(J(C), \mathbb{Z})$ can be represented by a genus 2 smooth holomorphic curve and so we can take $k_0 = 1$ in Theorem 1.A. As $\mathrm{Vol}(J(C), \Omega) = 1$, we have $N_0 = 2$.

3. For each $i = 1, 2$ choose a complex structure j_i on C_i with the following properties:

- j_i is tamed by σ_i.
- There exists a meromorphic function $f_i : (C_i, j_i) \to \mathbb{C}P^1$ with exactly two (distinct) poles, say $p_i, q_i \in C_i$.

In other words, we choose j_i so that (C_i, j_i) is hyperelliptic.

Denote by $\pi_i : C_1 \times C_2 \to C_i$ the projection on C_i, and let

$$D = a\pi_1^*(p_1 + q_1) + b\pi_2^*(p_2 + q_2) \in \mathrm{Div}(C_1 \times C_2)\ .$$

Consider the linear system $|D|$. It is not hard to see that the linear system $|D|$ has the following properties:

- $|D|$ is free of base points.
- $\dim |D| \geq 2$.
- The holomorphic map $\iota_D : C_1 \times C_2 \to \mathbb{C}P^{\dim |D|}$ associated to the linear system $|D|$ has a 2-dimensional image (in fact, ι_D is generically one one-to-one).

It follows from Bertini's theorem that there exists an irreducible and smooth curve $\Sigma \in |D|$. Next we claim that $\text{genus}(\Sigma) \geq 1$. To see this note that since the projection of Σ to both C_1 and C_2 consists of more than a point, then Σ cannot be rational unless $C_1 = C_2 = S^2$. In this case however, it easily follows from the adjunction formula that $\text{genus}(\Sigma) = (2a-1)(2b-1) \geq 1$.

Clearly $\text{PD}([\Sigma]) = 2[a\sigma_1 \oplus b\sigma_2]$, hence by Theorem 1.A we obtain that $(C_1 \times C_2, \sigma_1 \oplus \sigma_2)$ admits full symplectic packing by N equal balls for every $N \geq \Sigma \cdot \Sigma = 8ab$. □

Remark. Similar arguments to the above imply the following refinement of the third statement of Corollary 1.B: *Same notations and assumptions as in* 3 *of Corollary* 1.B, *except that now assume that the integers* a, b *are at least* 2. *Then we have* $N_0 = 2ab$.

Manifolds which admit full packing by any number of balls. In [Bi 1] we detected several symplectic 4-manifolds, such as hyperelliptic (Kähler) surfaces, which admit full symplectic packing by N equal balls for every $N \geq 1$. This seems at first glance as a contradiction to a theorem of Gromov [Gr] which implies that it is impossible to fill more than half of the volume of the standard 4-dimensional ball via packing with two equal balls. A more careful consideration shows that the two results are not contradictory. In fact, the following interesting phenomenon occurs. Let (M, Ω) be a closed symplectic 4-manifold which admits full symplectic packing by N equal balls for every $N \geq 1$. Fix any λ_0 such that

$$\tfrac{1}{2}\,\text{Vol}(M, \Omega) < 2\text{Vol}\, B(\lambda_0) < \text{Vol}(M, \Omega) \ .$$

By our assumption on (M, Ω) there exists a symplectic packing

$$\varphi_0 : \ B(\lambda_0) \coprod B(\lambda_0) \to (M, \Omega) \ .$$

Next, consider an increasing sequence r_n with $\lim_{n\to\infty} \text{Vol}\ B(r_n) = \text{Vol}(M, \Omega)$. By our assumptions, for each n there exist a symplectic embedding $\phi_n : B(r_n) \to (M, \Omega)$. However, despite the fact that $\lim_{n\to\infty} \text{Vol}(\text{Image}\ \phi_n) = \text{Vol}(M, \Omega)$, Gromov's theorem implies that for any choice of the ϕ_n we will always have

$$\text{Image}\ \phi_n \not\supseteq \text{Image}\ \varphi_0 \ ,$$

that is, the image of the two balls $B(\lambda_0) \coprod B(\lambda_0)$ will always go out of the images of all the balls $B(r_n)$ no matter how tight the packings ϕ_n are.

Our list of manifolds from [Bi 1] that admit this sort of behavior consisted of *hyperelliptic surfaces*, and the surfaces of *Enriques, Dolgachev* and *Barlow*, all viewed as Kähler surfaces. Using the techniques of the present paper one can easily produce more, even simpler, examples. Here is a typical one.

Example. Let $C \subset \mathbb{C}P^2$ be an algebraic smooth curve of degree $n \geq 3$ (hence, of genus at least 1). Choose $n^2 - 1$ distinct points $p_1, \dots, p_{n^2-1} \in C$ and let $\Theta : (V, J) \to \mathbb{C}P^2$ be the (algebraic) blow-up of $\mathbb{C}P^2$ at these points, where J denotes the (integrable) complex structure on the blow-up, induced from $\mathbb{C}P^2$. Denote by $L \in H_2(V, \mathbb{Z})$ the homology class of (the proper transform of) a projective line which does not pass through any of the p_q's and by $E_1, \dots, E_{n^2-1}$ the homology classes of the exceptional divisors over $p_1, \dots, p_{n^2-1}$, respectively.

Consider $\bar{C}$, the proper transform of C in V. Clearly

$$[\bar{C}] = nL - \sum_{q=1}^{n^2-1} E_q \ .$$

It follows from a Theorem of Nagata [N] and from the Nakai-Moishezon criterion that the cohomology class $\mathrm{PD}[\bar{C}] \in H^2(V, \mathbb{Z})$ admits a J-Kähler representative, say Ω.

As $\bar{C} \cdot \bar{C} = 1$ and $\mathrm{genus}(\bar{C}) \geq 1$ it follows from Theorem 1.A that (V, Ω) admits full symplectic packing by N equal balls for every $N \geq 1$.

6. Proofs of the inflation and pseudo-holomorphic lemmas

We begin with proving the inflation lemma (Lemma 2.1.A)

6.1. Proof of the inflation lemma

The proof is a modification of the ones appearing in [McD 3, McD 5].

Suppose that $C \cap Z = \{p_1, \dots, p_n\}$. Choose a smooth function $f : C \to \mathbb{R}$ which has the following properties:

1. $f \geq 0$.
2. f vanishes on (disjoint) neighborhoods, say $\mathfrak{V}(p_1), \dots, \mathfrak{V}(p_n)$, of the points $p_1, \dots, p_n$.
3. $\int_C f \omega_C = C \cdot C$, where $\omega_C = \Omega|_{TC}$.

Consider the normal symplectic bundle of C in M, $\pi_C : N \to C$. This is a symplectic rank-2 vector bundle and so can be viewed as a complex line bundle over C. Choose a Hermitian metric on N and let γ be a connection on the unit circle bundle $P \subset N$, which satisfies

$$d\gamma = -(\pi_C|_P)^*(f\omega_C) \ .$$

Denote by $r : N \to \mathbb{R}$ the radial distance function and for every $d \geq 0$ let $\mathfrak{U}(d)$ be the sub disc-bundle of radius d, namely, $\mathfrak{U}(d) = \{x \in N | r(x) < d\}$. Consider now the following closed 2-form on N:

$$\omega = \pi_C^*\omega_C + d(\pi r^2 \gamma) \ .$$

As in [McD 3, McD 5], it is not hard to see that ω is well defined, non-degenerate near the zero section and restricts to ω_C on the zero section. Here and henceforth we shall identify C with the zero section of $N \to C$.

It follows from the symplectic neighborhood theorem that there exist neighborhoods $\mathfrak{U}_1$ of C in M, and $\mathfrak{U}_2$ of C in N and a symplectomorphism $F : (\mathfrak{U}_1, \Omega) \to (\mathfrak{U}_2, \omega)$ which is the identity on C (ie F takes C to C and $F|_C = \mathbb{1}$). Thus, without loss of generality we may replace Ω by ω and Z by $F(Z \cap \mathfrak{U}_1)$ and define the needed form ρ on $\mathfrak{U}_2$ in such a way that it is compactly supported inside a small neighborhood of C in $\mathfrak{U}_2$. To simplify notations, we continue to write Z for $F(Z \cap \mathfrak{U}_1)$.

The first step towards constructing the form ρ is the observation that there exists a neighborhood $\mathfrak{U}'$ of C in $\mathfrak{U}_2$ such that for every $a, b \geq 0$ the 2-form

$$\omega - ad\gamma + brdr \wedge \gamma \qquad \text{is non-degenerate on } \mathfrak{U}' \ . \tag{1}$$

Note that the form $rdr \wedge \gamma$ extends to the zero section. To prove the existence of the neighborhood $\mathfrak{U}'$, we use the following computation:

$$\begin{aligned} \omega - ad\gamma + brdr \wedge \gamma = {} & (1 - \pi r^2 f \circ \pi_C + af \circ \pi_C)\pi_C^*\omega_C \\ & + (2\pi + b)rdr \wedge \gamma \ . \end{aligned} \tag{$*$}$$

Now, choose a positive number r_0 which satisfies $\pi r_0^2 \max_{x \in C} f(x) < 1$, and take $\mathfrak{U}'$ to be any neighborhood of C in $\mathfrak{U}_2$ which lies within radial distance smaller than r_0, say $\mathfrak{U}' = \mathfrak{U}(r_0/2)$. It immediately follows that the forms in $(*)$ are non-degenerate in $\mathfrak{U}'$ for every $a, b \geq 0$.

Next, consider the 2-form $\eta = rdr \wedge \gamma$. We claim that there exist neighborhoods $\mathfrak{W}_1, \ldots, \mathfrak{W}_n \subset Z$ of $p_1, \ldots, p_n$ in Z such that for every $1 \leq i \leq n$, $\eta|_{T\mathfrak{W}_i}$ is an area form which gives the same orientation on $T\mathfrak{W}_i$ as $\omega|_{T\mathfrak{W}_i}$.

Indeed, it is enough to check this for the points $p_1, \dots, p_n$, namely to show that $\eta|_{T_{p_i}Z}$ is non-degenerate and gives the same orientation on $T_{p_i}Z$ as $\omega|_{T_{p_i}Z}$. To see this, first note that $T_{p_i}\mathfrak{U}' = T_{p_i}C \oplus N_{p_i}$, where N_{p_i} is the fiber of N over p_i. Let pr be the projection $T_{p_i}\mathfrak{U}' \to N_{p_i}$. Each of the three spaces $T_{p_i}C, N_{p_i}$ and $T_{p_i}Z$, being symplectic with respect to ω, inherits an orientation. Since N_{p_i} intersects positively $T_{p_i}C$ and, by assumption, $T_{p_i}Z$ intersects positively and transversally $T_{p_i}C$ it is not hard to see that $pr|_{T_{p_i}Z} : T_{p_i}Z \to N_{p_i}$ is an orientation preserving isomorphism. It easily follows now that $\eta|_{T_{p_i}Z}$ is non-degenerate and that the orientation induced by it on $T_{p_i}Z$ agrees with the one induced by $\omega|_{T_{p_i}Z}$.

Now we are ready to construct the form ρ. Choose $\epsilon > 0$ small enough so that:

1. $\mathfrak{U}(\epsilon) \cap Z \subset \mathfrak{W}_1 \cup \cdots \cup \mathfrak{W}_n$.
2. $\pi_C(\mathfrak{U}(\epsilon) \cap Z) \subset \mathfrak{V}(p_1) \cup \cdots \cup \mathfrak{V}(p_n)$.
3. $\mathfrak{U}(\epsilon) \subset \mathfrak{U}'$.

Let $g(r)$ be a non-negative and non-increasing smooth function of r which equals to $1 - \pi r^2$ near $r = 0$ and vanishes for $r \geq \epsilon/2$. Finally, let ρ be the form

$$\rho = -d\big(g(r)\gamma\big) \ .$$

Notice that ρ is compactly supported in $\mathfrak{U}(\epsilon) \subset \mathfrak{U}_2$.

Let us check now that ρ indeed has the needed properties. The first property is that for $t > 0, s \geq 0$ the forms $\omega_{s,t} = t\omega + s\rho$ are non-degenerate. This follows from (1) above and from our choices of $\mathfrak{U}(\epsilon)$ and $\mathfrak{U}'$, because

$$\omega_{s,t} = t\Big(\omega - \frac{s}{t}\, g'(r)dr \wedge \gamma - \frac{s}{t}\, g(r)d\gamma\Big), \quad g' \leq 0 \ ,$$

and g is supported inside $\mathfrak{U}(\epsilon) \subset \mathfrak{U}'$.

It remains to show that Z remains symplectic with respect to $\omega_{s,t}$. To see this it suffices to show that ρ restricts to an ω-semi-positive form on Z, namely that there exists a non-negative function h on Z with $\rho|_{TZ} = h\omega|_{TZ}$. Indeed we have $\rho = -g'(r)dr \wedge \gamma - g(r)d\gamma$. Now $d\gamma = -\pi_C^*(f\omega_C)$ and by construction this identically vanishes on $Z \cap \mathfrak{U}(\epsilon)$. Thus,

$$\rho|_{TZ} = \big(-g'(r)dr \wedge \gamma\big)|_{TZ} = -\frac{g'(r)}{r}\eta|_{TZ} \ .$$

As $g'(r) \leq 0$, our claim follows from what we have just proved regarding the orientations induced by $\eta|_{TZ}$ and $\omega|_{TZ}$, and the fact that

$$(\mathrm{supp}\ \rho) \cap Z \subset \mathfrak{U}(\epsilon) \cap Z \subset \mathfrak{W}_1 \cup \cdots \cup \mathfrak{W}_n.$$

□

6.2. Proof of the pseudo-holomorphic lemma (2.2.B)

In order to simplify notations we shall give the proof for the case $n = 1$, that is, assume that we have only one submanifold $Z = Z_1$. The proof of the general case is essentially the same.

Fix an almost complex structure $J_0 \in \mathcal{J}$ for which Z is J_0-holomorphic and let $\mathcal{J}' \subset \mathcal{J}$ be the space of all $J \in \mathcal{J}$ which coincide with J_0 near Z. Finally, let $\mathfrak{U} \subset M$ be a tubular neighborhood of Z in M.

Step 1. We claim that if $S \subset M$ is a J-holomorphic curve for some $J \in \mathcal{J}'$ then either $S = Z$ or $S \not\subseteq \mathfrak{U}$.

Indeed, suppose that $S \subset \mathfrak{U}$ and let $[S]_{\mathfrak{U}}, [Z]_{\mathfrak{U}} \in H_2(\mathfrak{U}, \mathbb{Z})$ be the homology classes of S and Z, respectively. As $H_2(\mathfrak{U}, \mathbb{Z}) = H_2(Z, \mathbb{Z})$ and the latter is 1-dimensional, $[S]_{\mathfrak{U}}$ and $[Z]_{\mathfrak{U}}$ must be proportional, say $[S]_{\mathfrak{U}} = a[Z]_{\mathfrak{U}}$ for some a. Since Z, S are both symplectic we must have $a > 0$. This implies that $[S]_{\mathfrak{U}} \cdot [Z]_{\mathfrak{U}} = a[Z]_{\mathfrak{U}} \cdot [Z]_{\mathfrak{U}} = aZ \cdot Z < 0$, hence by positivity of intersections $S = Z$.

Step 2. We now consider moduli spaces of J-holomorphic curves for J's in $\mathcal{J}'$. Fix $g \geq 0$, and $B \in H_2(M, \mathbb{Z})$, $B \neq [Z]$ and consider for $J \in \mathcal{J}'$ the space $\mathcal{M}(B, J, g)$ of all somewhere injective J-holomorphic curves which represent the class B and are parameterized by all possible Riemann surfaces of genus g. More precisely, $\mathcal{M}(B, J, g)$ consists of all pairs (u, j), where $j \in \mathcal{T}_g$, the Teichmüller space of a closed oriented surface Σ_g of genus g, and $u : (\Sigma_g, j) \to (M, J)$ is a somewhere injective (j, J)-holomorphic map with $u_*[\Sigma_g] = B$. Since $B \neq Z$, step 1 shows that for $J \in \mathcal{J}'$ any J-holomorphic curve in the class B must go out of $\mathfrak{U}$ and so the transversality argument from [M-S 2] extend to prove that for generic $J \in \mathcal{J}'$ the spaces $\mathcal{M}(B, J, g)$ are smooth manifolds of dimension $2(c_1(B) + g - 1) + \dim G_g$ (provided that they are not empty). Here, G_g stands for the reparameterization group (of a generic (Σ_g, j)), and c_1 for the first Chern class of (TM, J).

Step 3. Given a homology class $B \in H_2(M, \mathbb{Z})$ let us write

$$k(B) = \frac{B \cdot B + c_1(B)}{2} \quad .$$

We claim that there exists a second category subset $\mathscr{J}'_{\text{gen}} \subset \mathscr{J}'$ such that for every $J \in \mathscr{J}'_{\text{gen}}$ the following holds: for every A-cusp configuration

$$A = \sum_{j=1}^{l} m_j A_j, \quad l \geq 2, m_j \geq 1 \quad \text{or} \quad l = 1, m_1 \geq 2$$

the set of all points $(x_1, \ldots, x_{k(A)}) \in M^{k(A)}$ which lie on a J-holomorphic A-cusp curve of this configuration has at least codimension 2 in $M^{k(A)}$. In particular, for every $J \in \mathscr{J}'_{\text{gen}}$ there exists a second category subset $G_J \subset M^{k(A)}$ such that for every $(x_1, \ldots, x_{k(A)}) \in G_J$ there are no J-holomorphic A-cusp curves, of any configuration, which pass through all of $x_1, \ldots, x_{k(A)}$. Notice that we regard multiply covered curves as cusp curves too by allowing $l = 1, m_1 \geq 2$.

Let us defer the proof of this claim for a while and show first how the statement of the lemma follows from it.

Step 4. By assumption there exists a second category subset $\mathscr{J}_{\text{gen}} \subset \mathscr{J}$ such that for every $J \in \mathscr{J}_{\text{gen}}$ the class A is J-effective and simple. It follows that we can choose $J' \in \mathscr{J}'_{\text{gen}}$, a sequence $J_n \in \mathscr{J}_{\text{gen}}$ and a $k(A)$-tuple of distinct points $(y_1, \ldots, y_{k(A)}) \in G_{J'}$ such that:

- $J_n \to J'$.
- For every n there exists a smooth, connected and reduced J_n-holomorphic curve $C_n \subset M$ with $[C_n] = A$, which passes through $y_1, \ldots, y_{k(A)}$.

Note that by the adjunction formula, the genus of C_n, say g, is independent of n.

By our construction, the sequence C_n cannot have any subsequence which converges to a cusp curve (this includes multiply covered curves, as remarked in Step 3 above). It follows from Gromov compactness theorem that C_n must have a subsequence which converges to a J'-holomorphic curve $C \subset M$ of the same genus, g, and in the homology class A. Clearly C is connected and, by construction, non-multiply covered. Finally, the adjunction formula implies that C is smooth.

In order to obtain transversality between C and Z one needs to perform a local arbitrarily C^1-small perturbation of C around each non-transversal intersection point of C with Z, and then alter J' in a spherical shell around each of these points. This is all possible by the methods of [McD 2], (see also [McD 1] and [Mi-Wh]). Roughly speaking, this is done as follows: let $x \in C \cap Z$ be a point in which C does not intersect Z transversally. Fix a neighborhood $\mathfrak{W}$ of x in M so that $\mathfrak{W} \cap C \cap Z = \{x\}$. It follows from [McD 2] that arbitrarily C^1-close to the identity, $1_{\mathfrak{W}} : \mathfrak{W} \to \mathfrak{W}$, there exists a diffeomorphism $F : \mathfrak{W} \to \mathfrak{W}$ and an open subset $\mathfrak{V}_F \subset \mathfrak{W}$ (which depends on F) with the following properties:

1. F has compact support in $\mathfrak{W}$.
2. $F(C \cap \mathfrak{W}) \pitchfork \overline{\wedge}$.
3. $F(C \cap \mathfrak{W}) \cap Z \subset \mathfrak{V}_F$.
4. $F(C \cap \mathfrak{W})$ is J'-holomorphic in $\mathfrak{V}_F$.

Thus, by taking F to be C^1-close enough to $\mathbb{1}$, $F(C)$ will still be a symplectic submanifold and the only thing we have to do is to alter J' appropriately in a neighborhood of $F(C)$ inside $\mathfrak{W} \setminus \mathfrak{V}_F$. Note that we need not change J' neither outside $\mathfrak{W}$ nor in a small enough neighborhood of Z. Finally, remark that since F can be assumed to be arbitrarily C^1-close to $\mathbb{1}$, the perturbed J' may be assumed to be arbitrarily C^0-close to the original J'.

Proof of Step 3. To complete the proof it remains to prove the claim stated in Step 3. For this end, we begin by observing that for any exceptional sphere class $E \in \mathscr{E}$ we must have $A \cdot E \geq -1$. Indeed, let $E \in \mathscr{E}$. It is well known that for generic $J \in \mathscr{J}$ there exists a J-holomorphic sphere in the class E (see [M-P] or [Bi 1, Bi 2]). As A is assumed to be J-effective and simple we may realize both A and E by smooth, connected and reduced J-holomorphic representatives for some $J \in \mathscr{J}$. By positivity of intersections we obtain $A \cdot E \geq -1$ with equality if and only if $A = E$.

Next, notice that it is enough to prove that for each possible configuration of A-cusp curves there exists a second category subset of $\mathscr{J}'$ for which no J-holomorphic cusp curve of that configuration can pass through $k(A)$ generic points in M. The reason is that there are only countable number of possible A-cusp configurations and the intersection of a countable number of second category subsets of $\mathscr{J}'$ will still be of second category. This intersection will be our needed $\mathscr{J}'_{\text{gen}}$.

Henceforth we fix an A-cusp configuration, say

$$A = \sum_{j=1}^{l} m_j A_j \ , \tag{$*$}$$

and assume that for generic $J \in \mathscr{J}'$ it can be realized by a J-holomorphic cusp curve. There are two cases to consider:

1. None of the A_j's is equal to $[Z]$ or a multiple of it.
2. One of the A_j's equals to $[Z]$ or a multiple of it.

Consider the first case. In view of Step 1, if $D = (D^{(1)}, \dots, D^{(l)})$ is a J-holomorphic cusp curve with $[D^{(j)}] = A_j$ then for every j, $D^{(j)}$ must go out of the neighborhood $\mathfrak{U}$ of Z. Noting that outside $\mathfrak{U}$ no restrictions were put on $\mathscr{J}'$ it follows that all the arguments from [McD 4] extend essentially without any change to prove that the maximal number of points through which a cusp curve of type D may pass is strictly smaller than $k(A)$. More precisely, since $A \cdot E \geq -1$ for every $E \in \mathscr{E}$, Lemma 2.10 of [McD 4] extends to case of generic $J \in \mathscr{J}'$ to show that the class A is good in the sense of Taubes, ie any J-holomorphic curve (possibly reducible or disconnected) in the class A which passes through $k(A)$ generic points does not have multiply covered components of negative self intersection. Then, the proof of Theorem 1.2 of [McD 4] extends to the case of generic $J \in \mathscr{J}'$ to prove that a J-holomorphic curve in the class A (again, possibly reducible or disconnected) which passes through $k(A)$ generic points must in fact consist of disjoint components and so it cannot be a cusp curve, unless $l = 1, m_1 \geq 2$ and $D^{(1)}$ is a torus of self intersection zero. This cannot occur in our situation because it implies that $A = m_1[D^{(1)}], A \cdot A = 0$ and $c_1(A) = 0$ (by adjunction), which contradicts our assumptions on A. This concludes the first case.

Let us consider now the second case in which we assume that one of the A_j's is $[Z]$ itself or a multiple of $[Z]$. As $Z \cdot Z < 0$, for generic $J \in \mathscr{J}'$ any J-holomorphic cusp curve can have at most one component with homology class $[Z]$ and this component must coincide, set-theoretically, with Z itself (of course, it might be a multiple cover of Z). Furthermore, all the other components must go outside of the neighborhood $\mathfrak{U}$ of Z. Thus, we can represent the A-cusp configuration $(*)$ as

$$A = \sum_{i=1}^{h} r_i B_i + m[Z], \quad m, r_i \geq 1, \ h \geq 1 \ ,$$

where none of the B_i's is a multiple of $[Z]$. Note that we may assume that $h \geq 1$ since otherwise $A = m[Z]$ with $m \geq 2$. But then because $Z \cdot Z < 0$ and $A \cdot Z \geq -c_1([Z])$ we have $A \cdot A = mA \cdot Z \geq -mc_1([Z]) = -c_1(A)$ and so $-2 \geq 2A \cdot A \geq A \cdot A - c_1(A) \geq -2$, where the very last inequality follows from the adjunction formula for A. It follows that $A \cdot A = -1$ and so $m = 1$; contradiction. Thus we may assume that $h \geq 1$.

The component $m[Z]$ of A cannot move ($Z \cdot Z < 0$), and so in order to prove our assertion we have to show that for generic $J \in \mathscr{J}'$ the maximal number of generic points through which a J-holomorphic cusp curve of the configuration $A - m[Z] = \sum_{i=1}^{h} r_i B_i$ can pass is strictly less than $k(A)$. For this purpose we write $\bar{A} = \sum_{i=1}^{h} r_i B_i$ and let us divide the set of indices $\{1, \ldots, h\}$ into three disjoint types:

1. $I_1 \subset \{1, \ldots, h\}$ consisting of i's such that B_i is an exceptional sphere class and $r_i \geq 2$.
2. $I_2 \subset \{1, \ldots, h\}$ consisting of i's such that B_i is an exceptional sphere class and $r_i = 1$.
3. $I_3 \subset \{1, \ldots, h\}$ consisting of all the other i's.

With this partition we have

$$\bar{A} = \sum_{i \in I_1} r_i B_i + \sum_{i \in I_2} B_i + \sum_{i \in I_3} r_i B_i \ . \tag{$**$}$$

Note that it might happen that some of the above indices sets are empty.

We claim that if for generic $J \in \mathscr{J}'$ there exists a J-holomorphic curve in one of the classes B_i with $i \in I_3$, then

$$B_i \cdot B_i \geq 0 \quad \text{and} \quad k(B_i) \geq 0 \ . \tag{1}$$

This will imply that by taking $J \in \mathscr{J}'$ to be generic we may, from now on, assume that (1) holds for every $i \in I_3$. Indeed, suppose that $B_i \cdot B_i < 0$ for some $i \in I_3$, and that there exists a J-holomorphic curve in the class B_i for generic $J \in \mathscr{J}'$. Since B_i is not $[Z]$ itself or a multiple of it, we have by Steps 1 and 2 that for generic $J \in \mathscr{J}'$ the moduli space of non-parameterized J-holomorphic curves in the class B_i has dimension $\dim \mathscr{M}(B_i, J, g_i)/G_g = 2(c_1(B_i) + g_i - 1)$, where g_i is the genus of the parameterizing Riemann surface. As $B_i \cdot B_i < 0$, we obtain by adjunction that $c_1(B_i) < 2 - 2g_i$ and so $\dim \mathscr{M}(B_i, J, g_i)/G_{g_i} \leq -2g_i$. For this number to be non-negative, we must assume that $g_i = 0$. But then $\dim \mathscr{M}(B_i, J, g_i)/G_{g_i} = 2(c_1(B_i) - 1)$ and again for this to be non-negative we have to assume that $c_1(B_i) \geq 1$. In total we have: $g_i = 0, B_i \cdot B_i < 0$ and $c_1(B_i) \geq 1$ and so by adjunction $B_i \cdot B_i = -1$. In other words, B_i is an exceptional sphere class and so $i \in I_1 \cup I_2$ rather than $i \in I_3$. This contradiction shows that $B_i \cdot B_i \geq 0$. Finally, $k(B_i) \geq 0$, because

$$2k(B_i) \geq 2(c_1(B_i) + g - 1) = \dim \mathscr{M}(B_i, J, g_i)/G_{g_i} \geq 0 \ .$$

Now we are ready to compare the maximal number of generic points through which a pseudo-holomorphic A-cusp curve can pass, with $k(A)$. Clearly, for generic $J \in \mathscr{J}'$ a cusp A-curve of the above configuration cannot pass through more than $\sum_{i \in I_3} k(B_i)$ points (recall that $Z \cdot Z < 0$ and that $B_i \cdot B_i = -1$ for $i \in I_1 \cup I_2$ and so these components cannot move). Now,

$$2\sum_{i\in I_3} k(B_i) \le 2\sum_{i\in I_3} r_i k(B_i) \le \sum_{i\in I_3} \left(c_1(r_iB_i) + r_i^2 B_i\cdot B_i\right)$$

$$\le c_1\left(\sum_{i\in I_3} r_iB_i\right) + \left(\sum_{i\in I_3} r_iB_i\right)\cdot\left(\sum_{i\in I_3} r_iB_i\right)$$

$$\le c_1\left(\sum_{i\in I_3} r_iB_i + \sum_{i\in I_2} B_i\right) + \left(\sum_{i\in I_3} r_iB_i + \sum_{i\in I_2} B_i\right)\cdot\left(\sum_{i\in I_3} r_iB_i + \sum_{i\in I_2} B_i\right)$$

$$= 2k\left(\sum_{i\in I_3} r_iB_i + \sum_{i\in I_2} B_i\right) \ . \qquad (2)$$

Here we have used the inequality $B_i \cdot B_j \ge 0$ for $i \ne j$ (by positivity of intersections), and the fact that for $i \in I_2$, B_i is an exceptional sphere class and so $B_i \cdot B_i = -1 = -c_1(B_i)$.

Following McDuff (see [McD 4] Section 1.3) let us define the following number:

$$k'(\bar{A}) = k(\bar{A}) + \frac{1}{2}\sum_{i\in I_1}\left(m_{B_i}(\bar{A})^2 - m_{B_i}(\bar{A})\right),$$

where $\quad m_{B_i}(\bar{A}) = \max(-\bar{A}\cdot B_i, 0)$.

In the proof of Proposition 3.1 of [McD 4] McDuff proves that if for some almost complex structure J there exists a J-holomorphic cusp curve of the configuration $(**)$ then

$$k\left(\sum_{i\in I_2} B_i + \sum_{i\in I_3} r_iB_i\right) \le k'(\bar{A}) \ . \qquad (3)$$

We remark that our definition of $k'(\bar{A})$ is slightly different from McDuff's. We sum over all exceptional sphere classes which appear as components of $\bar{A}$ while McDuff sums over all possible exceptional sphere classes. Nevertheless, this difference turns out to be irrelevant with respect to (3), and it is easy to see that McDuff's proof from Proposition 3.1 of [McD 4] applies word by word to prove (3). Note also that no genericity assumptions on J are needed in order to prove this inequality.

Combining inequality (2) with (3) we obtain

$$\sum_{i\in I_3} k(B_i) \le k'(\bar{A}) \ .$$

Our proof will be complete if we show that $k'(\bar{A}) < k(A)$. To prove this, suppose that $E_1, \ldots, E_s$ are all the classes in $\{B_i\}_{i \in I_1 \cup I_2}$ for which $\bar{A} \cdot E_j < 0$, and for every $1 \leq j \leq s$ put $n_j = -\bar{A} \cdot E_j > 0$. Note that s might be 0. We claim that for every $1 \leq j \leq s$, $A \cdot E_j \geq 0$. Indeed, we have already proved that $A \cdot E_j \geq -1$ with equality if and only if $A = E_j$. But $A = E_j$ is impossible since $A - E_j$ must be J-effective for it contains $m[Z]$ as a component. Here (unlike in Definition 2.2A) by J-effective we just mean that the relevant class can be represented by a J-holomorphic cusp curve. Thus, $A \cdot E_j \geq 0$ for every $1 \leq j \leq s$ and we immediately obtain that

$$0 < n_j \leq m(Z \cdot E_j) \ . \tag{4}$$

Next, observe that $\bar{A} - \sum_{j=1}^{s} n_j E_j$ must be J-effective because any J-representative of a J-holomorphic $\bar{A}$-cusp curve of configuration $(**)$ must contain E_j with multiplicity at least n_j. Thus, for generic $J \in \mathscr{J}'$ we have

$$\left(\bar{A} - \sum_{j=1}^{s} n_j E_j\right) \cdot Z \geq 0 \ . \tag{5}$$

A straightforward computation leads to

$$\begin{aligned} 2k(A) - 2k'(\bar{A}) &= m\bar{A} \cdot Z - \sum_{j=1}^{s} n_j^2 + m(A \cdot Z + c_1(Z)) + \sum_{j=1}^{s} n_j \geq \\ & m\left(\bar{A} \cdot Z - \sum_{j=1}^{s} n_j (E_j \cdot Z)\right) + m(A \cdot Z + c_1(Z)) + \sum_{j=1}^{s} n_j \\ &= m\left(\bar{A} - \sum_{j=1}^{s} n_j E_j\right) \cdot Z + m(A \cdot Z + c_1(Z)) + \sum_{j=1}^{s} n_j \ . \end{aligned}$$

Condition 1 of the lemma and inequalities (4), (5) imply that the last three summands are non-negative and so either $2k(A) > 2k'(\bar{A})$ which is what we want to prove, or $2k(A) = 2k'(\bar{A})$ which might happen only if $s = 0$ and $\bar{A} \cdot Z = 0$. But obviously $\bar{A} \cdot Z > 0$ because cusp curves are assumed to be connected and we explicitly assumed the existence of a cusp curve of the configuration $A = \bar{A} + m[Z], \bar{A} \neq 0$, and with the components of $\bar{A}$ being distinct from Z. This rules out the case $2k(A) = 2k'(\bar{A})$. The proof of the lemma is finally complete. □

7. Proof of Proposition 4.A

The proof is based on the following lemma:

Lemma 7.A. *Let $N \to \Sigma$ be a complex line bundle of non-negative degree over a symplectic (real) surface (Σ^2, τ). Then, there exists a smooth family of S^1-invariant symplectic forms $\{\omega^{(\lambda)}\}_{0<\lambda< \int_\Sigma \tau / \deg \mathrm{N}}$* on $S = \mathbb{P}(N \oplus \mathbb{C})$, *with the following properties for every* $0 < \lambda < \frac{\int_\Sigma \tau}{\deg N}$: 1

1. $Z_0, Z_\infty \subset (S, \omega^{(\lambda)})$ *are symplectic.*
2. $(pr_\Sigma^* \tau)|_{TZ_0} = \omega^{(\lambda)}|_{TZ_0}$, where $pr_\Sigma : S \to \Sigma$ *is the obvious projection.*
3. $\int_{Z_0} \omega^{(\lambda)} = \int_\Sigma \tau$, $\int_{Z_\infty} \omega^{(\lambda)} = \int_\Sigma \tau - \lambda \deg(N)$, $\int_F \omega^{(\lambda)} = \lambda$, *where F denotes the homology class of a fiber of the projection* $pr_\Sigma : S \to \Sigma$.
4. *For every neighborhood* $\mathfrak{U}$ of Z_0 in S *there exists* $0 < \lambda' < \lambda$ *and a symplectic embedding* $\varphi : (S \backslash Z_\infty, \omega^{(\lambda')}) \to (\mathfrak{U}, \omega^{(\lambda)})$ *which takes Z_0 to itself identically* (ie $\varphi|_{Z_0} = \mathbb{1}$).

Before proving this lemma, let us see how it implies Proposition 4.A.

Proof of Proposition 4.A. Put $\tau = \Omega|_{T\Sigma}$. Fix λ between 0 and $\frac{\int_\Sigma \tau}{\deg(N_{\Sigma/M})}$ and consider the form $\omega^{(\lambda)}$ on $S = \mathbb{P}(N_{\Sigma/M} \oplus \mathbb{C})$ as defined by Lemma 7.A. Since $Z_0 \cdot Z_0 = \Sigma \cdot \Sigma$ and $\int_{Z_0} \tau = \int_\Sigma \Omega$, then by the symplectic neighborhood theorem there exists a neighborhood $\mathfrak{U}$ of Z_0 in S and a symplectic embedding

$$\phi : (\mathfrak{U}, \omega^{(\lambda)}) \to (M, \Omega)$$

which takes Z_0 to Σ. By Lemma 7.A there exists $0 < \lambda' < \lambda$ and a symplectic embedding

$$\varphi : (S \backslash Z_\infty, \omega^{(\lambda')}) \to (\mathfrak{U}, \omega^{(\lambda)}) \qquad \text{with} \quad \varphi(Z_0) = Z_0 \ .$$

The form $\omega_0 = \omega^{(\lambda')}$ and the embedding $\phi_0 = \phi \circ \varphi$ are what we need. □

We now turn to the

Proof of Lemma 7.A. Let $P \subset N$ be the principle S^1-bundle associated to N (with respect to some Hermitian metric on N). The group S^1 acts diagonally on $P \times \mathbb{C}P^1$ by

[1] In case deg $N = 0$ the family of forms will just be parameterized as $\{\omega^{(\lambda)}\}_{0<\lambda}$.

$$t \cdot (p, [z_0 : z_1]) = (e^{-2\pi i t} \cdot p, [e^{2\pi i t} z_0 : z_1]), \qquad t \in S^1 = \mathbb{R}/\mathbb{Z} \ .$$

Let $\pi : P \times \mathbb{C}P^1 \to P \times_{S^1} \mathbb{C}P^1$ be the quotient map. Note that $P \times_{S^1} \mathbb{C}P^1$ still fibers over Σ. It is easy to see that the map

$$P \times \mathbb{C}P^1 \ni (p, [z_0 : z_1]) \mapsto [z_0 p : z_1] \in S = \mathbb{P}(N \oplus \mathbb{C})$$

descends to a diffeomorphism $P \times_{S^1} \mathbb{C}P^1 \to S$ which preserves the fibers and takes $\pi(P \times [0 : 1])$ to Z_0 and $\pi(P \times [1 : 0])$ to Z_∞. In view of this, we shall identify for the rest of the proof $P \times_{S^1} \mathbb{C}P^1$ with S and denote $\pi(P \times [0 : 1]), \pi(P \times [1 : 0])$ by Z_0, Z_∞ respectively.

Let λ be a positive real number such that $\lambda \deg(N) < \int_\Sigma \tau$, and consider the following form on $P \times \mathbb{C}P^1$:

$$\tilde{\omega}^{(\lambda)} = pr_\Sigma^* \tau + \lambda d(h\alpha) + \lambda pr_{\mathbb{C}P^1}^* \sigma_{\mathrm{std}} \ ,$$

where $h : \mathbb{C}P^1 \to [0, 1]$ is the Hamiltonian function $h([z_0 : z_1]) = |z_0|^2/(|z_0|^2 + |z_1|^2)$, σ_{std} is the standard Kähler form of $\mathbb{C}P^1$ normalized so that $\int_{\mathbb{C}P^1} \sigma_{\mathrm{std}} = 1$, and α is a connection 1-form on P which satisfies

$$\alpha(X) = 1, \quad d\alpha = -\frac{\deg(N)}{\int_\Sigma \tau} pr_\Sigma^* \tau \ .$$

Here, X is the vector field on P generated by the S^1 action, namely $X_{(p)} = \frac{d}{dt}\big|_{t=0} (e^{2\pi i t} \cdot p)$.

A straightforward computation shows that $\tilde{\omega}^{(\lambda)}$ descends to a symplectic form, $\omega^{(\lambda)}$, on $P \times_{S^1} \mathbb{C}P^1$ which has properties 1–3 claimed in the statement of lemma. We refer the reader to Chapter 5 of [M-S 1] for more details on the model forms $\tilde{\omega}^{(\lambda)}$, however note that our convention of signs is somewhat opposite to theirs.

Let us prove now the fourth property claimed by the lemma. For this end denote by $D(r) = \{w \in \mathbb{C} \,\big|\, |w| < r\}$ the open 2-disc of radius $r > 0$ and let S^1 act on $P \times D(r)$ by $t \cdot (p, w) = (e^{-2\pi i t} p, e^{2\pi i t} w)$. Consider the map

$$j_{(\lambda)} : P \times_{S^1} D\big(\sqrt{\lambda}\big) \to \big(P \times_{S^1} \mathbb{C}P^1\big) \setminus Z_\infty \ ,$$

induced by

$$D\big(\sqrt{\lambda}\big) \ni w \mapsto \left[w : \sqrt{\lambda - |w|^2}\right] \in \mathbb{C}P^1 \ .$$

A simple calculation shows that the following closed 2-form on $P \times D(\sqrt{\lambda})$

$$pr_{\Sigma}^{*}\tau + d(|w|^{2}\alpha) + \frac{1}{\pi} pr_{D}^{*}\omega_{\text{std}}$$

descends to the form $j_{(\lambda)}^{*}\omega^{(\lambda)}$ on $P \times_{S^1} D(\sqrt{\lambda})$. Here, pr_D is the projection $P \times D(\sqrt{\lambda}) \to D(\sqrt{\lambda})$ and ω_{std} stands for the standard symplectic form on $D(\sqrt{\lambda}) \subset \mathbb{C} \approx \mathbb{R}^2$.

Now, given a neighborhood $\mathfrak{U}$ of Z_0 in S, choose $0 < \lambda' < \lambda$ small enough so that

$$j_{(\lambda)}\big(P \times_{S^1} D(\sqrt{\lambda'})\big) \subset \mathfrak{U} \ .$$

Take φ to be:

$$j_{(\lambda)} \circ j^{-1}{}_{(\lambda')} : \big((P \times_{S^1} \mathbb{C}P^1)\backslash Z_{\infty}, \omega^{(\lambda')}\big) \hookrightarrow \big(\mathfrak{U}, \omega^{(\lambda)}\big) \ . \qquad \square$$

Acknowledgements. This work was initiated while I was participating in the IAS/Park City summer session on symplectic geometry at Utah in July 1997. During my stay there I have had many fruitful discussions with various people and I would like to thank the organizers of this conference for the invitation and support.

I am especially thankful to Wladek Lorek, Leonid Polterovich, Dietmar Salamon and Brian White for quite a few discussions and useful remarks, especially regarding pseudo-holomorphic curves. I am indebted to Yasha Eliashberg for the encouragement and for attracting my attention to very interesting points which I was not aware of. Special thanks to Dusa McDuff, from whom I had originally learned about Gromov invariants, for stimulating discussions, and to Robert Lazarsfeld for the help with the algebraic geometry and the examples.

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