PROPAGATION IN HAMILTONIAN DYNAMICS AND RELATIVE SYMPLECTIC HOMOLOGY

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Abstract

The main result asserts the existence of noncontractible periodic orbits for compactly supported time-dependent Hamiltonian systems on the unit cotangent bundle of the torus or of a negatively curved manifold whenever the generating Hamiltonian is sufficiently large over the zero section. The proof is based on Floer homology and on the notion of a relative symplectic capacity. Applications include results about propagation properties of sequential Hamiltonian systems, periodic orbits on hypersurfaces, Hamiltonian circle actions, and smooth Lagrangian skeletons in Stein manifolds.

1. Introduction

1.1. Stable propagation

Let $M := U^*\mathbb{T}^n$ be the open unit cotangent bundle of the n-dimensional Euclidean torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We express the elements of M in terms of the canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, where $q_i \equiv q_i + 1$. Thus we identify M with $\mathbb{T}^n \times \mathbb{D}^n$, where $\mathbb{D}^n = \{|p| < 1\}$ is the open unit ball in $\mathbb{R}^n$, and $|v|$ stands for the Euclidean norm of a vector $v \in \mathbb{R}^n$. Consider the space $\mathscr{H}$ of all smooth compactly supported functions on $[0, 1] \times M$. Every function $H \in \mathscr{H}$ gives rise to the Hamiltonian system on M:

$$\dot{q} = \frac{\partial H}{\partial p}(t, q, p), \qquad \dot{p} = -\frac{\partial H}{\partial q}(t, q, p). \tag{1}$$

The flow h_t which sends any initial condition $x(0) = (q(0), p(0))$ to the solution $x(t) = (q(t), p(t))$ at time t is called the *Hamiltonian flow* (or *isotopy*) generated by H, and the time-1 map h_1 is called the *Hamiltonian diffeomorphism* generated by H. The set of all Hamiltonian diffeomorphisms of M form a group denoted by $\mathscr{D}$.

Let $f_* = \{f_k\}_{k=1,2,\ldots}$ be an infinite sequence of Hamiltonian diffeomorphisms. We consider f_* as a dynamical system as follows. Put $f^{(k)} = f_k \cdots f_1$. This sequence is called the *evolution* of f_*. The orbit $\{x_k\}_{k\in\mathbb{N}}$ of a point $x \in M$ is defined by

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$x_k = f^{(k)}x$. Classical dynamical systems (iterations of a single map f) correspond to constant sequences $f_k \equiv f$. Sequential systems arise naturally as perturbations of the classical ones. In a number of interesting situations, stability has been observed in the sense that these perturbations inherit dynamical properties of the original system (see [PR]). In this paper we present a new stability phenomenon of this type—stable propagating behaviour—which we are going to describe next.

Every Hamiltonian diffeomorphism h has a *canonical* lift $\widetilde{h}$ to the universal cover $\widetilde{M} = \mathbb{R}^n \times \mathbb{D}^n$ of M. Throughout we denote

$$Q := \left\{(q, p) \in \widetilde{M} \,\middle|\, |p| < 1,\ \max_i |q_i| \le \frac{1}{2}\right\};$$

this is a fundamental domain of the covering.

Definition
A sequential system f_* *propagates to infinity with speed* (at least) c if for every vector $v \in \mathbb{R}^n$ with $|v| \le c$ there exists a sequence of points $\widetilde{x}_k \in Q$ such that the sequence $(p_k, q_k) := \widetilde{f}^{(k)}(\widetilde{x}_k)$ satisfies

$$\lim_{k\to\infty} \frac{q_k}{k} = v.$$

In other words, the projection to $\mathbb{R}^n$ of $\widetilde{f}^{(k)}(Q)$ covers the Euclidean ball of radius kc up to an error that is small with respect to k.

Let us illustrate this notion in the following simple example. Consider a Hamiltonian function $H \in \mathscr{H}$ which depends only on momenta variables: $H = H(p)$. The lift of the corresponding Hamiltonian flow to the universal cover is given by $\widetilde{h}^t(q, p) = (q + t\nabla H(p), p)$. Therefore the projection of $\widetilde{h}^k(Q)$ to $\mathbb{R}^n$ coincides up to a bounded error with the set kI, where I denotes the image of the gradient map $p \mapsto \nabla H(p)$. Assume now that $H(0) > c$ for some $c > 0$. We claim that I contains the Euclidean ball of radius c centered at zero. Indeed, for every $v \in \mathbb{R}^n$ with $|v| \le c$ the function $F(p) := H(p) - pv$ satisfies $F(0) > c$ and $F \le c$ near $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$. Therefore F attains its maximum at a point $p_0 \in \mathbb{D}^n$. Hence $\nabla H(p_0) = v$, and the claim follows. This shows that in our example we have propagation with speed c.

The group $\mathscr{D}$ carries a remarkable *bi-invariant* metric ρ called *Hofer's metric* (see [H]). The corresponding geometry provides us with a suitable language for the study of various stable phenomena in Hamiltonian dynamics. Given a diffeomorphism $f \in \mathscr{D}$, write $\rho(\mathrm{id}, f)$ for $\inf(\max F - \min F)$ where the infimum is taken over all Hamiltonians $F \in \mathscr{H}$ generating f. Define Hofer's distance $\rho(f, g)$ between two elements $f, g \in \mathscr{D}$ as $\rho(\mathrm{id}, fg^{-1})$.

THEOREM A (Stable propagation)
Let $0 < a < c$, and suppose that h is a Hamiltonian diffeomorphism generated by

a Hamiltonian $H \in \mathscr{H}$ such that $H(t,q,0) \geq c$, for all t and q. Let f_* be any sequential system such that $\rho(f_i, h) < a$ for all $i \in \mathbb{N}$. Then f_* propagates to ∞ with speed $c - a$.

Theorem A is proved in Section 2.3. Interestingly enough, such a propagating behaviour may be completely destroyed by an appropriate arbitrarily C^∞-small dissipative perturbation even in the framework of classical dynamics. In Section 2.4 we elaborate this in the case of $n = 1$. We show that every Hamiltonian diffeomorphism h generated by the Hamiltonian $H = H(p)$ admits an arbitrarily small smooth (non-Hamiltonian!) perturbation f such that the images $\widetilde{f}^k(Q)$, $k \in \mathbb{N}$, of the set Q under the iterates of $\widetilde{f}$ remain in a compact part of $\widetilde{M}$.

Note that Theorem A does not provide any information about propagation of *individual* trajectories on the universal cover (and we doubt that such information is available at all in this generality). The situation improves when one considers sequences $f_* = \{f_i\}$, which roughly speaking are uniformly distributed with respect to some "nice" measure on $\mathscr{D}$ whose support is close to h in the sense of Hofer's metric. It turns out that such sequential systems have trajectories that propagate to ∞ with constant velocity. We refer the reader to Section 2.3 for the details.

1.2. Noncontractible closed orbits

The main tool for studying stable propagation as described in Section 1.1 is an existence result for noncontractible periodic solutions of compactly supported Hamiltonian systems under quite robust assumptions on the Hamiltonian functions. This result is based on Floer homology filtered by the symplectic action, and its proof is quite involved. A solution $x(t) = (q(t), p(t))$ of a Hamiltonian system generated by a function $H \in \mathscr{H}$ is called *periodic* if $p(1) = p(0)$ and $q(1) = q(0) + e$ for some integer vector $e \in \mathbb{Z}^n$. The lattice $\mathbb{Z}^n$ is identified in a natural way with the fundamental group of M; hence we refer to e as the *homotopy class* of the solution. An important quantity associated to a periodic solution is its *action*:

$$\mathscr{A}_H(x) = \int_0^1 \Big(H\big(t, q(t), p(t)\big) - \sum_{i=1}^n p_i(t)\dot{q}_i(t) \Big)\, dt.$$

Denote by $Z \subset M$ the zero section $\{p = 0\}$.

THEOREM B

For every compactly supported smooth Hamiltonian function $H \in \mathscr{H}$ and every $e \in \mathbb{Z}^n$ such that

$$|e| \leq c := \inf_{[0,1]\times Z} H, \tag{2}$$

the Hamiltonian system (1) has a periodic solution $x(t)$ in the homotopy class e with action $\mathscr{A}_H(x) \geq c$.

The result above is *sharp* in the following sense. First, the inequalities in Theorem B which guarantee the existence of periodic solutions in the class e cannot be improved. For instance, for every $\varepsilon > 0$ it is easy to produce a Hamiltonian of the form $H = H(|p|)$ whose restriction to $[0, 1] \times Z$ equals $1 - \varepsilon$ and such that all periodic solutions are contractible (such a function $H(|p|)$ can be obtained from $1 - |p|$ by an appropriate smoothing). Second, the zero section Z in the inequality (2) cannot be replaced by an arbitrary smooth section. Consider, for instance, an arbitrary C^∞-small perturbation $S = \{p = u(q)\}$ of Z such that the 1-form $u(q)dq$ on $\mathbb{T}^n$ is *not* closed. (In symplectic terms this means that the section S is non-Lagrangian; this is possible only when $n \geq 2$.)

THEOREM C
Given any $c > 0$, there exists a Hamiltonian function $H \in \mathscr{H}$ such that $H(t, x) \geq c$ for every $t \in [0, 1]$ and every $x \in S$ and such that every 1-periodic solution of (1) is contractible.

If one takes S as the graph of *an exact* 1-form on $\mathbb{T}^n$, the assertion of Theorem B remains valid with Z replaced by S. Thus the existence mechanism for noncontractible periodic solutions described in Theorem B is quite sensitive to the choice of a subset where the Hamiltonian is large enough. This phenomenon is studied below in terms of a *relative symplectic capacity* (see Sec. 3).

On the other hand, the statement of Theorem B is *robust* from the following viewpoint: if the restriction of a Hamiltonian H to $[0, 1] \times Z$ is bigger than a certain positive number, Theorem B guarantees that most of the periodic solutions persist under C^0-perturbations of the Hamiltonian. This robustness plays a crucial role in the study of stable propagation.

In Section 3 we prove Theorem B and its generalization where the torus is replaced by a hyperbolic manifold. The proof uses Floer homology for the action functional on the space of noncontractible loops. The main difficulty we have to go around is as follows. Since we deal with compactly supported Hamiltonians, noncontractible solutions are "nonessential" from the viewpoint of Floer homology—they may not persist under deformations of the Hamiltonian. In brief, the idea of the proof can be described as follows. One can squeeze the Hamiltonian function H between two more or less standard functions, $H_- \leq H \leq H_+$, where H_- and H_+ depend only on $|p|$. The filtered Floer homologies of H_- and H_+, as well as the natural morphism between them, can be computed explicitly. This morphism turns out to be nontrivial, and since it factors through the Floer homology of H, we get nontrivial information about the noncontractible periodic solutions corresponding to H. The calculations are quite involved; hence we introduce a convenient algebraic tool—relative symplectic homology—which helps us perform them in a more organized way.

Noncontractible orbits of Hamiltonian systems were studied earlier in a number of interesting situations. In a beautiful paper [GL], D. Gatien and F. Lalonde considered the following setting. Let L_0 and L_1 be two disjoint closed Lagrangian submanifolds of a symplectic manifold. Assume that H is an autonomous Hamiltonian function that is "small" on L_0 and "large" on L_1. It turns out that under certain additional assumptions of a topological nature one can prove the existence of noncontractible periodic orbits for the Hamiltonian flow generated by H. This result was the starting point for Theorem B. Another important idea of a symplectic capacity which is sensitive to the fundamental group is contained in M. Schwarz's work [Sc]. We develop it further in Section 3. Noncontractible orbits of autonomous Hamiltonians on cotangent bundles whose levels are starshaped were considered by a number of authors (see, e.g., [C1]). In contrast to our case, these closed orbits are homologically essential in the sense of Floer homology. Let us mention finally that the interest in noncontractible periodic orbits on cotangent bundles comes from classical mechanics, where one considers Hamiltonians $H(t, q, p)$ which are convex with respect to the momenta variable p. In this case, the existence of closed orbits in given homotopy classes can be derived with methods from the classical calculus of variations.

2. Stable propagation in sequential Hamiltonian dynamics

In this section we study stable propagation along the lines mentioned in the introduction. The main tool is Theorem B.

2.1. Preliminaries on Hamiltonian diffeomorphisms

Let M be an open manifold (i.e., M is connected, noncompact, and has no boundary), and denote by

$$\pi : \widetilde{M} \to M$$

its universal cover. Let $\mathrm{Diff}_0(M)$ denote the group of compactly supported diffeomorphisms of M which are isotopic to the identity by isotopies with compact support in $[0, 1] \times M$. Then every $h \in \mathrm{Diff}_0(M)$ has a canonical lift $\widetilde{h} : \widetilde{M} \to \widetilde{M}$ to the universal cover. To see this, choose a compactly supported isotopy $[0, 1] \to \mathrm{Diff}_0(M) : [0, 1] \mapsto h_t$ from $h_0 = \mathrm{id}$ to $h_1 = h$. Given $\widetilde{x}_0 \in \widetilde{M}$, lift the path $[0, 1] \to M : t \mapsto h_t(\pi(\widetilde{x}_0))$ to a path $[0, 1] \to \widetilde{M} : t \mapsto \widetilde{x}(t)$ such that $\widetilde{x}(0) = \widetilde{x}_0$, and define $\widetilde{h}(\widetilde{x}_0) := \widetilde{x}(1)$. The following remark shows that this definition is independent of the choice of the isotopy $t \mapsto h_t$.

Remark 2.1.1
Let $t \mapsto \phi_t$ and $t \mapsto \psi_t$ be two compactly supported isotopies such that $\phi_1 = \psi_1 = h \in \mathrm{Diff}_0(M)$. Given a point $x \in M$, consider the paths $t \mapsto \phi_t(x)$ and $t \mapsto \psi_t(x)$ connecting x to $h(x)$. We claim that they are homotopic with fixed endpoints. To see this, choose a path $[0,1] \to M : s \mapsto \gamma(s)$ such that $\gamma(0) = x$ and $\gamma(1)$ lies outside the support of the isotopies. Looking at $\varphi_t(\gamma(s))$ and $\psi_t(\gamma(s))$, it is easy to see that paths $t \mapsto \phi_t(x)$ and $t \mapsto \psi_t(x)$ are both homotopic with fixed endpoints to the path Γ obtained by going first from x to $\gamma(1)$ along γ and then from $\gamma(1)$ to $h(x)$ along $h(\gamma^{-1})$. Here γ^{-1} stands for the reverse of γ.

Now suppose that M is equipped with a symplectic form ω, and denote by $\mathscr{D} \subset \mathrm{Diff}_0(M)$ the group of Hamiltonian diffeomorphisms that are generated by compactly supported Hamiltonian functions on $[0,1] \times M$. Given a compact subset $A \subset M$ and a real number c, let us denote by $\mathscr{D}_c = \mathscr{D}_c(M, A) \subset \mathscr{D}$ the subset of Hamiltonian diffeomorphisms that are generated by compactly supported Hamiltonian functions $H \in C_0^\infty([0,1] \times M)$ with $\inf_{[0,1]\times A} H \geq c$. As before, denote by ρ the Hofer metric on $\mathscr{D}$.

PROPOSITION 2.1.2
Let $c > a$ be positive numbers, and let $f, g \in \mathscr{D}$ be Hamiltonian diffeomorphisms with $f \in \mathscr{D}_c$ and $\rho(f, g) < a$. Then $g \in \mathscr{D}_{c-a}$.

Proof
This is an immediate consequence of the following product formula. If the functions Φ_t and Ψ_t generate Hamiltonian flows ϕ_t and ψ_t, respectively, then the product flow $\phi_t \circ \psi_t$ is generated by the Hamiltonian $\Phi_t + \Psi_t \circ \phi_t^{-1}$. □

In what follows we also deal with time-periodic Hamiltonians, namely, functions H satisfying $H(t, \cdot) = H(t+1, \cdot)$. It is useful to think of these Hamiltonians as smooth functions $H : S^1 \times M \to \mathbb{R}$, where we identify $S^1 \cong \mathbb{R}/\mathbb{Z}$. The following proposition shows that every $h \in \mathscr{D}_c$ can be generated by a periodic Hamiltonian bounded below by c on $S^1 \times A$.

PROPOSITION 2.1.3 (Periodic Hamiltonians)
Let M be an open symplectic manifold, and let $A \subset M$ be a compact subset. Let h be a Hamiltonian diffeomorphism of M generated by a Hamiltonian $H \in C_0^\infty([0,1] \times M)$ with $\inf_{[0,1]\times A} H \geq c$. Then there exists a Hamiltonian $\overline{H} \in C_0^\infty(S^1 \times M)$ with $\inf_{S^1 \times A} \overline{H} \geq c$ that generates h.

Proof
Let h_t denote the Hamiltonian isotopy generated by H, and let f_t denote the Hamiltonian flow generated by a time-independent compactly supported function $F : M \to \mathbb{R}$. Let $\tau : [0, 1] \to [0, 1]$ be a smooth nondecreasing function that equals zero near $t = 0$ and equals 1 near $t = 1$. Then the Hamiltonian isotopy

$$\overline{h}_t := f_{t-\tau(t)} \circ h_{\tau(t)}$$

is generated by the Hamiltonian function

$$\overline{H}_t := F + \tau'(t)(H_{\tau(t)} - F) \circ f_{\tau(t)-t}.$$

The function $\overline{H}_t$ equals F near $t = 0$ and $t = 1$ and hence defines a smooth Hamiltonian on $S^1 \times M$. Moreover, $\overline{h}_1 = h_1 = h$. If F is chosen to be equal to c in a neighbourhood of A, then f_t is equal to the identity on A and $H_{\tau(t)} - F$ is nonnegative on A; hence $\inf_A \overline{H}_t \geq c$ for every t. □

In the remainder of this section, we assume that $M = U^*\mathbb{T}^n$ is the open unit cotangent bundle and $A = Z = \mathbb{T}^n \subset U^*\mathbb{T}^n$ is the zero section. Given $e \in \mathbb{Z}^n$, we denote by T_e the deck transformation $(q, p) \to (q + e, p)$ of the covering. Note that 1-periodic solutions of the Hamiltonian system associated to H are in one-to-one correspondence with the fixed points of h. The homotopy class of the periodic solution $x(t)$ corresponding to the fixed point $x \in M$ (or, in brief, the *homotopy class of the fixed point* x) can be determined as follows. Pick a lift y of x. Then $\widetilde{h}(y) = T_e(y)$ for some $e \in \mathbb{Z}^n$, and this e is the homotopy class in question. (We emphasize that the class e depends on h alone regardless of the Hamiltonian isotopy generating it; see Rem. 2.1.1.) In this terminology Theorem B asserts that every diffeomorphism $h \in \mathscr{D}_c = \mathscr{D}_c(U^*\mathbb{T}^n, \mathbb{T}^n)$ has a fixed point in the homotopy classes e for every e such that $|e| \leq c$. It turns out that the same result allows us to get information on the fixed points of the iterates h^k for $k \in \mathbb{N}$.

PROPOSITION 2.1.4
If $h \in \mathscr{D}_c$, then $h^k \in \mathscr{D}_{kc}$ for every $c > 0$ and every $k \in \mathbb{N}$. In particular, h^k has a fixed point in every homotopy class $e \in \mathbb{Z}^n$ such that $|e| \leq kc$.

Proof
We have seen in Proposition 2.1.3 that, for every $h \in \mathscr{D}_c$, one can find a function $H : \mathbb{R} \times M \to \mathbb{R}$ which is 1-periodic in time, satisfies $\inf_{\mathbb{R}\times A} H \geq c$, and generates h as the time-1 map. It follows that the kth iterate h^k is generated by $kH(kt, x)$; hence $h^k \in \mathscr{D}_{kc}$, and hence the result follows from Theorem B. □

2.2. *A lemma from ergodic theory*

Let X be a compact metrizable topological space, and let $f : X \to X$ be a homeomorphism. Let $\mathscr{M}(f)$ denote the subset of f-invariant Borel probability measures.

LEMMA 2.2.1

Let $f : X \to X$ be a homeomorphism of a compact metric space, and let $u : X \to \mathbb{R}^n$ be a continuous function. Consider the linear projection

$$\mathscr{M}(f) \to \mathbb{R}^n : \mu \mapsto R(\mu) := \int u \, d\mu.$$

Let r be an extremal point of the compact convex set $K := R(\mathscr{M}(f)) \subset \mathbb{R}^n$. Then there exists an ergodic f-invariant Borel probability measure $\mu \in \mathscr{M}(f)$ such that $r = \int u \, d\mu$.

Proof

The set $\mathscr{M}_r(f) := \{\mu \in \mathscr{M}(f) \mid R(\mu) = r\}$ is a nonempty weak-$*$ compact convex subset of the dual space of $C^0(X)$. Hence, by the Krein-Milman theorem, it has an extremal point μ. We prove that μ is an extremal point of $\mathscr{M}(f)$. To see this, suppose that $\mu = (1-t)\mu_0 + t\mu_1$ such that $\mu_0, \mu_1 \in \mathscr{M}(f)$ and $0 < t < 1$. We must prove that $\mu_0 = \mu_1$. To see this, let $r_0 := R(\mu_0)$ and $r_1 := R(\mu_1)$. Then $r = (1-t)r_0 + tr_1$ and $r_0, r_1 \in K$. Since r is an extremal point of K, it follows that $r_0 = r_1 = r$ and hence $\mu_0, \mu_1 \in \mathscr{M}_r(f)$. Since μ is an extremal point of $\mathscr{M}_r(f)$, this implies that $\mu_0 = \mu_1$. Thus we have proved that μ is an extremal point of $\mathscr{M}(f)$, and hence it is an ergodic f-invariant Borel probability measure. □

2.3. *Stable propagation revisited*

Let us now return to the case where $M = U^*\mathbb{T}^n$ is the open unit cotangent bundle of the n-torus and $\mathscr{D}$ is the group of compactly supported Hamiltonian diffeomorphisms of M. For $a > 0$ and $h \in \mathscr{D}$, denote by

$$B(h, a) := \left\{ f \in \mathscr{D} \,\middle|\, \rho(f, h) < a \right\}$$

the open ball of radius a in Hofer's metric. Recall that Theorem A states the following. Let $c > a > 0$ be real numbers, let $h \in \mathscr{D}_c$ be a Hamiltonian diffeomorphism, and let f_* be a sequential system such that $f_i \in B(h, a)$ for every $i \in \mathbb{N}$. Then f_* propagates to ∞ with speed $c - a$. We now prove Theorem A, assuming Theorem B.

Proof of Theorem A

Write $f_i = h\psi_i$ with $\rho(\mathrm{id}, \psi_i) < a$. Then the evolution of f_* can be written as follows:

$$f^{(k)} = (h\psi_k h^{-1})(h^2\psi_{k-1}h^{-2}) \cdots (h^k\psi_1 h^{-k})h^k.$$

Combining the fact that ρ is bi-invariant with the triangle inequality, we get

$$\rho(f^{(k)}, h^k) < ka.$$

Now let $w \in \mathbb{R}^n$ be any vector such that $|w| \leq c - a$, and choose a sequence $v_k \in \mathbb{Z}^n$ such that

$$w = \lim_{k\to\infty} \frac{v_k}{k}, \quad |v_k| < k(c-a).$$

Since $h^k \in \mathcal{D}_{kc}$ (see Prop. 2.1.4), we obtain from Proposition 2.1.2 that $f^{(k)} \in \mathcal{D}_{k(c-a)}$. Hence, by Theorem B, there exists a point $\widetilde{x}_k = (q_k, p_k) \in Q$ so that $\widetilde{f}^{(k)}(\widetilde{x}_k) = (q_k + v_k, p_k)$. Since the sequence q_k is bounded, we have $\lim_{k\to\infty}(q_k + v_k)/k = w$, and hence f_* propagates to ∞ with speed $c - a$. □

Now we turn to the study of the longtime behaviour of *individual trajectories*. We start with the following general definition. Consider the trajectory $x_k = f^{(k)}(x)$ of a point $x \in M$ under a sequential system f_*. Let $\widetilde{x} \in \widetilde{M}$ be a lift of x, and consider the lifted trajectory $\widetilde{x}_k = \widetilde{f}^{(k)}(\widetilde{x})$ in the universal cover.

Definition
A trajectory x_k of a sequential system with a lift $\widetilde{x}_k = (q_k, p_k) \in \widetilde{M}$ is said to *propagate with velocity vector* $r \in \mathbb{R}^n$ if

$$\lim_{k\to\infty} \frac{q_k}{k} = r.$$

The Euclidean norm of r is called the *speed* of the trajectory x_k.

As an illustration, let us mention that every k-periodic orbit of a Hamiltonian diffeomorphism h in a nonzero homotopy class $e \in \mathbb{Z}^n$ propagates with velocity vector e/k. Note also that propagation in the universal cover $\widetilde{M}$ corresponds to "rotation" in M. In classical dynamics the velocity vector of a propagating orbit is called the rotation vector.

Consider now the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (which plays the role of the "parameter space" of the sequential system). Fix a vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ whose coordinates are independent over the rationals. Let $g : \mathbb{T}^d \to \mathcal{D}$ be a map that is continuous with respect to the C^0-topology on $\mathcal{D}$. For a point $y \in \mathbb{T}^d$, define a sequential system

$$f_*(y) := \{g(y + k\alpha)\}_{k\in\mathbb{N}}.$$

Roughly speaking, the sequence $f_*(y)$ is "random"—it is uniformly distributed on the subset $g(\mathbb{T}^d)$. The next result is an improvement of Theorem A for (almost all) such sequences.

THEOREM 2.3.1
Let $0 < a < c$ *be real numbers, and let* $h \in \mathscr{D}_c$ *be a Hamiltonian diffeomorphism. Let* $g : \mathbb{T}^d \to \mathscr{D}$ *be a continuous map whose image is contained in* $B(h, a)$*. Then, for a set of Lebesgue measure* 1 *of points* $y \in \mathbb{T}^d$*, the sequential system* $f_*(y)$ *has at least* $n+1$ *trajectories that propagate with speed at least* $c - a$*.*

In fact, there exists a compact convex set $K \subset \mathbb{R}^n$ *containing the closed ball of radius* $c - a$ *in* $\mathbb{R}^n$ *centered at the origin so that the following holds. For every extremal point* $r \in K$*, there exists a subset* $Y_r \subset \mathbb{T}^d$ *of Lebesgue measure* 1 *such that, for every* $y \in Y_r$*, the system* $f_*(y)$ *has a trajectory that propagates with velocity vector* r*.*

The basic feature of the irrational shift of the torus which is crucial for our purposes is unique ergodicity, which means that it has a unique invariant Borel probability measure (the Lebesgue measure). The assertion of Theorem 2.3.1 continues to hold, and the proof remains the same if one replaces the torus $\mathbb{T}^d$ by an arbitrary compact metric space Y, the Lebesgue measure by any Borel probability measure σ, and the irrational shift by any σ-preserving uniquely ergodic homeomorphism of Y.

Proof
Let $\overline{M}$ denote the closed unit cotangent bundle, and let $\widetilde{M}$ denote its universal cover. Write $Y := \mathbb{T}^d$ and $\phi(y) := y + \alpha$, and denote by σ the Lebesgue measure on Y. Consider the skew-product map

$$\overline{M} \times Y \to \overline{M} \times Y : (x, y) \mapsto S(x, y) := \big(g(y)(x), \phi(y)\big)$$

and its canonical lift $\widetilde{S}$ to $\widetilde{M} \times Y$. There exist functions $u, v : M \times Y \to \mathbb{R}^n$ so that

$$\widetilde{S}(q, p, y) = \big(q + u(q, p, y), v(q, p, y), \phi(y)\big).$$

Here we slightly abuse notation and write $u(q, p, y)$ instead of $u(x, y)$ whenever $\widetilde{x} = (q, p)$. The function u which measures displacement along the q-plane on $\widetilde{M}$ plays an important role below. Note that u has compact support in $M \times Y$.

Let $\mathscr{M}(S)$ denote the set of all S-invariant Borel probability measures on $\overline{M} \times Y$. This space is convex and weak-$*$ compact (see Sec. 2.2). For a measure $\mu \in \mathscr{M}$, define its *rotation vector* $R(\mu) \in \mathbb{R}^n$ by

$$R(\mu) := \int u \, d\mu \in \mathbb{R}^n.$$

The map $R : \mathscr{M}(S) \to \mathbb{R}^n$ is affine and continuous. Hence its image $K := R(\mathscr{M}(S))$ is a compact convex subset of $\mathbb{R}^n$. This set has all the properties formulated in the theorem.

We prove that K contains a ball of radius $c - a$. Pick a vector $v \in \mathbb{R}^n$ with $|v| \leq c - a$, and pick a point $y \in Y$. Then, by Theorem A, there exists a sequence $x_k \in M$ such that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} u\big(S^i(x_k, y)\big) = v.$$

Denote by $\delta_{i,k}$ the Dirac measure on $\overline{M} \times Y$ concentrated at $S^i(x_k, y)$, and consider the sequence of measures μ_k on $\overline{M} \times Y$ defined by

$$\mu_k := \frac{1}{k} \sum_{i=0}^{k-1} \delta_{i,k}.$$

By Alaoglu's theorem, this sequence has a limit point $\mu = \lim_{\nu\to\infty} \mu_{k_\nu}$ with respect to the weak-$*$ topology. Since

$$\lim_{k\to\infty} \Big(\int F \circ S \, d\mu_k - \int F \, d\mu_k \Big) = \lim_{k\to\infty} \frac{F(S^k(x_k, y)) - F(x_k, y)}{k} = 0$$

for every continuous function $F : \overline{M} \times Y \to \mathbb{R}$, the limit point μ is S-invariant. Moreover, it satisfies

$$R(\mu) = \int u \, d\mu = \lim_{\nu\to\infty} \int u \, d\mu_{k_\nu} = \lim_{\nu\to\infty} \frac{1}{k_\nu} \sum_{i=0}^{k_\nu - 1} u\big(S^i(x_{k_\nu}, y)\big) = v.$$

Hence $v \in K$.

We prove that the extremal points of K satisfy the requirements of the theorem. Let r be an extremal point of K. Then, by Lemma 2.2.1, there exists an ergodic measure $\mu_r \in \mathscr{M}(S)$ such that $R(\mu_r) = r$. By Birkhoff's ergodic theorem, there exists a Borel set $Z_r \subset \overline{M} \times Y$ such that $\mu_r(Z_r) = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} u\big(S^i(z)\big) = \int u \, d\mu_r = r$$

for every $z \in Z_r$. Note that $Z_r \subset M \times Y$. Let $Y_r \subset Y$ be the image of Z_r under the obvious projection $\overline{M} \times Y \to Y$, and let σ_r be the pushforward of the measure μ_r (i.e., $\sigma_r(B) := \mu_r(\overline{M} \times B)$ for every Borel set $B \subset Y$). Then σ_r is a ϕ-invariant Borel probability measure on Y and so, by unique ergodicity, equals σ. Hence

$$\sigma(Y_r) = \sigma_r(Y_r) = \mu_r(\overline{M} \times Y_r) \geq \mu_r(Z_r) = 1.$$

Moreover, for every $y \in Y_r$ there exists a point $x \in M$ such that $(x, y) \in Z_r$. By definition of the set Z_r, the trajectory of the point x under the system $f_*(y)$ propagates with velocity vector r. □

Let us mention that the idea of detecting propagating trajectories by looking at limits of invariant measures sitting at noncontractible periodic orbits goes back to J. Mather's work [M].

2.4. A dissipative counterexample

Assume $n = 1$ so that $M = S^1 \times (-1, 1)$ is the annulus. Let $H : S^1 \times (-1, 1) \to \mathbb{R}$ be any compactly supported function that depends only on the momenta variable and has an arbitrarily large value at $p = 0$. Then the corresponding Hamiltonian diffeomorphism h has the form

$$h(q, p) = \big(q + H'(p), p\big),$$

where H' denotes the derivative of H. Recall that Q denotes the fundamental domain $Q := [-1/2, 1/2] \times (-1, 1)$ in the universal cover $\widetilde{M} = \mathbb{R} \times (-1, 1)$. We present an example of an arbitrarily small smooth perturbation f of h such that the images $\widetilde{f}^k(Q)$ under the iterates of $\widetilde{f}$ remain in a compact part of $\widetilde{M}$; thus the propagating behaviour disappears. Assume, without loss of generality, that the support of $H = H(p)$ is contained in the open interval $(-1/2, 1/2)$, and let $\gamma := \max_p |H'(p)|$.

Let $u : [-1, 1] \to [-1, 1]$ be an orientation-preserving diffeomorphism such that $u(s) > s$ for $-3/4 < s < 3/4$ and $u(s) = s$ for $|s| \geq 3/4$. Note that u can be chosen arbitrarily close to the identity. Choose $N \in \mathbb{N}$ such that $u^N(-2/3) > 2/3$.

Define a map $\phi : M \to M$ by $\phi(q, p) := (q, u(p))$, and let $f := \phi h$. For a point $x = (q_0, p_0) \in Q$, consider its orbit $(q_i, p_i) := \widetilde{f}^i(x)$. We claim that $|q_i - q_0| \leq (N + 1)\gamma$ for all $x \in Q$ and $i \in \mathbb{Z}$. This universal estimate means the absence of propagation.

To prove this, note that h preserves the p-coordinate and ϕ preserves the q-coordinate of each point. Moreover, the sequence p_i is nondecreasing. If $|p_0| \geq 3/4$, then $(p_i, q_i) = (p_0, q_0)$ for all $i \in \mathbb{Z}$. If $|p_0| < 3/4$, then $\lim_{i\to\pm\infty} p_i = \pm 3/4$. In this case, let $j_0 \in \mathbb{Z}$ be the largest integer such that $p_{j_0} < -2/3$, and let $j_1 > j_0$ be the smallest integer such that $p_{j_1} > 2/3$. Then $|j_1 - j_0| \leq N + 2$ and $q_i = q_{j_0}$ for $i \leq j_0$ and $q_i = q_{j_1}$ for $i \geq j_1$. For every $i \in [j_0, j_1 - 1]$, we have $|q_{i+1} - q_i| = |H'(p_i)| \leq \gamma$. Hence, for all $j, j' \in \mathbb{Z}$ such that $j' > j$,

$$|q_{j'} - q_j| \leq \sum_{i=j}^{j'-1} |q_{i+1} - q_i| \leq \sum_{i=j_0}^{j_1-1} |q_{i+1} - q_i| \leq (N + 1)\gamma.$$

3. Relative symplectic topology

Below we discuss existence and nonexistence results for noncontractible closed orbits in a more general topological framework. The main player in this section is a relative symplectic capacity—a symplectic invariant that provides a convenient language for thinking about these results. Using this language, we prove Theorems B and C as stated in the introduction.

3.1. Symplectic action

This is a preparatory section in which we set notation. Let (M, ω) be an open symplectic manifold. We assume throughout that the symplectic form ω is exact and fix a 1-form $\lambda \in \Omega^1(M)$ such that

$$d\lambda = \omega.$$

Let $S^1 := \mathbb{R}/\mathbb{Z}$, and denote the free loop space of M by $LM := C^\infty(S^1, M)$. We denote the set of free homotopy classes of loops in M by $\widetilde{\pi}_1(M)$, and for $x \in LM$ we write $[x] \in \widetilde{\pi}_1(M)$ for its free homotopy class. Given a subset $\alpha \subset \widetilde{\pi}_1(M)$, we write

$$L_\alpha M := \left\{x \in LM \,\middle|\, [x] \in \alpha\right\}.$$

We mostly consider single elements $\alpha \in \widetilde{\pi}_1(M)$; however, for some of our applications it is useful to consider more general subsets of $\widetilde{\pi}_1(M)$. We denote the space of compactly supported time-dependent 1-periodic Hamiltonian functions on M by $\mathscr{H}(M) := C_0^\infty(S^1 \times M)$. We do not distinguish in notation between the function $H \in \mathscr{H}(M)$ and its lift $H : \mathbb{R} \times M \to \mathbb{R}$. For $H \in \mathscr{H}(M)$ and $t \in \mathbb{R}$, we define $H_t : M \to \mathbb{R}$ by $H_t(x) := H(t, x)$. Every Hamiltonian function $H \in \mathscr{H}(M)$ determines a 1-periodic family of Hamiltonian vector fields $X_t = X_{t+1} \in \mathrm{Vect}(M, \omega)$ via $\iota(X_t)\omega = -dH_t$. The space of 1-periodic solutions of the corresponding Hamiltonian differential equation representing a class in the set $\alpha \subset \widetilde{\pi}_1(M)$ is denoted by

$$\mathscr{P}(H; \alpha) := \left\{x \in L_\alpha M \,\middle|\, \dot{x}(t) = X_t(x(t))\right\}. \tag{3}$$

The elements of $\mathscr{P}(H; \alpha)$ can be interpreted as the critical points of the symplectic action $\mathscr{A}_H : L_\alpha M \to \mathbb{R}$, defined by

$$\mathscr{A}_H(x) := \int_0^1 \big(H_t(x(t)) - \lambda(\dot{x}(t))\big)\, dt \tag{4}$$

for $x \in L_\alpha M$. The sole purpose of the 1-form λ is to fix a normalization of the symplectic action (which otherwise is well defined only up to an additive constant). Some of the invariants discussed in this paper do not depend on this normalization, and this is pointed out at the appropriate places. However, we do not indicate the dependence of $\mathscr{A}_H$ on λ in the notation.

Remark 3.1.1

The notation in this section differs from the one used in Section 1.2, where the Hamiltonian functions are not required to be periodic in the t-variable. Proposition 2.1.3

shows that this does not affect the class of Hamiltonian diffeomorphisms to which the theory applies. It also does not affect the value of the symplectic action at the periodic orbits. To see this, fix a compactly supported, but not necessarily periodic, Hamiltonian function $H \in C_0^\infty([0,1] \times M)$, and let $[0,1] \times M \to M : (t,x) \mapsto h_t(x)$ be the Hamiltonian isotopy generated by H. Fix any point $x_0 \in M$, and consider the path $x : [0,1] \to M$ given by

$$x(t) := h_t(x_0).$$

Let $\gamma : [0,1] \to M$ be a smooth path such that $\gamma(0)$ lies outside of the support of H and $\gamma(1) = x_0$. Differentiating the function $s \mapsto \mathscr{A}_H(x_s)$, where $x_s(t) := h_t(\gamma(s))$, we find

$$\mathscr{A}_H(x) = \int_0^1 (h^*\lambda - \lambda)\big(\dot\gamma(s)\big)\, ds.$$

Thus the action of x depends only on x_0, h, and λ.

3.2. A relative symplectic capacity

Let M be an open symplectic manifold with symplectic form $\omega = d\lambda$, and let $A \subset M$ be a compact subset. In this section we define a relative symplectic capacity

$$C(M,A) : 2^{\widetilde{\pi}_1(M)} \times [-\infty, \infty) \to [0,\infty]$$

for the pair (M,A). This capacity has the following feature: given $\alpha \subset \widetilde{\pi}_1(M)$ and $a \in \mathbb{R} \cup \{-\infty\}$ such that $C(M,A;\alpha,a) < \infty$, every $H \in \mathscr{H}(M)$ with $\inf_{S^1 \times A} H > C(M,A;\alpha,a)$ must have a 1-periodic orbit representing one of the homotopy classes in α with symplectic action $\mathscr{A}_H(x) \geq a$. Moreover, $C(M,A;\alpha,a)$ is the optimal bound for the existence of such an orbit. More precisely, for $c > 0$ let

$$\mathscr{H}_c(M,A) := \Big\{ H \in \mathscr{H}(M) \,\Big|\, \inf_{S^1 \times A} H \geq c \Big\}.$$

For a subset $\alpha \subset \widetilde{\pi}_1(M)$ and a number $a \geq -\infty$, we define the *relative symplectic capacity* $C(M,A;\alpha,a) \geq 0$ by

$$\begin{aligned} &C(M,A;\alpha,a) \\ &\qquad := \inf \Big\{ c > 0 \,\Big|\, \forall H \in \mathscr{H}_c(M,A), \exists x \in \mathscr{P}(H;\alpha) \text{ such that } \mathscr{A}_H(x) \geq a \Big\}. \end{aligned}$$

We use the convention that $\inf \emptyset = \infty$. The infimum in the definition of $C(M,A;\alpha,a)$ is always achieved (see Prop. 3.3.4).

In order to relax the notation, we identify the single element subset $\{\alpha\} \subset \widetilde{\pi}_1(M)$ with $\alpha \in \widetilde{\pi}_1(M)$. Also, for $a = -\infty$ we abbreviate

$$\begin{aligned} C(M,A;\alpha) &:= C(M,A;\alpha,-\infty) \\ &= \inf \Big\{ c > 0 \,\Big|\, \mathscr{P}(H;\alpha) \neq \emptyset \text{ for every } H \in \mathscr{H}_c(M,A) \Big\}. \end{aligned}$$

Note that the invariant $C(M, A; \alpha)$ is independent of the choice of λ. It turns out that $C(M, A; \alpha)$ is *finite* in some interesting cases. To begin, let us consider the case of $A = \text{pt}$. Then the invariant $C(M, A; \alpha)$, with $\alpha = \widetilde{\pi}_1(M)$, is analogous to the Hofer-Zehnder capacity in [HZ]. The difference is that H. Hofer and E. Zehnder considered nonnegative and time-independent Hamiltonian functions. Lalonde [L] suggested considering more general subsets A.

Here are some examples in which our capacity can be computed. Let X be a closed (i.e., compact without boundary) connected Riemannian manifold. Denote by

$$U^*X := \left\{ v \in T^*X \,\middle|\, |v| < 1 \right\} \subset T^*X$$

the open unit cotangent bundle. This manifold is equipped with the canonical symplectic form $\omega_{\text{can}} = d\lambda_{\text{can}}$ of T^*X. We identify X with the zero section of U^*X and $\widetilde{\pi}_1(X)$ with $\widetilde{\pi}_1(U^*X)$. Moreover, in the case of the standard torus $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, there is a natural isomorphism

$$\widetilde{\pi}_1(U^*\mathbb{T}^n) \cong \widetilde{\pi}_1(\mathbb{T}^n) \cong \mathbb{Z}^n.$$

We denote the Euclidean norm of an integer vector $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ by

$$|k| := \sqrt{\sum_{j=1}^{n} k_j^2}.$$

THEOREM 3.2.1

(i) *Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the flat equilateral torus with the metric induced by the standard metric on $\mathbb{R}^n$. Then, for every $k \in \mathbb{Z}^n \cong \widetilde{\pi}_1(U^*\mathbb{T}^n)$ and every $a \in \mathbb{R}$,*

$$C(U^*\mathbb{T}^n, \mathbb{T}^n; k, a) = \max\{|k|, a\}.$$

(ii) *Let X be a closed Riemannian manifold with negative sectional curvature. Then, for every $\alpha \in \widetilde{\pi}_1(X) \cong \widetilde{\pi}_1(U^*X)$ and every $a \in \mathbb{R}$,*

$$C(U^*X, X; \alpha, a) = \max\{\text{length}(\gamma_\alpha), a\},$$

where $\gamma_\alpha : S^1 \to X$ is the (unique up to time shift) closed geodesic that represents the homotopy class α.

3.3. Properties

In this section we establish some basic properties of the relative capacity and prove Theorems B and C, assuming Theorem 3.2.1(i). The following proposition is an immediate consequence of the definitions.

PROPOSITION 3.3.1 (Monotonicity)
Let $A_2 \subset A_1 \subset M_1 \subset M_2$, where M_1 is an open subset of M_2. Let $\iota_* : \widetilde{\pi}_1(M_1) \to \widetilde{\pi}_1(M_2)$ denote the map induced by the inclusion $\iota : M_1 \subset M_2$. Then, for $\alpha_i \subset \widetilde{\pi}_1(M_i)$ and $a_i \geq -\infty$, we have

$$\iota_*^{-1}(\alpha_2) \subset \alpha_1,\ a_1 \leq a_2 \implies C(M_1, A_1; \alpha_1, a_1) \leq C(M_2, A_2; \alpha_2, a_2).$$

In particular, if ι_* is injective, then for every $\alpha \subset \widetilde{\pi}_1(M_1)$ and $a_1 \leq a_2$ we have

$$C(M_1, A_1; \alpha, a_1) \leq C(M_2, A_2; \iota_*(\alpha), a_2).$$

Our next result provides a useful condition that guarantees that the relative capacity is trivial.

PROPOSITION 3.3.2 (Displacement)
Suppose there exists a compactly supported Hamiltonian isotopy $f_t : M \to M$, $0 \leq t \leq 1$, such that $f_0 = \mathrm{id}$, $f_1(A) \cap A = \emptyset$, and all 1-periodic orbits of f_t are contractible. Then, for every constant $c > 0$, there exists a Hamiltonian $H \in \mathscr{H}_c(M, A)$ such that $\mathscr{P}(H; \alpha) = \emptyset$ for every nontrivial class $\alpha \in \widetilde{\pi}_1(M)$. In particular, $C(M, A; \alpha) = \infty$ for every nontrivial class α.

Proof
Let $F \in \mathscr{H}$ be the Hamiltonian function generating f_t, and suppose that U is a neighbourhood of A such that $f_1(U) \cap U = \emptyset$. Choose a smooth function $G : M \to \mathbb{R}$ such that

$$\operatorname{supp} G \subset U, \qquad \inf_A G + \inf_{[0,1]\times M} F \geq c.$$

Let g_t denote the Hamiltonian flow of G, and consider the Hamiltonian flow $h_t := f_t g_t$. Since $f_1(U) \cap U = \emptyset$, the Hamiltonian diffeomorphisms h_1 and f_1 have the same fixed points. Hence all 1-periodic orbits of the flow h_t are contractible: they have the form $x(t) = x(t+1) = h_t(x_0) = f_t(x_0)$ for some $x_0 \in M \setminus U$. The same holds for the conjugate isotopy $\phi_t := f_1^{-1} h_t f_1$. Moreover, $\phi_1 = g_1 f_1 = \psi_1$, where $\psi_t = g_t f_t$. Hence, by Remark 2.1.1, all 1-periodic orbits of the isotopy ψ_t are contractible as well. Finally, observe that the flow ψ_t is generated by the Hamiltonian function $\Psi_t := G + F_t \circ g_t^{-1}$ which satisfies $\inf_A \Psi_t \geq c$ for $x \in A$. By Proposition 2.1.3, there is a periodic Hamiltonian in $\mathscr{H}_c(M, A)$ with the same time-1 map as Ψ_t, and, by Remark 2.1.1, the 1-periodic orbits of this periodic Hamiltonian are also contractible. □

Let us now look at the relative capacity from the geometric viewpoint. The group $\mathscr{D}$ of all compactly supported Hamiltonian diffeomorphisms of M carries the Hofer metric (see Sec. 1.1). Denote by $\mathscr{D}_c \subset \mathscr{D}$ the subset consisting of all Hamiltonian diffeomorphisms generated by functions from $\mathscr{H}_c(M, A)$. Recall that every $h \in \mathscr{D}$ has a canonical lift to the universal cover, and hence the homotopy class (in $\widetilde{\pi}_1(M)$) of a fixed point is independent of the choice of the compactly supported Hamiltonian generating h (see Rem. 2.1.1).

PROPOSITION 3.3.3 (Stability)

(i) *Suppose that $C(M, A; \alpha) \leq c < \infty$ for some nontrivial class $\alpha \in \widetilde{\pi}_1(M)$. Let $f \in \mathscr{D}_{c+a}$ be a Hamiltonian diffeomorphism, where $a > 0$. Then every compactly supported Hamiltonian diffeomorphism $h \in \mathscr{D}$ with $\rho(f, h) < a$ has a fixed point in the class α. In particular, $\rho(f, \mathrm{id}) \geq a$.*

(ii) *Suppose that*

$$\lim_{\substack{a>0\\a\to 0}} C(M, A; 0, a) = 0,$$

and let $c > 0$. Then $\rho(f, \mathrm{id}) \geq c$ for every $f \in \mathscr{D}_c$.

Proof

We prove (i). Let $f \in \mathscr{D}_{c+a}$ and $h \in \mathscr{D}$ such that $\rho(f, h) < a$. Then, by Proposition 2.1.2, $h \in \mathscr{D}_c$, and hence there exists a Hamiltonian isotopy h_t with $h_1 = h$ whose Hamiltonian function belongs to $\mathscr{H}_c(M, A)$. By definition of the relative capacity, this flow has a 1-periodic orbit in the class α. In particular, $h \neq \mathrm{id}$. This proves (i).

We prove (ii). Suppose, by contradiction, that $\rho(f, \mathrm{id}) < c$ for some $f \in \mathscr{D}_c$. Then, by Propositions 2.1.2 and 2.1.3, there exists a Hamiltonian function $H \in \mathscr{H}(M)$ that generates the identity and satisfies $\inf_{S^1 \times A} H > 0$. By assumption, there exists a constant $a > 0$ such that $C(M, A; 0, a) < \inf_{S^1 \times A} H$. Hence there exists a contractible periodic orbit $x \in \mathscr{P}(H; 0)$ with action $\mathscr{A}_H(x) \geq a$. On the other hand, since H generates the identity, every orbit is a contractible periodic orbit with action zero. This contradiction proves (ii). □

PROPOSITION 3.3.4

Let $\alpha \subset \widetilde{\pi}_1(M)$ and $a \geq -\infty$. Then every Hamiltonian $H \in \mathscr{H}(M)$ with $\inf_{S^1 \times A} H \geq C(M, A; \alpha, a)$ must have a 1-periodic orbit representing one of the homotopy classes in α with symplectic action $\mathscr{A}_H(x) \geq a$. In other words, the set $\{c > 0 \mid \forall H \in \mathscr{H}_c(M, A), \exists x \in \mathscr{P}(H; \alpha)$ such that $\mathscr{A}_H(x) \geq a\}$ is either empty or has a minimum.

Proof

Without loss of generality, we may assume that either $0 \notin \alpha$ or $a > 0$, since otherwise there is nothing to prove. (All points x outside the support of H are constant periodic orbits in the class zero and with action zero.) Choose a compact subset $K \subset M$ such that $A \subset \operatorname{Int} K$ and $S^1 \times K \supset \operatorname{supp} H$. Next, let $\sigma : M \to \mathbb{R}$ be a Hamiltonian function supported in K and such that $\sigma|_A > 0$. Consider the sequence of Hamiltonians $H_n := H + (1/n)\sigma$. Clearly, $\inf_{S^1 \times A} H_n > C(M, A; \alpha, a)$; hence, for every n, H_n has a 1-periodic orbit x_n with $[x_n] \in \alpha$ and $\mathscr{A}_{H_n}(x_n) \geq a$. Note that $x_n \subset K$ for every n due to the assumption that either $0 \notin \alpha$ or $a > 0$.

Now H_n converges to H in the C^∞-topology, and K is compact. Hence, replacing H_n by a suitable subsequence if necessary, we obtain a sequence of periodic orbits x_n which converges to a periodic orbit x of H. If follows that, for n sufficiently large, the loops x_n all represent the same homotopy class. Hence $[x] = \lim_{n\to\infty}[x_n] \in \alpha$ and $\mathscr{A}_H(x) = \lim_{n\to\infty} \mathscr{A}_{H_n}(x_n) \geq a$. $\square$

Proof of Theorem B

Due to Proposition 2.1.3 and the discussion on homotopy classes following its proof, we may assume our Hamiltonians to be defined on $S^1 \times U^*\mathbb{T}^n$ rather than on $[0, 1] \times U^*\mathbb{T}^n$. Let $H : S^1 \times U^*\mathbb{T}^n \to \mathbb{R}$ be a compactly supported Hamiltonian function, and let $k \in \mathbb{Z}^n$ be an integer vector such that

$$|k| \leq c := \inf_{S^1 \times \mathbb{T}^n} H.$$

By Theorem 3.2.1(i), $C(U^*\mathbb{T}^n, \mathbb{T}^n; k, c) = c$. It follows from the definition of our capacity and Proposition 3.3.4 that there exists a periodic solution $x \in \mathscr{P}(H; k)$ of (1) such that x represents the homotopy class k and $\mathscr{A}_H(x) \geq c$. This proves the assertion of Theorem B for periodic Hamiltonian functions. The assertion in the nonperiodic case follows from the periodic case, Proposition 2.1.3, and Remark 3.1.1. $\square$

Proof of Theorem C

Let $S \subset U^*\mathbb{T}^n$ be a non-Lagrangian section, and let $c > 0$ be any real number. It is shown in [P] and [LS] that there exists a Hamiltonian function $H : U^*\mathbb{T}^n \to \mathbb{R}$ such that the vector field X_H is nowhere tangent to S:

$$x \in S \implies X_H(x) \notin T_x S.$$

We may assume, without loss of generality, that H has compact support. Now let $\phi_t : U^*\mathbb{T}^n \to U^*\mathbb{T}^n$ denote the flow generated by X_H. Then there exists an $\varepsilon > 0$ such that

$$0 < t < \varepsilon \implies \phi_t(S) \cap S = \emptyset.$$

If δ is sufficiently small, then the only possible 1-periodic solutions of the Hamiltonian flow $t \mapsto \phi_{\delta t}$ are constant and $\phi_\delta(S) \cap S = \emptyset$. Hence, by Proposition 3.3.2, there exists, for every $c > 0$, a Hamiltonian function $F \in \mathscr{H}_c(U^*\mathbb{T}^n, S)$ such that $\mathscr{P}(F; k) = \emptyset$ for every nonzero integer vector $k \in \mathbb{Z}^n$. □

3.4. Existence of closed orbits on hypersurfaces

As a by-product of our study of the relative capacity, we obtain existence of closed orbits on hypersurfaces in various situations (see [BPS] for the proofs).

Definition

Let X be a smooth closed manifold. We say that a nontrivial free homotopy class $\alpha \in \widetilde{\pi}_1(X)$ is *symplectically essential* if there exists a domain $U \subset T^*X$ containing the zero section such that $C(U, X; \iota_*(\alpha)) < \infty$. Here $\iota_* : \widetilde{\pi}_1(X) \to \widetilde{\pi}_1(U)$ is the map induced by the inclusion $\iota : X \to U$ of the zero section into U.

Example

Let X be either $\mathbb{T}^n$ or a closed negatively curved manifold. It follows from Theorem 3.2.1 that every nontrivial homotopy class $\alpha \in \widetilde{\pi}_1(X)$ is symplectically essential.

THEOREM 3.4.1

Let X be a smooth closed manifold, and let $H : T^*X \to \mathbb{R}$ be a proper Hamiltonian bounded from below. Suppose that the sublevel set $\{H < c\}$ contains the zero section. Then for every nontrivial symplectically essential class $\alpha \in \widetilde{\pi}_1(X)$, there exists a dense subset $S_\alpha \subset (c, \infty)$ with the property that for every $s \in S_\alpha$ the level set $\{H = s\}$ contains a closed orbit whose projection to the zero section is in the class α^{-1}.

Let (M, ω) be a symplectic manifold, and let $U \subset M$ be a relatively compact domain with smooth *convex* boundary $Q = \partial\overline{U}$. Recall that this means, by definition, that there exists a Liouville vector field Y (namely, $\mathscr{L}_Y\omega = \omega$), defined on a neighbourhood of Q in M, such that Y points outside of U along Q. Note that, in this case, Q is a hypersurface of contact type since the vector field Y gives rise to a contact form $\lambda_Q = (\iota(Y)\omega)|_{TQ}$ on Q with $d\lambda_Q = \omega|_{TQ}$.

Denote by $\mathscr{L}_Q = \ker(\omega|_{TQ}) \subset TQ$ the characteristic line field of Q. The Reeb vector field R of λ_Q is a nonvanishing section of $\mathscr{L}_Q$ and so defines an orientation on $\mathscr{L}_Q$. We call this the *canonical orientation* of $\mathscr{L}_Q$. It induces an orientation on each leaf of the characteristic foliation of Q (namely, the foliation corresponding to $\mathscr{L}_Q$).

COROLLARY 3.4.2

Let X be a closed manifold, and let $U \subset T^*X$ be a relatively compact domain containing the zero section and with smooth convex boundary $Q = \partial\overline{U}$. Let $\mathscr{L}_Q$ be equipped with its canonical orientation. Then for every nontrivial symplectically es-

sential homotopy class $\alpha \in \widetilde{\pi}_1(X)$, the characteristic foliation of Q has a closed leaf $x \subset Q$ with $j_*[x] = \alpha$, where $j_* : \widetilde{\pi}_1(Q) \to \widetilde{\pi}_1(X)$ is the map induced by the composition $j : Q \subset T^*X \xrightarrow{\pi} X$.

Remark 3.4.3
We were informed by C. Viterbo that a similar result should also follow from the variational techniques of [HV].

3.5. Topological applications
Definition (Manifolds of type $\mathscr{F}$)
Let X be a smooth closed manifold. We say that X is of *type* $\mathscr{F}$ if there exist two *nontrivial* free homotopy classes $\alpha_1, \alpha_2 \in \widetilde{\pi}_1(X)$ such that

(i) α_1 and α_2 are not *positively proportional*; namely, there exist no $k_1, k_2 \in \mathbb{N}$ with $\alpha_1^{k_1} = \alpha_2^{k_2}$;

(ii) α_1 and α_2 are both symplectically essential; it is easy to see that this is equivalent to existence of one domain $U \subset T^*X$ containing the zero section and such that both $C(U, X; \iota_*(\alpha_1))$ and $C(U, X; \iota_*(\alpha_2))$ are finite; here ι_* is the map induced by the inclusion of zero section X into U.

Examples
(i) Let X be a closed negatively curved manifold. Then X is of type $\mathscr{F}$. To see this, first note that X is not simply connected because the universal cover is diffeomorphic to Euclidean space. Now let $\alpha \in \widetilde{\pi}_1(X)$ be a nontrivial free homotopy class, and denote by γ the unique (up to parametrization) closed geodesic representing the class α. We claim that α and α^{-1} are not positively proportional. Suppose otherwise that there exist positive integers k and ℓ such that $\alpha^k = \alpha^{-\ell}$. Then the kth and ℓth iterates γ^k and $\gamma^{-\ell}$ of γ and γ^{-1}, respectively, are two different geodesics representing the same free homotopy class. This is impossible for negatively curved manifolds.

Now let $\alpha_1, \alpha_2 \in \widetilde{\pi}_1(X)$ be two nontrivial positively nonproportional classes. By Theorem 3.2.1(ii), $C(U^*X, X; \alpha_1)$ and $C(U^*X, X; \alpha_2)$ are finite; hence X is of type $\mathscr{F}$.

(ii) It follows from Theorem 3.2.1(i) that $\mathbb{T}^n$ is also of type $\mathscr{F}$.

3.5.1. Applications to Hamiltonian circle actions
THEOREM 3.5.1.1
Let (M, ω) be a compact symplectic manifold (possibly with boundary) with $\dim_{\mathbb{R}} M \geq 4$, and let $L \subset \mathrm{Int}(M)$ be a compact connected Lagrangian submanifold. Suppose that $M \setminus L$ admits a Hamiltonian circle action with a surjective moment map $\mu : M \setminus L \to [r, R)$ that is proper onto its image (R may possibly be ∞). Then L

cannot be of type $\mathscr{F}$. In particular, L cannot be diffeomorphic to $\mathbb{T}^n$, $n \geq 2$, or to any negatively curved manifold.

Note that Theorem 3.5.1.1, as stated, is not true when $\dim_{\mathbb{R}} M = 2$. For example, take $M = U^*S^1$, and let $L = S^1$ be the zero section. Then $U^*S^1 \setminus S^1$ is diffeomorphic to $S^1 \times \big((-1, 0) \cup (0, 1)\big)$ and has a circle action with moment map $\mu(q, p) = -|p|$ which satisfies all the assumptions of Theorem 3.5.1.1.

The crucial difference between dimension two and higher ones is that in dimension two, $M \setminus L$ might be disconnected, whereas in higher dimension this never happens. Indeed, the following 2-dimensional version of Theorem 3.5.1.1 holds.

THEOREM 3.5.1.2
Let (M, ω) be a compact symplectic 2-manifold (possibly with boundary), and let $S^1 \approx L \subset \mathrm{Int}(M)$ be an embedded circle. Suppose that $M \setminus L$ admits a Hamiltonian circle action with a surjective moment map $\mu : M \setminus L \to [r, R)$ that is proper onto its image (R may possibly be ∞). Then $M \setminus L$ is disconnected.

Let us remark that in Theorems 3.5.1.1 and 3.5.1.2 it is impossible to drop the assumption that μ is proper onto its image and that its image is a half-open interval. Indeed, take $M = \mathbb{T}^2 \approx S^1 \times S^1$ and $L = \mathrm{pt} \times S^1$. Then $M \setminus L$ is diffeomorphic to $S^1 \times (0, 1)$ and so is connected. It has an obvious circle action whose moment map $\mu : S^1 \times (0, 1) \to (0, 1)$ is the projection onto the second factor. However, there is no Hamiltonian circle action on $S^1 \times (0, 1)$ such that the image of the moment map is a half-open interval and the moment map is proper. One can easily produce higher dimension examples as well, for example, multiplying M by another $\mathbb{T}^2$-factor and L by S^1.

We prove only Theorem 3.5.1.1. The proof of Theorem 3.5.1.2 can be found in [BPS].

Proof of Theorem 3.5.1.1
By Darboux's theorem, there exist an open relatively compact neighbourhood U_0 of L in T^*L, an open neighbourhood W_0 of L in M, and a symplectomorphism $f : (U_0, \omega_{\mathrm{can}}) \to (W_0, \omega)$ taking $L \subset U_0$ identically to $L \subset W_0$.

Note that $M \setminus U_0$ is compact, and let

$$R_0 := \max_{M \setminus W_0} \mu, \qquad U := f^{-1}(M \setminus \{\mu \leq R_0\}) \subset U_0.$$

Then $U \setminus L$ carries a circle action as well, with moment map $\mu \circ f : U \setminus L \to (R_0, R)$ which is proper onto its image. By reducing U_0 if necessary, we may assume that U is connected. Note that $U \setminus L$ is also connected because the codimension of L in U is at least two. It follows that any two S^1-orbits in $U \setminus L$ represent positively proportional homotopy classes in $\widetilde{\pi}_1(U)$.

Now let $R_0 < R_1 < R_2 < R$, and choose a smooth function $\sigma : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $\sigma(r) = 0$ for every $r \le R_1$;
- $\sigma(r) = c$ for every $r \ge R_2$, where $c > 0$ is a constant that is determined later;
- $\sigma'(r) > 0$ for every $R_1 < r < R_2$.

Let us now define a compactly supported Hamiltonian $H : U \to \mathbb{R}$ by putting $H := \sigma \circ \mu \circ f$ on $U \setminus L$, and extending H to be c on L. Note that the vector field X_H is everywhere tangent to the orbits of the circle action on $U \setminus L$, and moreover along $\mu^{-1}(R_1, R_2)$, X_H points in the direction induced by the circle action (because $\sigma' > 0$ there). Since any two orbits of the circle action represent positively proportional homotopy classes in $\widetilde{\pi}_1(U)$, it follows that any two nonconstant closed orbits of X_H must also be positively proportional in $\widetilde{\pi}_1(U)$.

Suppose now that L is of type $\mathscr{F}$. We get a contradiction by showing that, once the constant c from the definition of σ is chosen to be large enough, X_H must carry two closed orbits whose classes are positively nonproportional. Indeed, if L is of type $\mathscr{F}$, then there exist two positively nonproportional classes $\alpha_1, \alpha_2 \in \widetilde{\pi}_1(L)$ and a domain U' containing the zero section such that both capacities $C(U', L; \iota'_*(\alpha_1))$ and $C(U', L; \iota'_*(\alpha_2))$ are finite, where ι'_* is the map induced by the inclusion of the zero section L into U'. By reducing the size of the Darboux neighbourhood W_0 if necessary, we may assume that $U \subset U'$. Denote by $\iota : L \to U$ and $j : U \to U'$ the obvious inclusions, and denote by $\pi : U \to L$ the obvious projection. Then $\pi_* \iota_* = \mathrm{id}$ and hence $j_*{}^{-1}(\iota'_*(\alpha_i)) \subset \pi_*^{-1}(\alpha_i)$ for $i = 1, 2$. Hence, by monotonicity (see Prop. 3.3.1), we have

$$C\big(U, L; \pi_*{}^{-1}(\alpha_1)\big) \le C\big(U_1, L; \iota'_*(\alpha_1)\big) < \infty,$$
$$C\big(U, L; \pi_*{}^{-1}(\alpha_2)\big) \le C\big(U_2, L; \iota'_*(\alpha_2)\big) < \infty.$$

Now choose the constant c in the definition of σ to be larger than both of $C(U, L; \pi_*{}^{-1}(\alpha_1))$ and $C(U, L; \pi_*{}^{-1}(\alpha_2))$. Since $H|_L = c$, the Hamiltonian H must have two periodic orbits $x_1, x_2 \subset U$ with $\pi_*[x_1] = \alpha_1$ and $\pi_*[x_2] = \alpha_2$. As α_1 and α_2 are not positively proportional, neither are the classes $[x_1]$ and $[x_2]$. This contradicts the fact, established above, that any two periodic orbits of X_H represent proportional homotopy classes in $\widetilde{\pi}_1(U)$. □

3.5.2. Applications to Stein manifolds

Let (W, J) be a Stein manifold. Recall that a smooth function $\varphi : W \to \mathbb{R}$ is called *plurisubharmonic* if the 2-form

$$\omega_\varphi := -d(d\varphi \circ J)$$

is a J-positive symplectic form, that is, if $\omega_\phi(v, Jv) > 0$ for every nonzero tangent vector $v \in TW$. We denote by $g_\varphi(\cdot,\cdot) := \omega_\varphi(\cdot, J\cdot)$ the associated Kähler metric. Let $\varphi : W \to \mathbb{R}$ be an *exhausting plurisubharmonic function*; namely, in addition to being plurisubharmonic, φ is also proper, bounded from below, and has no critical points outside some compact subset of W. Let

$$X_\varphi := \operatorname{grad}_{g_\varphi} \varphi$$

be the gradient vector field of φ with respect to the metric g_φ. Then a simple computation shows that $\mathscr{L}_{X_\phi}\omega_\phi = \omega_\phi$, and hence the flow $X^t_\phi : W \to W$ of X_ϕ satisfies $(X^t_\varphi)^*\omega_\varphi = e^t\omega_\varphi$. Denote by Δ_φ the union of all the stable submanifolds of the flow of X_φ:

$$\Delta_\varphi := \bigcup_{p\in \mathrm{Crit}(\varphi)} W^s_p(X_\varphi). \tag{5}$$

We call Δ_φ the *associated skeleton* of φ. When φ is Morse, each stable submanifold in the union (5) is isotropic with respect to ω_φ (see [EG]); in particular, $\dim \Delta_\varphi \leq (1/2)\dim_{\mathbb{R}} W$. It is known that, for a generic exhausting plurisubharmonic function φ, Δ_φ has the structure of an isotropic CW-complex (see [B]). In rare situations it may even happen that some φ's (Morse or not) have smooth skeletons.

We now turn to a special class of Stein manifolds, namely, those obtained from removing an ample divisor from a smooth algebraic variety. More precisely, let (M, J) be a closed algebraic manifold, and let $\Sigma \subset M$ be a smooth ample divisor. It is well known that $(M \setminus \Sigma, J)$ is an affine variety and, in particular, Stein. The following theorem deals with topological restrictions on the possible smooth manifolds that may arise as skeletons of Stein manifolds of the type just mentioned.

THEOREM 3.5.2.1
Let (M, J) be a closed algebraic manifold, and let $\Sigma \subset M$ be a smooth ample divisor. If the Stein manifold $(M \setminus \Sigma, J)$ admits an exhausting plurisubharmonic function $\varphi : M \setminus \Sigma \to \mathbb{R}$ whose skeleton Δ_φ is a smooth connected Lagrangian submanifold, then Δ_φ cannot be of type $\mathscr{F}$. In particular, Δ_φ cannot be diffeomorphic to $\mathbb{T}^n$, $n \geq 2$, or to any closed negatively curved manifold.

Outline of the proof
Put $W := M \setminus \Sigma$, and endow W with the complex structure J. The proof has two steps.

Step 1. For every exhausting plurisubharmonic function $\varphi : W \to \mathbb{R}$, there exists an open relatively compact domain $W_0 \subset W$ with the following properties:
(i) $W_0 \supset \Delta_\varphi$;

(ii) the boundary $P := \partial \overline{W}_0$ is smooth, connected, and convex with respect to ω_φ;
(iii) the leaves of the characteristic foliation (with respect to ω_φ) on P are all orbits of a free circle action.

We present here only the main ideas of the proof of this statement. Full details can be found in [BPS]. The idea is the following. Since W is the complement of a smooth ample divisor, it is possible to endow W with an exhausting plurisubharmonic function φ_0 for which the above statement holds (see [B]). In order to pass from φ_0 to any given plurisubharmonic function φ, we modify φ_0 and φ at ∞ to obtain new plurisubharmonic functions $\overline{\varphi}_0$ and $\overline{\varphi}$ for which the vector fields $X_{\overline{\varphi}_0}$ and $X_{\overline{\varphi}}$ are complete. Then by the theory of Y. Eliashberg and M. Gromov [EG], the symplectic forms $\omega_{\overline{\varphi}_0}$ and $\omega_{\overline{\varphi}}$ are diffeomorphic. Hence the statement being true for φ_0 is true also for φ.

Step 2. Assume on the contrary that $\varphi : W \to \mathbb{R}$ is an exhausting plurisubharmonic function with skeleton Δ_φ which is a connected smooth manifold of type $\mathscr{F}$.

By Darboux's theorem, there exist a neighbourhood $V(\Delta_\varphi)$ of Δ_φ and a symplectic embedding $g : (V(\Delta_\varphi), \omega_\varphi) \to (T^*\Delta_\varphi, \omega_{\text{can}})$ which takes $\Delta_\varphi \subset V(\Delta_\varphi)$ identically onto the zero section $\Delta_\varphi \subset T^*\Delta_\varphi$.

Let $X_\varphi := \operatorname{grad}_{g_\varphi} \varphi$, denote by X_φ^t the flow of X_φ, and recall that $(X_\varphi^t)^*\omega_\varphi = e^t \omega_\varphi$. Let $W_0 \subset W$ be the domain defined by step 1, and let $W_1 \subset W$ be a relatively compact domain containing $\overline{W}_0$. From the definition of Δ_φ, it follows that for $T > 0$ large enough we have $\Delta_\varphi \subset X_\varphi^{-T}(W_1) \subset V(\Delta_\varphi)$.

Denote by $Y = p(\partial/\partial p)$ the standard Liouville vector field on $T^*\Delta_\varphi$. Then its flow Y^t satisfies $(Y^t)^*\omega_{\text{can}} = e^t \omega_{\text{can}}$. Hence

$$Y^T \circ g \circ X_\varphi^{-T} : (W_1, \omega_\varphi) \to (T^*\Delta_\varphi, \omega_{\text{can}})$$

is a symplectic embedding. It follows that

$$U := Y^T \circ g \circ X_\varphi^{-T}(W_0) \subset T^*\Delta_\varphi$$

is an open relatively compact domain containing the zero section and with convex boundary $Q := Y^T \circ g \circ X_\varphi^{-T}(P)$. Moreover, by property (iii) in step 1, the leaves of the characteristic foliation on Q coincide with the orbits of a free circle action on Q. Since Q is connected, it follows that any two leaves of the characteristic foliation on Q represent positively proportional homotopy classes. On the other hand, Δ_φ is of type $\mathscr{F}$ and so, by Corollary 3.4.2, the characteristic foliation of Q contains two closed leaves whose homotopy classes are positively nonproportional. This is a contradiction. □

4. A homological capacity

In order to study and compute the relative capacity $C(M, A; \alpha)$, we define another quantity $\widehat{C}(M, A; \alpha) \geq C(M, A; \alpha)$ which captures the existence of *homologically essential* periodic orbits in given homotopy classes. Here the term "homologically essential" refers to Floer homology. The homological capacity $\widehat{C}(M, A; \alpha)$ is defined in purely Floer-homological terms. It is easier to compute and enjoy some nice functorial properties. We begin with a brief discussion of convex boundaries.

4.1. Convex boundaries

Let $(\overline{M}, \omega)$ be a compact connected symplectic manifold with convex boundary, and denote $M := \overline{M} \setminus \partial \overline{M}$. Recall (see [EG]; see also Sec. 3.4 above) that the boundary is called *convex* if there exist a vector field $X \in \mathrm{Vect}(\overline{M})$ and a neighbourhood U of $\partial \overline{M}$ such that X points out on the boundary and is dilating on U; namely, $\mathscr{L}_X \omega = \omega$ on U. Let ϕ_t denote the flow of X, suppose that $U = \{\phi_t(x) \mid x \in \partial M, -\varepsilon < t \leq 0\}$, and denote by $\xi := \ker(\iota(X)\omega|_{T\partial\overline{M}})$ the contact structure on the boundary determined by X and ω. Under these hypotheses (the existence of X and U), there is an ω-compatible almost complex structure J on $\overline{M}$ such that

$$J\xi = \xi, \qquad \omega\big(X(x), J(x)X(x)\big) = 1, \qquad D\phi_t(x)J(x) = J\big(\phi_t(x)\big)D\phi_t(x)$$

for all $x \in \partial \overline{M}$ and $t \in (-\varepsilon, 0]$. Such an almost complex structure is called *convex near the boundary*.

Consider the function $f : U \to \mathbb{R}$ given by

$$f\big(\phi_t(x)\big) := e^t$$

for $x \in \partial \overline{M}$ and $-\varepsilon < t \leq 0$. A simple computation shows that its gradient with respect to the metric $\omega(\cdot, J\cdot)$ is X, and hence $-d(df \circ J) = \omega$, which implies that f is plurisubharmonic on U. Let $u : \Omega \to U$ be a nonconstant J-holomorphic curve, defined on a connected open subset $\Omega \subset \mathbb{C}$. Then the function $f \circ u : \Omega \to \mathbb{R}$ is subharmonic and, by the mean-value inequality, cannot have a strict interior maximum. Hence a nonconstant J-holomorphic curve in $\overline{M}$ cannot intersect $\partial \overline{M}$ at an interior point of its domain Ω.

Remark 4.1.1

The space of almost complex structures on $\overline{M}$ which are convex near the boundary is connected. To see this, fix first the dilating vector field X on a neighbourhood of $\partial \overline{M}$. Then the space of ω-compatible almost complex structures on the symplectic bundle $\xi = \ker(\iota(X)\omega|_{T\partial\overline{M}})$ is connected (see [MS]); hence the space of ω-compatible almost complex structures J as above is also connected. Finally, we may allow X to vary since the space of vector fields that are dilating near $\partial \overline{M}$ and point out on $\partial \overline{M}$ is convex and hence connected.

4.2. The setting

From now on, our standing hypotheses are that $(\overline{M}, \omega)$ is a compact connected symplectic manifold with convex boundary $\partial\overline{M}$ and $A \subset M := \overline{M} \setminus \partial\overline{M}$ is a compact subset. We assume that the symplectic form is exact and fix a 1-form $\lambda \in \Omega^1(\overline{M})$ such that $d\lambda = \omega$. We call λ an *ω-primitive*.

As in Section 3.1, we denote by $\mathscr{H} := \mathscr{H}(M) := C_0^\infty(S^1 \times M)$ the space of smooth compactly supported Hamiltonian functions on $S^1 \times M$, and we denote by $\mathscr{D} \subset \mathrm{Diff}_0(M)$ the group of Hamiltonian diffeomorphisms of M generated by functions from $\mathscr{H}$. For $c > 0$ we denote by $\mathscr{H}_c = \mathscr{H}_c(M, A)$ the subspace of all Hamiltonian functions $H \in \mathscr{H}$ which satisfy $\inf_{S^1 \times A} H \geq c$, and we denote by $\mathscr{D}_c = \mathscr{D}_c(M, A)$ the set of all Hamiltonian diffeomorphisms that are generated by functions from $\mathscr{H}_c$.

4.3. Floer homology

Floer homology is an essential ingredient in the definition of our invariants. The purpose of this section is to summarize the main building blocks of this theory needed for our applications. The reader is referred to [F4], [FH2], [CFH], and [V] for a detailed foundation of the subject (see also [S] for a general exposition).

Fix a *nontrivial* free homotopy class $\alpha \in \widetilde{\pi}_1(M)$, and recall from Section 3.1 that $\mathscr{P}(H;\alpha) \subset L_\alpha M$ denotes the set of 1-periodic solutions of the Hamiltonian system associated to $H \in \mathscr{H}$ which represents the class α. These periodic solutions are the critical points of the symplectic action functional $\mathscr{A}_H : L_\alpha M \to \mathbb{R}$ defined by (4). The set of critical values of $\mathscr{A}_H$ is called the *action spectrum* and is denoted by

$$\mathscr{S}(H;\alpha) := \mathscr{A}_H\big(\mathscr{P}(H;\alpha)\big) = \big\{\mathscr{A}_H(x) \,\big|\, x \in L_\alpha M,\ \dot{x}(t) = X_{H_t}(x(t))\big\}.$$

Here $X_{H_t} \in \mathrm{Vect}(M)$ is given by $\iota(X_{H_t})\omega = -dH_t$. Now let $-\infty \leq a < b \leq \infty$, and denote by $\mathscr{P}^{[a,b)}(H;\alpha)$ the set of 1-periodic solutions of the Hamiltonian system associated to H which represents the class α and whose action lies in the interval $[a, b)$:

$$\mathscr{P}^{[a,b)}(H;\alpha) := \mathscr{P}^b(H;\alpha) \setminus \mathscr{P}^a(H;\alpha),$$
$$\mathscr{P}^a(H;\alpha) := \big\{x \in \mathscr{P}(H;\alpha) \,\big|\, \mathscr{A}_H(x) < a\big\}.$$

Suppose that $H \in \mathscr{H}$ is a Hamiltonian function that satisfies the following hypothesis:

$$a, b \notin \mathscr{S}(H;\alpha) \textit{ and every } 1\textit{-periodic orbit } x \in \mathscr{P}(H;\alpha) \textit{ is nondegenerate.} \qquad \text{(H)}$$

Then the Floer homology group $\mathrm{HF}^{[a,b)}(H;\alpha)$ is defined as the homology of a chain complex over $\mathbb{Z}_2$ generated by the 1-periodic orbits in $\mathscr{P}^{[a,b)}(H;\alpha)$.* It is useful to

*We use the convention that the complex generated by the empty set is zero.

think of this chain complex as the quotient

$$\mathrm{CF}^{[a,b)}(H;\alpha) := \mathrm{CF}^{b}(H;\alpha)/\,\mathrm{CF}^{a}(H;\alpha), \quad \mathrm{CF}^{a}(H;\alpha) := \bigoplus_{x \in \mathscr{P}^{a}(H;\alpha)} \mathbb{Z}_2 x.$$

The Floer boundary operator is defined as follows. Let $J_t = J_{t+1} \in \mathscr{J}(M,\omega)$ be a t-dependent smooth family of ω-compatible almost complex structures on $\overline{M}$ such that J_t is convex and independent of t near the boundary $\partial\overline{M}$ (see Sec. 4.1). Consider the Floer differential equation

$$\partial_s u + J_t(u)\big(\partial_t u - X_{H_t}(u)\big) = 0. \tag{6}$$

For a smooth solution $u : \mathbb{R} \times S^1 \to M$ of (6), define its energy to be

$$E(u) := \int_0^1 \int_{-\infty}^{\infty} |\partial_s u|^2 \, ds \, dt.$$

Then if $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) with finite energy, then the limits

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t), \qquad \lim_{s \to \pm\infty} \partial_s u(s,t) = 0 \tag{7}$$

exist and are uniform in the t-variable. Moreover, $x^{\pm} \in \mathscr{P}(H)$, and we have

$$E(u) = \mathscr{A}_H(x^-) - \mathscr{A}_H(x^+). \tag{8}$$

The following observations allow us to define Floer homology groups in the present situation.

(i) Since every periodic solution $x \in \mathscr{P}(M;\alpha)$ is nonconstant (the class α is nontrivial) and J is convex near the boundary, there exists an open set $U \subset M$ such that $M \setminus U$ is compact and $u(\mathbb{R} \times S^1) \cap U = \emptyset$ for every finite-energy solution of (6).

(ii) By (i) and the energy identity (8), the space of finite-energy solutions of (6) is compact with respect to C^∞-convergence on compact sets; that is, only the splitting into a finite sequence of adjacent Floer connecting orbits can occur in the limit. We use here the fact that solutions of (6) are genuinely pseudo-holomorphic near the boundary of $\overline{M}$ due to the compact support of H and our assumptions on J. Therefore solutions cannot touch the boundary in view of the maximum principle.

(iii) For a generic family of almost complex structures $J = \{J_t\}$ (which are convex and independent of t on U), the linearized operator for equation (6) is surjective for every finite-energy solution of (6) in the homotopy class α (see [FHS]). Such a family of almost complex structures is called *regular*, and the space of regular families of almost complex structures is denoted by $\mathscr{J}_{\mathrm{reg}}(H;\alpha)$.

For every $J \in \mathscr{J}_{\rm reg}(H;\alpha)$ and every pair $x^\pm \in \mathscr{P}(H;\alpha)$, the space $\mathscr{M}(x^-, x^+; H, J)$ of solutions of (6) and (7) is a smooth manifold whose dimension near a solution u of (6) and (7) is given by the difference of the Conley-Zehnder indices (see [SZ]) of x^- and x^+ (relative to u). The subspace of solutions of index 1 is denoted by $\mathscr{M}^1(x^-, x^+; H, J)$. For $J \in \mathscr{J}_{\rm reg}(H;\alpha)$, it follows from (i) and (ii) that the quotient $\mathscr{M}^1(x^-, x^+; H, J)/\mathbb{R}$ (modulo time shift) is a finite set for every pair $x^\pm \in \mathscr{P}(H;\alpha)$. The Floer boundary operator $\partial^{H,J}$ on $\mathrm{CF}^b(H;\alpha)$ is defined by

$$\partial^{H,J} x := \sum_{y \in \mathscr{P}^b(H;\alpha)} \#\big(\mathscr{M}^1(x, y; H, J)/\mathbb{R}\big)\, y$$

for every $x \in \mathscr{P}^b(H;\alpha)$. That this is indeed a boundary operator is proved as in A. Floer's original work [F4]. The energy identity shows that $\mathrm{CF}^a(H;\alpha)$ is a subcomplex; namely, it is invariant under the Floer boundary operator. We thus get an induced boundary operator $[\partial^{H,J}]$ on the quotient $\mathrm{CF}^{[a,b)}(H;\alpha)$. We denote the homology of the quotient complex by

$$\mathrm{HF}^{[a,b)}(H, J;\alpha) := \frac{\ker([\partial^{H,J}] : \mathrm{CF}^{[a,b)}(H;\alpha) \to \mathrm{CF}^{[a,b)}(H;\alpha))}{\mathrm{im}([\partial^{H,J}] : \mathrm{CF}^{[a,b)}(H;\alpha) \to \mathrm{CF}^{[a,b)}(H;\alpha))}.$$

These Floer homology groups are independent of the choice of the almost complex structure $J = \{J_t\}_{t\in S^1}$ in the sense that for any two almost complex structures $J_0, J_1 \in \mathscr{J}_{\rm reg}(H;\alpha)$ there is a natural isomorphism

$$\tau_{J_1 J_0} : \mathrm{HF}^{[a,b)}(H, J_0;\alpha) \to \mathrm{HF}^{[a,b)}(H, J_1;\alpha).$$

If the two almost complex structure agree near the boundary, then this follows from the standard arguments, as in Floer's original paper [F4]. (Choose a homotopy of almost complex structures $\{J_{s,t}\}$ from J_0 to J_1, independent of s and t near the boundary, and use the solutions of equation (9) with $H_{s,t} = H_t$ to construct the isomorphism between the two Floer homology groups; see also [S], [SZ].) To show that the Floer homology groups are also independent of the choice of the *convex* almost complex structure near the boundary, one can use the fact that the space of convex almost complex structures near the boundary is connected (see Rem. 4.1.1) and that the Floer chain complex associated to a regular almost complex structure remains unchanged under sufficiently small perturbations of J. The upshot is that the Floer homology groups are independent of J up to natural isomorphisms. For this reason we sometimes drop the argument J and refer to $\mathrm{HF}^{[a,b)}(H;\alpha) := \mathrm{HF}^{[a,b)}(H, J;\alpha)$ as the *Floer homology associated to H.*

4.4. Homotopy invariance

Following the work of Floer and Hofer [FH2], K. Cieliebak, Floer, and Hofer [CFH], and Viterbo [V], we describe the local isomorphisms of Floer homology in a given interval of the action spectrum. Consider the space

$$\mathscr{H}^{a,b}(M;\alpha) := \left\{H \in \mathscr{H}(M) \,\middle|\, a, b \notin \mathscr{S}(H;\alpha)\right\}$$

of all Hamiltonians $H \in \mathscr{H}$ which do not contain a and b in their action spectrum. We consider the space $\mathscr{H}$ with the strong Whitney C^∞-topology. Note that the action spectrum $\mathscr{S}(H;\alpha)$ is compact for every H and is lower semicontinuous as a multi-valued function of H (i.e., for every open neighbourhood $V \subset \mathbb{R}$ of $\mathscr{S}(H;\alpha)$, there exists a neighbourhood $\mathscr{U} \subset \mathscr{H}$ of H such that $\mathscr{S}(H';\alpha) \subset V$ for every $H' \in \mathscr{U}$). Hence the set $\mathscr{H}^{a,b}(M;\alpha)$ is open in $\mathscr{H}$. We now explain why the Floer homology groups $\mathrm{HF}^{[a,b)}(H;\alpha)$ are independent of H in every component of $\mathscr{H}^{a,b}(M;\alpha)$.

Fix a Hamiltonian function $H \in \mathscr{H}^{a,b}(M;\alpha)$, and choose a (convex) neighbourhood $\mathscr{U}$ of H such that $\mathscr{U} \subset \mathscr{H}^{a,b}(M;\alpha)$. Now suppose that $H^+, H^- \in \mathscr{U}$ satisfy (H), that is, suppose that all periodic solutions $x \in \mathscr{P}(H^\pm;\alpha)$ are nondegenerate. Connect H^- and H^+ by a smooth homotopy $\mathbb{R} \mapsto \mathscr{U} : s \mapsto H_s = \{H_{s,t}\}$ such that $H_{s,t} = H_t^-$ for $s \le -T$ and $H_{s,t} = H_t^+$ for $s \ge T$. Consider the equation

$$\partial_s u + J_{s,t}(u)\big(\partial_t u - X_{H_{s,t}}(u)\big) = 0, \tag{9}$$

where $s \mapsto \{J_{s,t}\}$ is a *regular homotopy* of families of almost complex structures. This means that $J_{s,t}$ satisfies the following conditions.

- $J_{s,t}$ is convex and independent of s and t near the boundary of $\overline{M}$.
- $J_{s,t} = J_t^-$ is regular for H_t^- for $s \le -T$.
- $J_{s,t} = J_t^+$ is regular for H_t^+ for $s \ge T$.
- The finite-energy solutions of (9) are transverse (i.e., the associated Fredholm operators are surjective) and hence form finite-dimensional moduli spaces.

The key observation is the energy identity

$$E(u) = \mathscr{A}_{H^-}(x^-) - \mathscr{A}_{H^+}(x^+) + \int_0^1 \int_{-\infty}^{\infty} (\partial_s H)\big(s,t,u(s,t)\big)\, ds\, dt \tag{10}$$

for every solution of (9) and (7). It follows from (10) that

$$\mathscr{A}_{H^+}(x^+) \le \mathscr{A}_{H^-}(x^-) + \int_{-\infty}^{\infty} \max_{S^1\times M} \partial_s H_s\, ds.$$

In particular, if the homotopy has the form $H_{s,t} := H_{0,t} + \beta(s)(H_t^+ - H_t^-)$ for a nondecreasing function $\beta : \mathbb{R} \to [0,1]$, we obtain $\partial_s H_s = \dot\beta(s)(H^+ - H^-)$ and hence

$$\mathscr{A}_{H^+}(x^+) \le \mathscr{A}_{H^-}(x^-) + \max_{S^1\times M}(H^+ - H^-). \tag{11}$$

Now choose $\varepsilon > 0$ such that

$$\mathscr{S}(H^\pm;\alpha) \cap [a-4\varepsilon, a+4\varepsilon] = \emptyset, \qquad \mathscr{S}(H^\pm;\alpha) \cap [b-4\varepsilon, b+4\varepsilon] = \emptyset,$$

and suppose that $\sup_{S^1\times M} |H^{\pm} - H| \le \varepsilon$. Then $\sup_{S^1\times M} |H^+ - H^-| \le 2\varepsilon$, and hence, by (11), the Floer chain map (see [F4], [FH2], [CFH], [V], [S], [SZ]) from $\mathrm{CF}(H^-;\alpha)$ to $\mathrm{CF}(H^+;\alpha)$ defined by the solutions of (9) preserves the subcomplexes CF^a and CF^b. The same applies to the Floer chain map from $\mathrm{CF}(H^+;\alpha)$ to $\mathrm{CF}(H^-;\alpha)$ and to the chain homotopy equivalence associated to a suitable homotopy of homotopies. Hence the solutions of (9) define a homomorphism $\mathrm{CF}^{[a,b)}(H^-;\alpha) \to \mathrm{CF}^{[a,b)}(H^+;\alpha)$ which induces an isomorphism of Floer homology whenever $H^{\pm}$ are sufficiently close to a given Hamiltonian function $H \in \mathscr{H}^{a,b}(M;\alpha)$.

Remark 4.4.1 (Local isomorphisms)
The above discussion shows that every Hamiltonian function $H \in \mathscr{H}^{a,b}(M;\alpha)$ has a neighbourhood $\mathscr{U}$ such that the Floer homology groups $\mathrm{HF}^{[a,b)}(H', J';\alpha)$, for every $H' \in \mathscr{U}$ that satisfies (H) and every regular almost complex structure $J' \in \mathscr{J}_{\mathrm{reg}}(H';\alpha)$, are naturally isomorphic. We can use these local isomorphisms to define the Floer homology groups $\mathrm{HF}^{[a,b)}(H;\alpha)$ for every Hamiltonian $H \in \mathscr{H}^{a,b}(M;\alpha)$ whether or not it satisfies (H).

Remark 4.4.2 (Contractible loops)
When $\alpha \in \widetilde{\pi}_1(M)$ is the homotopy class of the constant loops, we are not allowed to work with intervals $[a,b)$ which contain zero since the Hamiltonians we work with always have degenerate periodic orbits with action zero as they vanish at ∞. In this case, we are forced to work with either $0 < a < b \le \infty$ or $-\infty \le a < b < 0$.

Remark 4.4.3 (Composition)
We emphasize that the canonical isomorphism

$$\mathrm{HF}^{[a,b)}(H^-, J^-;\alpha) \to \mathrm{HF}^{[a,b)}(H^+, J^+;\alpha)$$

exists locally only when $H^{\pm}$ are sufficiently close to a given Hamiltonian function $H \in \mathscr{H}^{a,b}(M;\alpha)$. It is easy to construct Hamiltonian functions $H_0, H_1 \in \mathscr{H}^{a,b}(M;\alpha)$ such that $\mathrm{HF}^{[a,b)}(H_0;\alpha)$ is not isomorphic to $\mathrm{HF}^{[a,b)}(H_1;\alpha)$. If H_0 and H_1 belong to the same component of $\mathscr{H}^{a,b}(M;\alpha)$, then there is a smooth path $[0,1] \to \mathscr{H}^{a,b}(M;\alpha) : s \mapsto H_s$ connecting H_0 to H_1. Hence, in this case, $\mathrm{HF}^{[a,b)}(H_0;\alpha)$ is isomorphic to $\mathrm{HF}^{[a,b)}(H_1;\alpha)$. However, in general, the isomorphism cannot be defined directly in terms of the solutions of (9). It can be constructed only as a composition of isomorphisms

$$\mathrm{HF}^{[a,b)}(H_{s_i};\alpha) \to \mathrm{HF}^{[a,b)}(H_{s_{i+1}};\alpha)$$

for a regular homotopy, where each of these isomorphisms is defined in terms of the solutions of (9). Moreover, it is an open question if this composition along a loop $s \mapsto H_s$ with $H_0 = H_1$ is always the identity.

4.5. Monotone homotopies

Suppose that $H_0, H_1 \in \mathscr{H}^{a,b}(M;\alpha)$ satisfy

$$H_0(t,x) \geq H_1(t,x)$$

for all $(t,x) \in \mathbb{R}\times M$ as well as (H). Then there exists a homotopy $s \mapsto H_s$ from H_0 to H_1 such that $\partial_s H_s \leq 0$. We call such a homotopy of Hamiltonian functions *monotone*. In the monotone case, it follows from (10) that the Floer chain map $\mathrm{CF}(H_0;\alpha) \to \mathrm{CF}(H_1;\alpha)$, defined in terms of the solutions of (9), preserves the subcomplexes CF^a and CF^b. Hence every monotone homotopy $s \mapsto H_s$ induces a natural homomorphism

$$\sigma_{H_1H_0} : \mathrm{HF}^{[a,b)}(H_0;\alpha) \to \mathrm{HF}^{[a,b)}(H_1;\alpha).$$

We call such a homomorphism *monotone*. The standard arguments in Floer homology (see [F4], [FH2], [CFH], [V], [S], [SZ]) show that this homomorphism is independent of the choice of the monotone homotopy of Hamiltonians, used to define it, and that

$$\sigma_{H_2H_1} \circ \sigma_{H_1H_0} = \sigma_{H_2H_0}$$

whenever $H_0, H_1, H_2 \in \mathscr{H}^{a,b}(M;\alpha)$ satisfy $H_0 \geq H_1 \geq H_2$, and $\sigma_{HH} = \mathrm{id}$ for every $H \in \mathscr{H}^{a,b}(M;\alpha)$.

The homomorphism $\sigma_{H_1H_0}$ is, in general, neither injective nor surjective. For example, it may happen that during the homotopy the action of some periodic orbit of H_s leaves or enters the interval $[a,b)$. It turns out that this is the only possible reason for $\sigma_{H_1H_0}$ not to be an isomorphism. More precisely, we have the following proposition, which is an easy consequence of the theory developed in [FH1] and [CFH] (as outlined above) and appears in an explicit form in [V].

PROPOSITION 4.5.1

Let $-\infty \leq a < b \leq \infty$, $\alpha \in \widetilde{\pi}_1(M)$ be a nontrivial homotopy class, and let $H_0, H_1 \in \mathscr{H}^{a,b}(M;\alpha)$ be such that $H_0 \geq H_1$. Suppose that there exists a monotone homotopy $\{H_s\}_{0\leq s\leq 1}$ from H_0 to H_1 such that $H_s \in \mathscr{H}^{a,b}(M;\alpha)$ for every $s \in [0,1]$. Then $\sigma_{H_1H_0} : \mathrm{HF}^{[a,b)}(H_0;\alpha) \to \mathrm{HF}^{[a,b)}(H_1;\alpha)$ is an isomorphism. This continues to hold for the trivial homotopy class $\alpha = 0$ provided that $0 \notin [a,b]$.

Proof

The monotone homomorphism $\sigma_{H_{s_1}H_{s_0}}$ agrees with the local isomorphism of Section 4.4 whenever s_0 and s_1 are both sufficiently close to a number $s \in [0,1]$ such that $\mathscr{H}_s \in \mathscr{H}^{a,b}(H_s;\alpha)$. By assumption, we have $H_s \in \mathscr{H}^{a,b}(H_s;\alpha)$ for every $s \in [0,1]$. Hence we can write $\sigma_{H_1H_0}$ as a composition of finitely many isomorphisms of the form $\sigma_{H_{s_{i+1}}H_{s_i}}$, where $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$. □

4.6. Direct and inverse limits

The next step towards defining the relative capacity is to define two kinds of *symplectic homologies* especially suited for our purposes. The definitions of these invariants require the algebraic notions of direct and inverse limits. In this section we recall the basic definitions. (For more details, see [GM]; but note that below we use somewhat different conventions from theirs.)

Let $(I, \preceq)$ be a partially ordered set. Think of I as a category with precisely one morphism from i to j whenever $i \preceq j$. Let R be a commutative ring. A *partially ordered system* of R-modules over I is a functor from $(I, \preceq)$ into the category of R-modules. We write this functor as a pair (G, σ) where G assigns to each $i \in I$ an R-module G_i and σ assigns to each pair $i, j \in I$ with $i \preceq j$ a homomorphism $\sigma_{ji} : \mathrm{G}_i \to \mathrm{G}_j$ such that $\sigma_{kj} \circ \sigma_{ji} = \sigma_{ki}$ for $i \preceq j \preceq k$ and $\sigma_{ii} = \mathrm{id}$ is the identity map on G_i.

The partially ordered set $(I, \preceq)$ is called *upward directed* if for every pair $i, j \in I$ there exists an $\ell \in I$ such that $i \preceq \ell$ and $j \preceq \ell$. In this case, the functor (G, σ) is called a *directed system of R-modules*. The *direct limit* of such a directed system is defined as the quotient

$$\varinjlim \mathrm{G} := \varinjlim_{i \in I} \mathrm{G}_i := \left\{ (i, x) \,\middle|\, i \in I,\ x \in \mathrm{G}_i \right\} / \sim$$

where $(i, x) \sim (j, y)$ if and only if there exists an $\ell \in I$ such that $i \preceq \ell$, $j \preceq \ell$, and $\sigma_{\ell i}(x) = \sigma_{\ell j}(y)$. Since I is upward directed, this is an equivalence relation. The direct limit is an R-module with the operations $[i, x] + [j, y] := [\ell, \sigma_{\ell i}(x) + \sigma_{\ell j}(y)]$ for $\ell \in I$ such that $i \preceq \ell$ and $j \preceq \ell$ and $r[i, x] := [i, rx]$ for every $r \in R$. For $i \in I$ we denote by $\iota_i : \mathrm{G}_i \to \varinjlim \mathrm{G}$ the homomorphism given by $\iota_i(x) := [i, x]$. Then $\iota_i = \iota_j \circ \sigma_{ji}$ for $i \preceq j$. Despite the notation, ι_i need not be injective. Up to isomorphism, the direct limit is characterized by the following universal property. If H is any R-module and $\tau_i : \mathrm{G}_i \to H$ is a family of homomorphisms, indexed by $i \in I$, such that $\tau_i = \tau_j \circ \sigma_{ji}$ whenever $i \preceq j$, then there exists a unique homomorphism $\tau : \varinjlim \mathrm{G} \to H$ such that $\tau_i = \tau \circ \iota_i$ for every $i \in I$. (The homomorphism τ is given by $[i, x] \mapsto \tau_i(x)$.)

The partially ordered set $(I, \preceq)$ is called *downward directed* if for every pair $i, j \in I$ there exists a $k \in I$ such that $k \preceq i$ and $k \preceq j$. In this case, the functor (G, σ) is called an *inverse system of R-modules*. The *inverse limit* of such an inverse system is defined as

$$\varprojlim \mathrm{G} := \varprojlim_{i \in I} \mathrm{G}_i := \left\{ \{x_i\}_{i \in I} \in \prod_{i \in I} G_i \,\middle|\, i \preceq j \Longrightarrow \sigma_{ji}(x_i) = x_j \right\}.$$

For $j \in I$ we denote by $\pi_j : \varprojlim \mathrm{G} \to \mathrm{G}_j$ the obvious projection to the jth component. Then $\pi_j = \sigma_{ji} \circ \pi_i$ for $i \preceq j$. Despite the notation, π_j need not be surjective.

In most of our applications the partially ordered set $(I, \preceq)$ is *bidirected*, that is, both upward and downward directed. In this case, we call the functor (G, σ) a *bidirected system of R-modules*. The next lemma follows directly from the definitions.

LEMMA 4.6.1
Let $(I, \preceq)$ be a downward directed partially ordered set, and let $I' \subset I$ be an upward directed subset (with respect to the restriction of the partial order $\preceq$ to I'). Let (G, σ) be a partially ordered system of R-modules over I. Then there exists a unique homomorphism

$$T : \varprojlim_{i \in I} G_i \to \varinjlim_{i' \in I'} G_{i'}$$

such that the following diagram commutes for all $j', k' \in I'$ with $j' \preceq k'$:

$$\begin{array}{ccc} \varprojlim_{i \in I} \mathrm{G}_i & \xrightarrow{\quad T \quad} & \varinjlim_{i' \in I'} \mathrm{G}_{i'} \\ \Big\downarrow \pi_{j'} & & \Big\uparrow \iota_{k'} \\ \mathrm{G}_{j'} & \xrightarrow{\quad \sigma_{k'j'} \quad} & \mathrm{G}_{k'} \end{array}$$

In Lemma 4.6.1 we *do not* assume that for every $i \in I$ there exists an $i' \in I'$ such that $i \preceq i'$ (and indeed this condition is not satisfied in our application). If this holds, then the map T factors through π_i for every $i \in I$ and not just for $i \in I'$. However, there are examples where $\mathrm{G}_i = \{0\}$ for some $i \in I$ and $T \neq 0$.

4.7. Exhausting sequences

To compute direct and inverse limits, we introduce the notion of exhausting sequences. Let (G, σ) be a partially ordered system of R-modules over $(I, \preceq)$, and denote $\mathbb{Z}^{\pm} := \{\nu \in \mathbb{Z} \mid \pm \nu > 0\}$. A sequence $\{i_\nu\}_{\nu \in \mathbb{Z}^+}$ is called *upward exhausting* for (G, σ) if and only if the following holds:

- for every $\nu \in \mathbb{Z}^+$ we have $i_\nu \preceq i_{\nu+1}$, and $\sigma_{i_{\nu+1} i_\nu} : \mathrm{G}_{i_\nu} \to \mathrm{G}_{i_{\nu+1}}$ is an isomorphism;
- for every $i \in I$ there exists a $\nu \in \mathbb{Z}^+$ such that $i \preceq i_\nu$.

A sequence $\{i_\nu\}_{\nu \in \mathbb{Z}^-}$ is called *downward exhausting* for (G, σ) if and only if the following holds:

- for every $\nu \in \mathbb{Z}^-$ we have $i_{\nu-1} \preceq i_\nu$, and $\sigma_{i_\nu i_{\nu-1}} : \mathrm{G}_{i_{\nu-1}} \to \mathrm{G}_{i_\nu}$ is an isomorphism;
- for every $i \in I$ there exists a $\nu \in \mathbb{Z}^-$ such that $i_\nu \preceq i$.

The importance of such sequences is that they can be used to simplify computations of direct and inverse limits.

The proof of the next lemma is straightforward.

LEMMA 4.7.1
Let (G,σ) *be a partially ordered system of* R*-modules over* $(I,\preceq)$.

(i) *If* $\{i_\nu\}_{\nu\in\mathbb{Z}^+}$ *is an upward exhausting sequence for* (G,σ), *then* $(I,\preceq)$ *is upward directed and the homomorphism* $\iota_{i_\nu}:\mathrm{G}_{i_\nu}\to\varinjlim\mathrm{G}$ *is an isomorphism for every* $\nu\in\mathbb{Z}^+$.

(ii) *If* $\{i_\nu\}_{\nu\in\mathbb{Z}^-}$ *is a downward exhausting sequence for* (G,σ), *then* $(I,\preceq)$ *is downward directed and the homomorphism* $\pi_{i_\nu}:\varprojlim\mathrm{G}\to\mathrm{G}_{i_\nu}$ *is an isomorphism for every* $\nu\in\mathbb{Z}^-$.

4.8. Symplectic homology

The set $\mathscr{H}(M)$ of Hamiltonian functions on $S^1\times M$ with compact support is partially ordered by

$$H_0\preceq H_1 \iff H_0(t,x)\geq H_1(t,x),\quad \forall\,(t,x)\in S^1\times M.$$

This defines a bidirected partial order on $\mathscr{H}(M)$. Let $\alpha\in\widetilde{\pi}_1(M)$ be a nontrivial homotopy class, and let $a,b\in\mathbb{R}\cup\{\pm\infty\}$ such that $a<b$. As in Section 4.4, we denote by $\mathscr{H}^{a,b}(M;\alpha)$ the subset of all Hamiltonian functions $H\in\mathscr{H}(M)$ such that $a,b\notin\mathscr{S}(H;\alpha)$. In Section 4.5 we have seen that there is a natural homomorphism

$$\sigma_{H_1H_0}:\mathrm{HF}^{[a,b)}(H_0;\alpha)\to\mathrm{HF}^{[a,b)}(H_1;\alpha)$$

whenever $H_0,H_1\in\mathscr{H}^{a,b}(M;\alpha)$ satisfy $H_0\preceq H_1$. These homomorphisms define an inverse (in fact, bidirected) system of Floer homology groups over $(\mathscr{H}^{a,b}(M;\alpha),\preceq)$. The inverse limit of this system is called the *symplectic homology* of M in the homotopy class α for the action interval $[a,b)$. A version of this homology group was introduced in [FH2], [CFH] for the homotopy class of contractible loops and later on for general homotopy classes in [C2]. We denote it by

$$\underleftarrow{\mathrm{SH}}^{[a,b)}(M;\alpha):=\varprojlim_{H\in\mathscr{H}^{a,b}(M;\alpha)}\mathrm{HF}^{[a,b)}(H;\alpha).$$

Now fix a compact subset $A\subset M$ and a constant $c\in\mathbb{R}$. Consider the set $\mathscr{H}_c^{a,b}(M,A;\alpha)$ of all Hamiltonian functions $H\in\mathscr{H}^{a,b}(M;\alpha)$ which satisfy $H>c$ on $S^1\times A$, namely,

$$\mathscr{H}_c^{a,b}(M,A;\alpha):=\Big\{H\in\mathscr{H}^{a,b}(M;\alpha)\,\Big|\,\inf_{S^1\times A}H>c\Big\}.$$

This gives rise to a directed (in fact, bidirected) system of Floer homology groups over $(\mathscr{H}_c^{a,b}(M,A;\alpha),\preceq)$. The direct limit of this system is called the *relative symplectic homology* of the pair (M,A) at the *level* c in the homotopy class α for the action interval $[a,b)$. We denote it by

$$\underrightarrow{\mathrm{SH}}^{[a,b);c}(M,A;\alpha):=\varinjlim_{H\in\mathscr{H}_c^{a,b}(M,A;\alpha)}\mathrm{HF}^{[a,b)}(H;\alpha).$$

Remark 4.8.1
Since we have chosen to work with $\mathbb{Z}_2$-coefficients, all the Floer homology groups $\mathrm{HF}^{[a,b)}(H;\alpha)$ are, in fact, $\mathbb{Z}_2$-vector spaces. Consequently, the symplectic homologies $\underleftarrow{\mathrm{SH}}^{[a,b)}(M;\alpha)$ and $\underrightarrow{\mathrm{SH}}^{[a,b);c}(M,A;\alpha)$ also have the structure of $\mathbb{Z}_2$-vector spaces.

An important feature of absolute and relative symplectic homologies is the existence of a homomorphism between them which factors through Floer homology.

PROPOSITION 4.8.2
Let $\alpha \in \widetilde{\pi}_1(M)$ be a nontrivial homotopy class, and suppose that $-\infty \le a < b \le \infty$. Then, for every $c \in \mathbb{R}$, there exists a unique homomorphism

$$T_\alpha^{[a,b);c} : \underleftarrow{\mathrm{SH}}^{[a,b)}(M;\alpha) \to \underrightarrow{\mathrm{SH}}^{[a,b);c}(M,A;\alpha)$$

such that for any two Hamiltonian functions $H_0, H_1 \in \mathscr{H}_c^{a,b}(M,A;\alpha)$ with $H_0 \ge H_1$ the following diagram commutes:

$$\begin{array}{ccc}
\underleftarrow{\mathrm{SH}}^{[a,b)}(M;\alpha) & \xrightarrow{T_\alpha^{[a,b);c}} & \underrightarrow{\mathrm{SH}}^{[a,b);c}(M,A;\alpha) \\
\downarrow{\scriptstyle \pi_{H_0}} & & \uparrow{\scriptstyle \iota_{H_1}} \\
\mathrm{HF}^{[a,b)}(H_0;\alpha) & \xrightarrow{\sigma_{H_1,H_0}} & \mathrm{HF}^{[a,b)}(H_1;\alpha)
\end{array}$$

Here $\pi_{H_0} : \underleftarrow{\mathrm{SH}}^{[a,b)}(M;\alpha) \to \mathrm{HF}^{[a,b)}(H_0;\alpha)$ and $\iota_{H_1} : \mathrm{HF}^{[a,b)}(H_1;\alpha) \to \underrightarrow{\mathrm{SH}}^{[a,b);c}(M,A;\alpha)$ are the canonical homomorphisms introduced in Section 4.6. In particular, since $\sigma_{HH} = \mathrm{id}$ for every $H \in \mathscr{H}_c^{a,b}(M,A;\alpha)$, we have

$$\begin{array}{ccc}
\underleftarrow{\mathrm{SH}}^{[a,b)}(M;\alpha) & \xrightarrow{\quad T_\alpha^{[a,b);c}\quad} & \underrightarrow{\mathrm{SH}}^{[a,b);c}(M,A;\alpha) \\
 & {\scriptstyle \pi_H} \searrow \qquad \nearrow {\scriptstyle \iota_H} & \\
 & \mathrm{HF}^{[a,b)}(H;\alpha) &
\end{array}$$

The statements above continue to hold also for the trivial class $\alpha = 0$, provided that $0 \notin [a,b]$.

Proof
The proof follows at once from Lemma 4.6.1. □

4.9. The homological relative capacity

For every nontrivial homotopy class $\alpha \in \widetilde{\pi}_1(M)$ and every real number $c > 0$, we define the set

$$\mathcal{A}_c(M, A; \alpha) := \left\{a \in \mathbb{R} \,\middle|\, \text{the homomorphism } T_\alpha^{[a,\infty);c} \text{ does not vanish}\right\},$$

where $T_\alpha^{[a,\infty);c} : \underleftarrow{\mathrm{SH}}^{[a,\infty)}(M; \alpha) \to \underrightarrow{\mathrm{SH}}^{[a,\infty);c}(M, A; \alpha)$ is the homomorphism from Proposition 4.8.2.

For the trivial homotopy class $\alpha = 0 \in \widetilde{\pi}_1(M)$, we define $\mathcal{A}_c(M, A; 0)$ by the same formula except that we consider only real numbers $a > 0$ (for which $T_0^{[a,\infty);c} \neq 0$). The *homological relative capacity* of the pair (M, A) is the function

$$\widehat{C}(M, A) : \widetilde{\pi}_1(M) \times [-\infty, \infty) \to [0, \infty]$$

which assigns to the class $\alpha \in \widetilde{\pi}_1(M)$ and the number $a \geq -\infty$ the number

$$\widehat{C}(M, A; \alpha, a) := \inf\left\{c > 0 \,\middle|\, \sup \mathcal{A}_c(M, A; \alpha) > a\right\}.$$

Here we use the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. Roughly speaking, the quantity $\widehat{C}(M, A; \alpha, a)$ measures the minimal c for which the following is guaranteed: every Hamiltonian that is bigger than c on $S^1 \times A$ has a "homologically essential" 1-periodic orbit in the class α with action greater than or equal to a.

For $a = -\infty$ we abbreviate

$$\widehat{C}(M, A; \alpha) := \widehat{C}(M, A; \alpha, -\infty) = \inf\left\{c > 0 \,\middle|\, \mathcal{A}_c(M, A; \alpha) \neq \emptyset\right\}.$$

The latter quantity is independent of the ω-primitive λ, while $\widehat{C}(M, A; \alpha, a)$ does depend on this choice: the set $\mathcal{A}_c(M, A; \alpha) =: \mathcal{A}_c^\lambda(M, A; \alpha)$ depends on λ, but for two ω-primitives λ, λ' we have $\mathcal{A}_c^{\lambda'}(M, A; \alpha) = \mathcal{A}_c^\lambda(M, A; \alpha) - \int_\alpha (\lambda' - \lambda)$.

PROPOSITION 4.9.1

Let $\alpha \in \widetilde{\pi}_1(M)$, and let $a \in \mathbb{R}$. If $\widehat{C}(M, A; \alpha, a) < \infty$, then every compactly supported Hamiltonian H on $S^1 \times M$ with $H|_{S^1 \times A} \geq \widehat{C}(M, A; \alpha, a)$ has a 1-periodic orbit in the homotopy class α with action $\mathscr{A}_H(x) \geq a$. In particular,

$$\widehat{C}(M, A; \alpha, a) \geq C(M, A; \alpha, a).$$

Proof

Assume first that $\inf_{S^1 \times A} H > \widehat{C}(M, A; \alpha, a)$. Then, by definition of $\widehat{C}(M, A; \alpha, a)$, there exist two real numbers b and c such that

$$0 < c < \inf_{S^1 \times A} H, \qquad a < b, \qquad b \in \mathcal{A}_c(M, A; \alpha).$$

Hence, by definition of the set $\mathcal{A}_c(M, A; \alpha)$, the homomorphism

$$T_\alpha^{[b,\infty);c} : \underleftarrow{\mathrm{SH}}^{[b,\infty)}(M; \alpha) \to \underrightarrow{\mathrm{SH}}^{[b,\infty);c}(M, A; \alpha)$$

is nonzero. Now choose a sequence of Hamiltonian functions $H_i \in \mathscr{H}(M)$ such that H_i converges to H in the C^∞-topology, $b \notin \mathscr{S}(H_i; \alpha)$, and $\inf_{S^1 \times A} H_i > c$ for every i. Then $H_i \in \mathscr{H}_c^{b,\infty}(M, A; \alpha)$ and so, by Proposition 4.8.2, the nonzero homomorphism $T_\alpha^{[b,\infty);c}$ factors through the Floer homology group $\mathrm{HF}^{[b,\infty)}(H_i; \alpha)$ for every i. Hence there exists a sequence of periodic orbits $x_i \in \mathscr{P}(H_i; \alpha)$ such that $\mathscr{A}_{H_i}(x_i) > b$. Passing to a converging subsequence, we get a periodic orbit $x \in \mathscr{P}(H; \alpha)$ with $\mathscr{A}_H(x) \geq b > a$. This proves the assertion in the case of $\inf_{S^1 \times A} H > \widehat{C}(M, A; \alpha, a)$. If $\inf_{S^1 \times A} H = \widehat{C}(M, A; \alpha, a)$, the result follows by another approximation argument. □

5. Computation of the capacities

We are now in a position to compute, in certain cases, the relative symplectic homology of the unit cotangent bundle U^*X of a compact connected Riemannian manifold X without boundary. We always work with the Liouville form λ_{can} as a primitive of the canonical symplectic form ω_{can}. We consider the following two cases:

(T) $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the flat torus;

(N) X has negative sectional curvature.

In either case, we identify $\widetilde{\pi}_1(U^*X)$ with $\widetilde{\pi}_1(X)$, and in the case of the torus we identify $\widetilde{\pi}_1(\mathbb{T}^n)$ with $\mathbb{Z}^n$. More precisely, we identify $k \in \mathbb{Z}^n$ with the homotopy class of the loop $[0, 1] \to \mathbb{T}^n : t \mapsto tk + \mathbb{Z}^n$.

5.1. The main results

In this section we state the main results about the (relative) symplectic homology of open subsets of cotangent bundles and show how they can be used to establish Theorem 3.2.1(i) and (ii).

THEOREM 5.1.1

Assume (T) or (N), and consider the trivial class $\alpha = 0 \in \widetilde{\pi}_1(X)$. Then, for $a, c > 0$, we have

$$\underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X; 0) \cong H_*(X; \mathbb{Z}_2)$$

and

$$\underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X; 0) \cong \begin{cases} H_*(X; \mathbb{Z}_2) & \text{if } a \leq c, \\ 0 & \text{if } c < a. \end{cases}$$

Moreover, the homomorphism

$$T_0^{[a,\infty);c} : \underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X; 0) \to \underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X; 0)$$

is an isomorphism for $0 < a \leq c$*. In particular, for every* $a \in \mathbb{R}$*,*

$$\widehat{C}(U^*X, X; 0, a) = \max\{0, a\}.$$

THEOREM 5.1.2
Assume (T) or (N), and consider a nontrivial homotopy class $0 \neq \alpha \in \widetilde{\pi}_1(X)$*. Denote by* $\ell > 0$ *the (unique) length of the geodesics in the class* α*. Let* $P := \mathbb{T}^n$ *in the case of (T), and let* $P := S^1$ *in the case of (N). Then, for every* $a \in \mathbb{R}$ *and* $c > 0$*, we have*

$$\underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X; \alpha) \cong \begin{cases} 0 & \text{if } a < \ell, \\ H_*(P; \mathbb{Z}_2) & \text{if } a \geq \ell \end{cases}$$

and

$$\underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X; \alpha) \cong \begin{cases} H_*(P; \mathbb{Z}_2) & \text{if } 0 < a \leq c, \\ 0 & \text{if } a > c. \end{cases}$$

Moreover, the homomorphism

$$T_\alpha^{[a,\infty);c} : \underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X; \alpha) \to \underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X; \alpha)$$

is an isomorphism for $\ell \leq a \leq c$*. In particular, for every* $a \in \mathbb{R}$*,*

$$\widehat{C}(U^*X, X; \alpha, a) = \max\{\ell, a\}.$$

Theorem 5.1.2 is proved in Section 5.4. The proof of Theorem 5.1.1 is similar and hence omitted (see [BPS] for the details).

We are now in a position to prove Theorem 3.2.1.

Proof of Theorem 3.2.1
Assume that X satisfies (T) or (N). If $\alpha = 0$, we must prove that $C(U^*X, X; 0, a) = \max\{0, a\}$. To see this, note that every compactly supported Hamiltonian function $H \in \mathscr{H}(U^*X)$ has a contractible periodic orbit x with action $\mathscr{A}_H(x) = 0$ and hence $C(U^*X, X; 0, a) = 0$ whenever $a \leq 0$. If $a > 0$, then Theorem 5.1.1 asserts that $\widehat{C}(U^*X, X; 0, a) = a$ and hence

$$a = \widehat{C}(U^*X, X; 0, a) \geq C(U^*X, X; 0, a) \geq a.$$

Here the middle inequality follows from Proposition 4.9.1. To prove the right-hand inequality, let $0 < \delta < a$, and choose any Hamiltonian function $H = H(p)$ that depends only on the momenta variables and satisfies $\max H = a - \delta$. Then every contractible periodic orbit $x \in \mathscr{P}(H; 0)$ is (up to parametrization) a contractible geodesic and hence, since X satisfies (T) or (N), is constant and has action $\mathscr{A}_H(x) =$

$H(x) = a - \delta$. This shows that $C(U^*X, X; 0, a) \geq a - \delta$ for every $\delta > 0$. Thus we have proved that $C(U^*X, X; 0, a) = \max\{0, a\}$, as claimed.

Now assume $\alpha \neq 0$, and abbreviate $\ell := \ell(\gamma_\alpha)$ in the case of (N) and $\ell := |k|$ in the case of (T) with $\alpha = k \in \mathbb{Z}^n$. Then Theorem 5.1.2 asserts that $\widehat{C}(U^*X, X; \alpha, a) = \max\{\ell, a\}$ and hence

$$\max\{\ell, a\} = \widehat{C}(U^*X, X; \alpha, a) \geq C(U^*X, X; \alpha, a) \geq \max\{\ell, a\}$$

for every real number a. Again, the middle inequality follows from Proposition 4.9.1 and the rightmost inequality from an explicit construction of a Hamiltonian function. Namely, for any $\delta > 0$ choose a compactly supported function $f : [0, 1) \to \mathbb{R}$ such that

$$f(r) = \begin{cases} m - \delta & \text{for } r \text{ near } 0, \\ 0 & \text{for } r \text{ near } 1, \end{cases}$$

where $m := \max\{\ell, a\}$, and

$$f(r) < (1 - r)m, \qquad -m < f'(r) \leq 0$$

for every r. Now consider the compactly supported Hamiltonian function $H := f(|p|)$ on U^*X. Its 1-periodic solutions are reparametrized closed geodesics. The sphere bundle $|p| = r$ contains a periodic orbit x in the class α if and only if $f'(r) = -\ell$, and the action of this periodic orbit is

$$\mathscr{A}_H(x) = f(r) - rf'(r) < f(r) + rm < m$$

(see Lem. 5.3.2). If $a \leq \ell$, then $f'(r) > -\ell$ for all r; hence there is no 1-periodic solution of length ℓ and hence none in the class α. If $\ell \leq a$, then every 1-periodic solution has action $\mathscr{A}_H(x) < a$. In either case, there is no 1-periodic orbit in the class α with action at least a, and hence $C(U^*X, X; 0, a) \geq m - \delta$. Since this holds for every $\delta > 0$, we obtain $C(U^*X, X; 0, a) \geq m$, as claimed. □

5.2. Morse-Bott theory in Floer homology

Let us return to the general setting of Section 4.2, where $(\overline{M}, \omega)$ is a compact connected symplectic manifold with convex boundary, $\omega = d\lambda$ is an exact symplectic form, $M = \overline{M} \setminus \partial\overline{M}$, $\mathscr{H} = \mathscr{H}(M)$ denotes the space of compactly supported functions on $S^1 \times M$, and $\mathscr{J}$ denotes the space of 1-periodic ω-compatible almost complex structures $J_t = J_{t+1}$ on M.

A subset $P \subset \mathscr{P}(H)$ is called a *Morse-Bott manifold of periodic orbits* if the set $C_0 := \{x(0) \mid x \in P\}$ is a compact submanifold of M and $T_{x_0}C_0 = \ker(D\psi_1(x_0) - \mathbb{1})$ for every $x_0 \in C_0$.

Remark 5.2.1
The Morse-Bott condition can be reformulated as follows. First, a subset $P \subset \mathscr{P}(H)$ is a compact submanifold of the loop space LM if and only if the set $C_0 = \{x(0) \mid x \in P\}$ is a compact submanifold of M. Second, for every $x \in P$ the kernel of the linear map $D\psi_1(x(0)) - \mathbb{1}$ on $T_{x(0)}M$ is isomorphic to the space of periodic solutions of the following *linearized Hamiltonian differential equation* for vector fields $\xi(t) \in T_{x(t)}M$ along x:

$$\nabla_{\dot{x}}\xi = \nabla_{\xi} X_H(x), \qquad \xi(t+1) = \xi(t), \tag{12}$$

where ∇ stands for the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$. To see this, just note that ξ satisfies (12) if and only if $\xi(t) = D\psi_t(x(0))\xi(0)$ for all t. Note that every tangent vector of P is a solution of (12). The Morse-Bott condition can now be expressed in the form that P is a compact submanifold of LM and

$$T_x P = \left\{\xi \in \mathbb{C}^{\infty}(S^1, x^*TM) \,\middle|\, \xi \text{ satisfies (12)}\right\}.$$

We emphasize that the Hessian of the symplectic action functional $\mathscr{A}_H : LM \to \mathbb{R}$ at a critical point x is the linear operator $\xi \mapsto \nabla_{\xi}(\operatorname{grad} H) - (\nabla_{\xi} J)\dot{x} - J\nabla_{\dot{x}}\xi = J(\nabla_{\xi} X_H - \nabla_{\dot{x}}\xi)$ on $C^{\infty}(S^1, x^*TM)$ (equipped with the L^2-inner product). Hence the space of solutions of (12) is the kernel of the Hessian of $\mathscr{A}_H$ at x, and the Morse-Bott condition asserts that the kernel of the Hessian agrees with the tangent space of the critical manifold P.

THEOREM 5.2.2
Let $-\infty \le a < b \le \infty$, $\alpha \in \widetilde{\pi}_1(M)$, and $H \in \mathscr{H}^{a,b}(M;\alpha)$. Suppose that the set $P := \{x \in \mathscr{P}(H;\alpha) \mid a < \mathscr{A}_H(x) < b\}$ is a connected Morse-Bott manifold of periodic orbits. Then $\mathrm{HF}^{[a,b)}(H;\alpha) \cong H_*(P;\mathbb{Z}_2)$.

This is a version of M. Poźniak's theorem in [Po], which was originally proved in the context of Floer homology for Lagrangian intersections. In this section we explain the reduction of this theorem to Poźniak's original one.

In order to reformulate Floer homology in the Lagrangian setting, let us consider the symplectic manifold

$$\widetilde{M} := M \times M, \qquad \widetilde{\omega} := \omega \oplus (-\omega) = d\widetilde{\lambda}, \qquad \widetilde{\lambda} := \lambda \oplus (-\lambda).$$

Since $\widetilde{\omega}$ is exact, there are no nonconstant holomorphic spheres in $\widetilde{M}$ for any $\widetilde{\omega}$-compatible almost complex structure $\widetilde{J} \in \mathscr{J}(\widetilde{M}, \widetilde{\omega})$. Since $\widetilde{\lambda}$ vanishes on the diagonal $\Delta \subset M \times M$, there are also no nonconstant holomorphic disks with boundary in Δ. Hence the standard theory of Floer homology for Lagrangian intersections applies as in Floer's original work [F1], [F2], and [F3]. Given $H \in \mathscr{H}(M)$, define $\widetilde{H}_t : \widetilde{M} \to \mathbb{R}$ by

$$\widetilde{H}_t(x_0, x_1) := H_t(x_0) + H_{1-t}(x_1).$$

The Hamiltonian isotopy generated by $\widetilde{H}_t$ with respect to $\widetilde{\omega}$ is given by $\widetilde{\psi}_t(x_0, x_1) = (\psi_t(x_0), \psi_{1-t} \circ \psi_1^{-1}(x_1))$. Given $J \in \mathscr{J}(M)$, define $\widetilde{J}_t \in \mathscr{J}(\widetilde{M}, \widetilde{\omega})$ by

$$\widetilde{J}_t := \psi_t^* J_t \times \big(-(\psi_{1-t} \circ \psi_1^{-1})^* J_{1-t} \big) = (\widetilde{\psi}_t)^* \big(J_t \times (-J_{1-t}) \big) \tag{13}$$

for $0 \le t \le 1/2$. Given $u : \mathbb{R} \times S^1 \to M$, define $\widetilde{u} : \mathbb{R} \times [0, 1/2] \to \widetilde{M}$ by

$$\widetilde{u}(s,t) := \big(\psi_t^{-1}(u(s,t)), \psi_1 \circ \psi_{1-t}^{-1}(u(s, 1-t)) \big) = (\widetilde{\psi}_t)^{-1} \big(u(s,t), u(s, 1-t) \big).$$

Then u satisfies (6) if and only if $\widetilde{u}$ satisfies the Lagrangian boundary value problem

$$\partial_s \widetilde{u} + \widetilde{J}_t(\widetilde{u}) \partial_t \widetilde{u} = 0, \qquad \widetilde{u}(s, 0) \in \Delta, \qquad \widetilde{u}\Big(s, \frac{1}{2}\Big) \in \mathrm{graph}(\psi_1). \tag{14}$$

It satisfies (7) if and only if $\widetilde{u}$ satisfies

$$\lim_{s \to \pm\infty} \widetilde{u}(s,t) = \widetilde{x}^{\pm}, \qquad \lim_{s \to \pm\infty} \partial_s \widetilde{u}(s,t) = 0, \tag{15}$$

where $\widetilde{x}^{\pm} := (x^{\pm}(0), x^{\pm}(0)) \in \Delta \cap \mathrm{graph}(\psi_1)$. The solutions of (14) can be interpreted as the gradient flow lines of the action functional

$$\widetilde{\mathscr{A}}(\widetilde{x}) := -\int_0^{1/2} \widetilde{\lambda}\big(\dot{\widetilde{x}}(t)\big)\, dt$$

on the space $\widetilde{\mathscr{P}}$ of paths $\widetilde{x} : [0, 1/2] \to \widetilde{M}$ with endpoints $\tilde{x}(0) \in \Delta$ and $\tilde{x}(1/2) \in \mathrm{graph}(\psi_1)$ (with respect to the L^2-metric determined by $\widetilde{J}$). Note that the composition of $\widetilde{\mathscr{A}}$ with the map $LM \to \widetilde{\mathscr{P}} : x \mapsto \widetilde{x}$, given by $\widetilde{x}(t) := \big(\psi_t^{-1}(x(t)), \psi_1 \circ \psi_{1-t}^{-1}(x(1-t)) \big)$, agrees with $\mathscr{A}_H$. Note also that this map induces a bijection

$$\widetilde{\pi}_1(M) = \pi_0(LM) \to \pi_0(\widetilde{\mathscr{P}}) : \alpha \mapsto \widetilde{\alpha}.$$

Hence the solutions of (14) can be used to define the Floer homology groups of the pair $(\Delta, \mathrm{graph}(\psi_1))$ of Lagrangian submanifolds of $(\widetilde{M}, \widetilde{\omega})$. Moreover, the Floer homology groups defined by the solutions of (14) are independent of the choice of the (regular) almost complex structure $\widetilde{J}_t$ used to define them. More precisely, denote by $\widetilde{\mathscr{J}}$ the space of smooth functions $[0, 1/2] \to \mathscr{J}(\widetilde{M}, \widetilde{\omega}) : t \mapsto \widetilde{J}_t$ such that $\widetilde{J}_t = J \times (-J)$ near the boundary of $\widetilde{M}$, where $J \in \mathscr{J}(M, \omega)$ is convex. Given a Hamiltonian $H \in \mathscr{H}$ that satisfies (H), denote by $\widetilde{\mathscr{J}}_{\mathrm{reg}}(H)$ the set of all almost complex structures $\widetilde{J} \in \widetilde{\mathscr{J}}$ such that every finite-energy solution $\widetilde{u}$ of (14) is regular in the sense that the linearized operator along $\widetilde{u}$ is surjective. Then the solutions of (14) give rise to Lagrangian Floer homology groups $\mathrm{HF}^{[a,b)}(\Delta, \mathrm{graph}(\psi_1); \widetilde{J}, \widetilde{\alpha})$. Moreover, it follows as in [F1], [F2], and [F3] (and as outlined above) that these Floer homology groups are independent of the almost complex structure $\widetilde{J} \in \widetilde{\mathscr{J}}_{\mathrm{reg}}(H)$ used to define them. Note that if $J \in \mathscr{J}_{\mathrm{reg}}(H)$ and $\widetilde{J}$ is given by (13), then $\widetilde{J} \in \widetilde{\mathscr{J}}_{\mathrm{reg}}(H)$. Hence there is a natural isomorphism

$$\mathrm{HF}^{[a,b)}(H;\alpha) \cong \mathrm{HF}^{[a,b)}\left(\Delta, \mathrm{graph}(\psi_1); \widetilde{\alpha}\right)$$

for every $\alpha \in \widetilde{\pi}_1(M)$ and every Hamiltonian $H \in \mathscr{H}^{a,b}(M;\alpha)$, where $\widetilde{\alpha}$ is the image of α under the above homomorphism $\widetilde{\pi}_1(M) = \pi_0(LM) \to \pi_0(\widetilde{\mathscr{P}})$. The advantage of the Lagrangian approach in the present context is that we can use any (regular) family of $\widetilde{\omega}$-compatible almost complex structures $\{\widetilde{J}_t\}_{0\le t\le 1/2}$ to define the Floer homology groups, and we are not restricted to those arising from periodic families of almost complex structures on M via (13). Hence we can apply the results of Poźniak.

Let $\widetilde{J} \in \widetilde{\mathscr{J}}$ and $H \in \mathscr{H}$, and denote by $\mathbb{R} \to \mathrm{Diff}(M,\omega) : t \mapsto \psi_t$ the Hamiltonian isotopy generated by H. The *graph* of a loop $x \in LM$ is the set

$$\Gamma(x) := \left\{\left(t, \psi_t^{-1}(x(t)), \psi_1 \circ \psi_{1-t}^{-1}(x(1-t))\right) \,\middle|\, 0 \le t \le \frac{1}{2}\right\} \subset \left[0, \frac{1}{2}\right] \times \widetilde{M}.$$

For a subset $P \subset LM$, we write $\Gamma(P) := \bigcup_{x\in P}\Gamma(x)$, and for a map $\widetilde{u} : \mathbb{R} \times [0,1/2] \to \widetilde{M}$, we write

$$\Gamma(\widetilde{u}) := \left\{\left(t, \widetilde{u}(s,t)\right) \,\middle|\, s \in \mathbb{R},\ 0 \le t \le \frac{1}{2}\right\}.$$

A subset $P \subset \mathscr{P}(H)$ is called a *$\widetilde{J}$-isolated periodic set* if there exists an open neighbourhood $U \subset [0,1/2] \times \widetilde{M}$ of $\Gamma(P)$ such that the following hold:

(P1) the closure $\overline{U}$ is a compact subset of $[0,1/2] \times \widetilde{M}$;

(P2) if $\widetilde{u} : \mathbb{R} \times [0,1/2] \to \widetilde{M}$ is a finite-energy solution of (14) with $\Gamma(\widetilde{u}) \subset \overline{U}$, then there exists an $x \in P$ such that $u(s,t) = x(t)$ for every $(s,t) \in \mathbb{R}^2$.

An open neighbourhood $U \subset [0,1/2] \times \widetilde{M}$ of $\Gamma(P)$ which satisfies (P1) and (P2) is called *$\widetilde{J}$-isolating*. Note that every $\widetilde{J}$-isolated periodic set is compact.

LEMMA 5.2.3

Let $H \in \mathscr{H}(M)$. Then every Morse-Bott manifold $P \subset \mathscr{P}(H)$ of periodic orbits is a $\widetilde{J}$-isolated periodic set for every almost complex structure $\widetilde{J} \in \widetilde{\mathscr{J}}$.

Proof

We may assume, without loss of generality, that P is connected. Let $\widetilde{d}_t$ denote the distance function of the metric $\langle\cdot,\cdot\rangle_t := \widetilde{\omega}(\cdot, \widetilde{J}_t\cdot)$, and consider the open set

$$U := \left\{(t,\widetilde{x}) \,\middle|\, 0 \le t \le \frac{1}{2},\ \widetilde{x} \in \widetilde{M},\ \sup_{y\in P} \widetilde{d}_t\left(\widetilde{x}, \left(y\left(\frac{1}{2}-t\right), y\left(\frac{1}{2}+t\right)\right)\right) < \varepsilon\right\}$$
$$\subset \left[0, \frac{1}{2}\right] \times \widetilde{M}.$$

Let $\varepsilon > 0$ be sufficiently small. Then since C_0 is an isolated fixed point set for ψ_1, it follows that every $x \in \mathscr{P}(H)$ with $\Gamma(x) \subset \overline{U}$ is an element of P. Now the set U satisfies (P2) because every finite-energy solution $\widetilde{u} : \mathbb{R} \times [0, 1/2] \to \widetilde{M}$ of (14) with $\Gamma(\widetilde{u}) \subset \overline{U}$ is asymptotic to the set P as $s \to \pm\infty$. Since $\mathscr{A}_{\widetilde{H}} = \mathscr{A}_H$ is constant along P, it follows that every such solution $\widetilde{u}$ has energy $E(\widetilde{u}) = 0$ and hence has the form $\widetilde{u}(s, t) = \widetilde{x}(t) = (x(1/2 - t), x(1/2 + t)$ for some $x \in \mathscr{P}(H)$. □

LEMMA 5.2.4

Let $H \in \mathscr{H}$ and $J \in \widetilde{\mathscr{J}}$. Suppose that $P \subset \mathscr{P}(H)$ is a $\widetilde{J}$-isolated periodic set, and suppose that $U \subset [0, 1/2] \times \widetilde{M}$ is a $\widetilde{J}$-isolating neighbourhood of $\Gamma(P)$. Then there exist a compact neighbourhood $V \subset U$ of $\Gamma(P)$ and a constant $\delta > 0$ such that the following holds. If $\mathbb{R} \to \mathscr{H}(M) : s \mapsto H_s$ and $\mathbb{R} \to \mathscr{J}(\widetilde{M}) : s \mapsto \widetilde{J}_s$ are smooth homotopies such that

$$\|H_s - H\|_{C^2} + \|\widetilde{J}_s - \widetilde{J}\|_{C^1} + \|\partial_s H_s\|_{C^2} + \|\partial_s \widetilde{J}_s\|_{C^1} < \delta,$$

$\partial_s H_s = 0$, and $\partial_s \widetilde{J}_s = 0$ for $|s| \geq 1$, then every finite-energy solution $\widetilde{u} : \mathbb{R} \times [0, 1/2] \to \widetilde{M}$ of (14) with $(\widetilde{H}, \widetilde{J})$ replaced by $(\widetilde{H}_s, \widetilde{J}_s)$ satisfies

$$\Gamma(\widetilde{u}) \subset \overline{U} \implies \Gamma(\tilde{u}) \subset V.$$

Proof

Suppose, by contradiction, that there exist sequences

$$\mathbb{R} \to \mathscr{H} : s \mapsto H_s^\nu, \qquad \mathbb{R} \to \widetilde{\mathscr{J}} : s \mapsto \widetilde{J}_s^\nu, \qquad \widetilde{u}^\nu : \mathbb{R} \times [0, 1/2] \to \widetilde{M},$$

and $(s^\nu, t^\nu) \in \mathbb{R} \times [0, 1/2]$ such that the following hold:

(i) $\lim_{\nu\to\infty} \sup_{s\in\mathbb{R}} \left(\|H_s^\nu - H_s\|_{C^2} + \|\partial_s H_s^\nu\|_{C^2} + \|\widetilde{J}_s^\nu - \widetilde{J}\|_{C^1} + \|\partial_s \widetilde{J}_s^\nu\|_{C^1}\right) = 0$;

(ii) $\partial_s H_s^\nu = 0$ and $\partial_s \widetilde{J}_s^\nu = 0$ for $|s| \geq 1$;

(iii) $\widetilde{u}^\nu$ is a finite-energy solution of (14) with $(\widetilde{H}, \widetilde{J})$ replaced by $(\widetilde{H}_s^\nu, \widetilde{J}_s^\nu)$;

(iv) $\Gamma(\widetilde{u}_\nu) \subset \overline{U}$ and $\lim_{\nu\to\infty} \widetilde{u}^\nu(s^\nu, t^\nu) \in \partial U$.

Since there are no nonconstant $\widetilde{J}_t$-holomorphic spheres in $\widetilde{M}$ and no nonconstant $\widetilde{J}_t$-holomorphic disks with boundary in Δ, the first derivatives of the functions $\widetilde{u}_\nu$ are uniformly bounded. Hence, by Floer-Gromov compactness (see [F4], [G], [MS], [S]), there exists a subsequence, still denoted by $\widetilde{u}^\nu$, such that the shifted sequence $\widetilde{u}^\nu(s^\nu + s, t)$ converges in the C^1-topology on compact sets to a finite-energy solution $\widetilde{u} : \mathbb{R} \times [0, 1/2] \to \widetilde{M}$ of (14) such that $\Gamma(\widetilde{u}) \subset \overline{U}$. By taking a further subsequence, we may assume that $t^\nu \to t$, and hence $\widetilde{u}(0, t) = \lim_{\nu\to\infty} \widetilde{u}^\nu(s^\nu, t^\nu)$ satisfies $(t, \widetilde{u}(0, t)) \in \partial U \subset [0, 1/2] \times (M \setminus U)$. This contradicts (P2). □

Lemma 5.2.4 enables us to define the *local Floer homology* $\mathrm{HF}^{\mathrm{loc}}(H; P)$ of a $\widetilde{J}$-isolated periodic set $P \subset \mathscr{P}(H)$ as follows. Choose a $\widetilde{J}$-isolating neighbourhood

$U \subset S^1 \times M$ of $\Gamma(P)$, let $\delta > 0$ be as in Lemma 5.2.4, choose a Hamiltonian function H' such that all periodic solutions $x \in \mathscr{P}(H')$ are nondegenerate and $\|H' - H\|_{C^2} < \delta/4$, and choose a regular almost complex structure $\widetilde{J}' \in \widetilde{\mathscr{J}}_{\mathrm{reg}}(H')$ such that $\|\widetilde{J}' - \widetilde{J}\|_{C^1} < \delta/4$. Then, by Lemma 5.2.4, all the Floer connecting orbits of $(\widetilde{H}', \widetilde{J}')$ (i.e., solutions $\widetilde{u}'$ of (14) with $(\widetilde{H}, \widetilde{J})$ replaced by $(\widetilde{H}', \widetilde{J}')$) in $\overline{U}$ are actually contained in V. Denote the set of local periodic orbits of H' near P by

$$\mathscr{P}(H'; U) := \left\{x' \in \mathscr{P}(H') \,\middle|\, \Gamma(x') \subset U\right\},$$

and consider the local Floer chain complex

$$\mathrm{CF}^{\mathrm{loc}}(H'; U) := \bigoplus_{x' \in \mathscr{P}(H'; U)} \mathbb{Z}_2 x'.$$

The boundary operator $\partial^{H', \widetilde{J}'; U} : \mathrm{CF}^{\mathrm{loc}}(H'; U) \to \mathrm{CF}^{\mathrm{loc}}(H'; U)$ is defined by counting the index-1 solutions $\widetilde{u}'$ of (14), with $(\widetilde{H}, \widetilde{J})$ replaced by $(\widetilde{H}', \widetilde{J}')$, such that $\Gamma(\widetilde{u}') \subset U$. Since these solutions can never converge to the boundary of U, it follows that $\partial^{H', J'; U}$ is indeed a boundary operator and the local Floer homology is defined by

$$\mathrm{HF}^{\mathrm{loc}}(H', \widetilde{J}'; U) := H_*\big(\mathrm{CF}^{\mathrm{loc}}(H'; U), \partial^{H', \widetilde{J}'; U}\big).$$

The same arguments as in Floer's original theory (see [F1], [F2], [F3]) now show that this local Floer homology is independent (up to natural isomorphisms) of the isolating neighbourhood U and of the perturbations H' and $\widetilde{J}'$ used to define it. We write

$$\mathrm{HF}^{\mathrm{loc}}(H; P) := \mathrm{HF}^{\mathrm{loc}}(H', \widetilde{J}'; U)$$

for the local Floer homology in a $\widetilde{J}$-isolating neighbourhood U of $\Gamma(P)$. Strictly speaking, this is a *connected simple system* in the sense of Conley, namely, a small category whose objects are the triples $(H', \widetilde{J}'; U)$ of local perturbations and whose morphisms are the canonical (unique) isomorphisms between the local Floer homologies $\mathrm{HF}^{\mathrm{loc}}(H_0', \widetilde{J}_0', U_0)$ and $\mathrm{HF}^{\mathrm{loc}}(H_1', \widetilde{J}_1', U_1)$. The details of this construction were carried out by Poźniak [Po] in the context of Lagrangian intersections.

THEOREM 5.2.5 (Poźniak [Po])
Let $H \in \mathscr{H}(M)$, and suppose that $P \subset \mathscr{P}(H)$ is a connected Morse-Bott manifold of periodic orbits. Then $\mathrm{HF}^{\mathrm{loc}}(H; P) \cong H_*(P; \mathbb{Z}_2)$.

Proof
The local Floer homology of H near P is isomorphic to the local Floer homology of the pair of Lagrangian submanifolds $L_0 := \Delta \subset M \times M$ and $L_1 := \mathrm{graph}(\psi_1) \subset M \times M$ near their clean intersection $\Lambda := \{(x(0), x(0)) \,|\, x \in P\}$. Hence, by [Po, Th. 3.4.11], it is isomorphic to $H_*(\Lambda; \mathbb{Z}_2) \cong H_*(P; \mathbb{Z}_2)$. □

Proof of Theorem 5.2.2
Fix a 1-periodic almost complex structure $J \in \mathscr{J}$, and let $\widetilde{J} \in \widetilde{\mathscr{J}}$ be given by (13). Then, by Lemma 5.2.3, P is a $\widetilde{J}$-isolated periodic set. Let U be a $\widetilde{J}$-isolating neighbourhood of $\Gamma(P)$, and choose a sequence of regular perturbations (H^ν, J^ν) which agree with (H, J) in some neighbourhood of $\partial\overline{M}$ and converge to (H, J) in the C^2-norm. We claim that, for ν sufficiently large, all the Floer connecting orbits (i.e., solutions of (6)) for the pair (H^ν, J^ν) in the homotopy class α and the action interval $[a, b]$ are contained in U. Otherwise, there has to be a sequence u^ν of such connecting orbits passing through $M \setminus U$, and we can argue as in the proof of Lemma 5.2.4 that, in the limit $\nu \to \infty$, there must be a finite-energy solution of (6) for the pair (H, J) in the homotopy class α and the action interval $[a, b]$ which passes through $M \setminus U$. However, every such connecting orbit has the form $u(s, t) = x(t)$ for some $x \in P$ and so is contained in U. This contradiction proves the claim. Hence $\mathrm{HF}^{[a,b)}(H; \alpha) \cong \mathrm{HF}^{\mathrm{loc}}(H; P)$, and hence the result follows from Theorem 5.2.5. □

5.3. The main example
In this section we consider the case where $M = U^*X$ is the open unit cotangent bundle of a compact connected Riemannian n-manifold X without boundary which satisfies either (T) (i.e., X is a flat torus) or (N) (i.e., X has negative sectional curvature). We use the metric to identify the tangent bundle with the cotangent bundle and denote a point in U^*X by $x = (q, p)$, where $q \in X$ and $p \in T_qX$. Let $H : U^*X \to \mathbb{R}$ be a compactly supported Hamiltonian function of the form

$$H(q, p) = f(|p|),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f(r) = f(-r)$. The corresponding Hamiltonian differential equation has the form

$$\dot{q} = \frac{f'(|p|)}{|p|}p, \qquad \nabla_{\dot{q}}\, p = 0. \tag{16}$$

Here ∇ denotes the Levi-Civita connection. Since $|p|$ is constant, it follows that q is a geodesic for every solution (q, p) of (16). Moreover, since $f'(0) = 0$, the zero section $\{p = 0\}$ consists of constant solutions. There are other constant solutions $x(t) \equiv (q, p)$ whenever $f'(|p|) = 0$, but these are not relevant in the context of the present paper.

LEMMA 5.3.1
The set $P_0 := \left\{x = (p, q) : S^1 \to T^*X \mid \dot{q} \equiv 0,\ p \equiv 0\right\}$ *is a Morse-Bott manifold of periodic orbits for* H *if and only if* $f''(0) \neq 0$.

Proof
Since f is even, there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $f(r) = h(r^2)/2$. Then $h'(r^2) = f'(r)/r$ and $h'(0) = f''(0)$. So equation (16) reads

$$\dot{q} = h'(|p|^2)p, \qquad \nabla_{\dot{q}} p = 0.$$

By Remark 5.2.1, P_0 is a Morse-Bott manifold if and only if the space of periodic solutions of the linearized equation is equal to the tangent space of P_0 for every $x = (q, p) \in P_0$. This means that the space of periodic solutions of the linearized equation has the same dimension as P_0. Now the linearized equation at a constant solution with $p \equiv 0$ has the form

$$\frac{d}{dt}\hat{q} = h'(0)\hat{p} = f''(0)\hat{p}, \qquad \frac{d}{dt}\hat{p} = 0.$$

If $f''(0) \neq 0$, then the space of periodic solutions of this equation has dimension $n = \dim P_0$, and if $f''(0) = 0$, it has dimension $2n$. □

Lemma 5.3.1 continues to hold for any compact Riemannian manifold X. However, in the case of (T) or (N), every nonconstant closed geodesic is not contractible. Let us now consider a nonzero homotopy class $0 \neq \alpha \in \widetilde{\pi}_1(X)$, and let us denote by ℓ the (unique) length of the closed geodesics in the class α. The space of solutions to equation (16) which represent the class α consists of $\{(q(t), p(t))\}$, where

$$\begin{cases} q(t) \text{ is a geodesic in the class } \alpha, \text{ parametrized so that } |\dot{q}| \equiv \ell; \\ p(t) = \pm\frac{r}{\ell}\dot{q}(t), \text{ where } r > 0 \text{ is such that } f'(r) = \pm\ell. \end{cases} \tag{17}$$

Given $r > 0$ with $f'(r) = \pm\ell$, we denote

$$P^{\pm}(r, \alpha) := \left\{ x = (q, p) : S^1 \to U^*X \;\middle|\; p(t), q(t) \text{ satisfy (17)} \right\}.$$

In the case of (T), the space $P^{\pm}(r, \alpha)$ is diffeomorphic to X, and in the case of (N), it is diffeomorphic to S^1.

LEMMA 5.3.2
Assume X satisfies (T) or (N). Let $\alpha \neq 0$, let $\ell, r > 0$ with $f'(r) = \pm\ell$, and let $P^{\pm}(r, \alpha)$ be as above. Then $P^{\pm}(r, \alpha)$ is a Morse-Bott manifold of periodic orbits for H if and only if $f''(r) \neq 0$. Moreover, $\mathscr{A}_H(x) = f(r) - rf'(r) = f(r) \mp r\ell$ for every $x \in P^{\pm}(r, \alpha)$.

Proof
As in the proof of Lemma 5.3.1, it follows from Remark 5.2.1 that $P^{\pm}(r, \alpha)$ is a Morse-Bott manifold if and only if the space of periodic solutions of the linearized

equation has the same dimension as $P^{\pm}(r,\alpha)$, namely, n in the case of (T) and 1 in the case of (N). We begin by linearizing equation (16). Given a path $x:\mathbb{R}\to LU^*X: s\mapsto x_s=(q_s,p_s)$, we represent a variation of x by a pair $\hat{x}=(\hat{q},\hat{p})$ of periodic vector fields along q via $\hat{q}:=\partial_s q_s$ and $\hat{p}:=\nabla_s p_s$. Since $\partial_s|p|=|p|^{-1}\langle p,\nabla_s p\rangle$, the linearized equation has the form

$$\nabla_t\hat{q}=\frac{f'(r)}{r}\Big(\hat{p}-\Big\langle\frac{p}{r},\hat{p}\Big\rangle\frac{p}{r}\Big)+f''(r)\Big\langle\frac{p}{r},\hat{p}\Big\rangle\frac{p}{r},\qquad \nabla_t\hat{p}+\frac{f'(r)}{r}R(\hat{q},p)p=0. \quad (18)$$

Here $R\in\Omega^2(X,\mathrm{End}(TX))$ denotes the Riemann curvature tensor. Note that in the case of $f(r)=r^2/2$ we have $\nabla_t\hat{q}=\hat{p}$, and so (18) is equivalent to the standard Jacobi equation $\nabla_t\nabla_t\hat{q}+R(\hat{q},p)q=0$.

The periodic solutions of equation (18) form the kernel of the Hessian of the symplectic action (see Rem. 5.2.1). Taking the pointwise inner product of the second equation in (18) with p and using $\nabla_t p=0$, we find that $\langle p,\hat{p}\rangle$ is constant. Hence, taking the L^2-inner product of the first equation in (18) with p and using the fact that $|p|=r$, we find that every periodic solution of (18) satisfies

$$f''(r)\langle p,\hat{p}\rangle\equiv 0.$$

Moreover, taking the L^2-inner product of the second equation in (18) with $\hat{q}$ and using the first equation, we find that every periodic solution of (18) satisfies

$$\int_0^1\Big(|\hat{p}|^2-\Big\langle\frac{p}{r},\hat{p}\Big\rangle^2-\langle R(\hat{q},p)p,\hat{q}\rangle\Big)\,dt=0.$$

Now suppose that X has nonpositive sectional curvature, and suppose that $f''(r)\neq 0$. Then $\hat{p}=0$ and $\nabla_t\hat{q}=0$ for every periodic solution of (18). Hence the space of periodic solutions of (18) has dimension n in the torus case (namely, $\hat{q}$ is uniquely determined by $\hat{q}(0)\in T_{q(0)}X$) and has dimension one in the negative curvature case (namely, $\hat{q}$ is a scalar multiple of p). In both cases, it follows that the kernel of the Hessian of the symplectic action at every point $x\in P^{\pm}(r,\alpha)$ has the same dimension as $P^{\pm}(r,\alpha)$ and hence is equal to the tangent space of $P^{\pm}(r,\alpha)$ at x. This is equivalent to the Morse-Bott nondegeneracy condition (see Rem. 5.2.1). If, on the other hand, $f''(0)=0$, then the dimension of the space of periodic solutions of (18) is $n+1$ in the torus case and is 2 in the negative curvature case. □

5.4. Proof of Theorem 5.1.2

Fix a nontrivial homotopy class $\alpha\in\widetilde{\pi}_1(X)$, and let ℓ denote the length of the geodesics in this class. Moreover, fix a real number $c>0$, and choose a smooth family of real functions $\{f_s(r)\}_{s\in\mathbb{R}}$, defined for $r\in\mathbb{R}$, with the following properties (see Fig. 1).

(i) For all s and r, $f_s(-r) = f_s(r)$.

(ii) For every $s \in \mathbb{R}$,

$$f_s(0) > c, \qquad \lim_{s \to -\infty} f_s(0) = c, \qquad \lim_{s \to \infty} f_s(0) = \infty.$$

(iii) For all s and r, we have $\partial_s f_s(r) \geq 0$.

(iv) If $s \geq 1$, then

$$f_s(r) = \begin{cases} f_s(0) & \text{if } 0 \leq r \leq 1 - 3/8s, \\ 0 & \text{if } r \geq 1 - 1/8s, \end{cases}$$

$f_s'(r) \leq 0$ for all $r \geq 0$ and

$$\begin{cases} f_s''(r) < 0 & \text{if } 1 - 3/8s < r < 1 - 2/8s, \\ f_s''(r) > 0 & \text{if } 1 - 2/8s < r < 1 - 1/8s, \end{cases}$$

(v) If $s \leq -1$, then

$$f_s(r) = \begin{cases} f_s(0) & \text{if } 0 \leq r \leq 1/8|s|, \\ s & \text{if } 3/8|s| \leq r \leq 1 - 3/8|s|, \\ 0 & \text{if } r \geq 1 - 1/8|s|, \end{cases}$$

$f_s'(r) \leq 0$ for $r \leq 1/2$, $f_s'(r) \geq 0$ for $r \geq 1/2$, and

$$\begin{cases} f_s''(r) < 0 & \text{if } 1/8|s| < r < 2/8|s|, \\ f_s''(r) > 0 & \text{if } 2/8|s| < r < 3/8|s|. \end{cases}$$

(vi) For every $s \in \mathbb{R}$ such that $f_s(0) > \ell$, there exist real numbers $r_s' > r_s > 0$ such that

$$f_s'(r_s) = f_s'(r_s') = -\ell, \qquad f_s''(r_s) < 0, \qquad f_s''(r_s') > 0,$$

and $f_s'(r) \neq -\ell$ for every $r \in [0, \infty) \setminus \{r_s, r_s'\}$.

(vii) For every $s \in \mathbb{R}$, the only possible points $r > 0$ with $f_s'(r) = \ell$ must satisfy $f_s(r) < 0$.

It is not hard to show that such a family of functions $\{f_s(r)\}_{s \in \mathbb{R}}$ indeed exists. Now define $H_s : U^*X \to \mathbb{R}$ by

$$H_s(q, p) := f_s(|p|).$$

Consider first periodic orbits of H_s which belong to one of the sets $P^+(r, \alpha)$ $(r > 0)$, as defined by (17). We claim that the corresponding action is negative. Indeed, at such a value of $r > 0$, we have $f_s'(r) = \ell$, and the action is $f_s(r) - \ell r$, which is negative due to (vii).

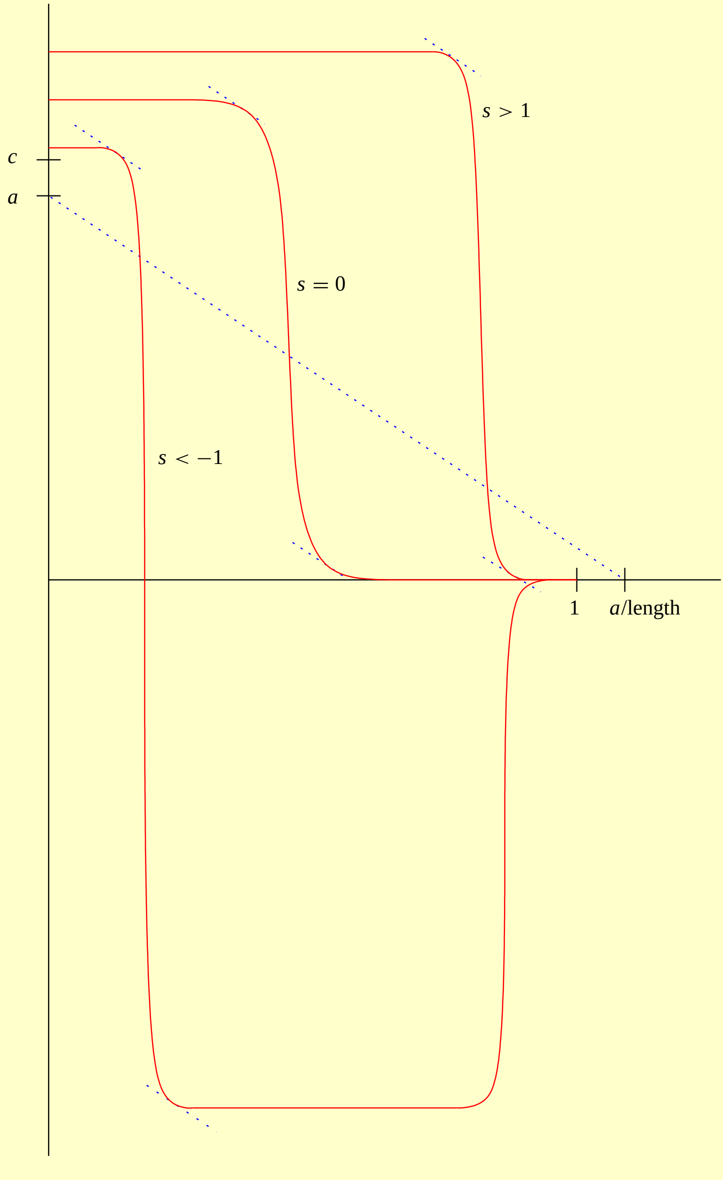

Figure 1. A family of profile functions

Next, denote by $P_s := P^-(r_s, \alpha)$ and $P_s' := P^-(r_s', \alpha)$ the other two components of the set of periodic solutions in the class α as defined by (17). Then P_s and P_s' are both diffeomorphic to $\mathbb{T}^n$ in the case of (T) and to S^1 in the case of (N). Moreover, by (vi) and Lemma 5.3.2, they are Morse-Bott manifolds of periodic orbits for H_s for every $s \in \mathbb{R}$, and the values of the symplectic action functional on these two critical manifolds are

$$\mathscr{A}_{H_s}(P_s) = f_s(r_s) + r_s\ell, \qquad \mathscr{A}_{H_s}(P_s') = f_s(r_s') + r_s'\ell.$$

Fix a real number a, and denote $P := \mathbb{T}^n$ in the case of (T) and $P := S^1$ in the case of (N). We prove Theorem 5.1.2 in five steps.

Step 1. If $a < \ell$, then $\underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X;\alpha) = 0$.

By (iv), $f_s'(r) \le 0$ for every $s \ge 1$, $r > 0$; hence for $s \ge 1$ there are no periodic orbits of the type $P^+(r,\alpha)$. Thus for $s \ge 1$ the only families of periodic orbits are P_s and P_s'. Since both r_s and r_s' converge to 1 as $s \to \infty$, it follows that the sets P_s and P_s' both have action bigger than a for s sufficiently large. Hence

$$\mathrm{HF}^{[a,\infty)}(H_s;\alpha) \cong \mathrm{HF}^{[-\infty,\infty)}(H_s;\alpha) = 0$$

for s sufficiently large. The last equation holds because $\alpha \ne 0$, so $\mathrm{HF}^{[-\infty,\infty)}(H;\alpha)$ is independent of H, and there is a Hamiltonian function with only contractible 1-periodic orbits. Now step 1 follows from Lemma 4.7.1(ii).

Step 2. If $a \ge \ell$, then $\underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X;\alpha) \cong H_*(P;\mathbb{Z}_2)$. Moreover, the homomorphism

$$\pi_s : \underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X;\alpha) \to \mathrm{HF}^{[a,\infty)}(H_s;\alpha)$$

is an isomorphism whenever $f_s(0) > a$.

As $a > \ell > 0$, we may ignore all periodic orbits of the type $P^+(r,\alpha)$ and consider only the families P_s, P_s'. The numbers r_s and r_s' are the critical points of the function $f_{s,\ell} : [0,1] \to \mathbb{R}$ given by

$$f_{s,\ell}(r) := f_s(r) + r\ell. \tag{19}$$

By (vi), the point r_s is a strict local maximum, and the point $r_s' > r_s$ is a strict local minimum. Suppose that $f_s(0) > a$. Then $f_{s,\ell}(0) = f_s(0) > a$ and $f_{s,\ell}(1) = \ell \le a$; hence it follows that $f_{s,\ell}(r_s) > a$ and $f_{s,\ell}(r_s') < a$. This means that

$$\mathscr{A}_{H_s}(P_s) > a, \qquad \mathscr{A}_{H_s}(P_s') < a.$$

Hence, by Theorem 5.2.2, $\mathrm{HF}^{[a,\infty)}(H_s;\alpha) \cong H_*(P;\mathbb{Z}_2)$, and, by Proposition 4.5.1, the monotone homomorphism $\sigma_{H_{s_1}H_{s_0}} : \mathrm{HF}^{[a,\infty)}(H_{s_0};\alpha) \to \mathrm{HF}^{[a,\infty)}(H_{s_1};\alpha)$ is an isomorphism whenever $f_{s_i}(0) > a$ for $i = 0, 1$ and $s_1 \le s_0$. Step 2 follows from Lemma 4.7.1(ii).

Step 3. If $a > c > 0$, then $\underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X;\alpha) = 0$.

As $a > 0$, we can again ignore all orbits of type $P^+(r,\alpha)$. Since both r_s and r_s' converge to zero as $s \to -\infty$, it follows that the sets P_s and P_s' both have action less than a for $-s$ sufficiently large. Hence $\mathrm{HF}^{[a,\infty)}(H_s;\alpha) = 0$ for $-s$ sufficiently large. Hence step 3 follows from Lemma 4.7.1(i).

Step 4. If $0 < a \le c$, then $\underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X;\alpha) \cong H_*(P;\mathbb{Z}_2)$. Moreover, the homomorphism

$$\iota_s : \mathrm{HF}^{[a,\infty)}(H_s;\alpha) \to \underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X;\alpha)$$

is an isomorphism for $s \ll -1$.

Since $a > 0$, we may ignore as in previous steps orbits of type $P^+(r,\alpha)$. Let $f_{s,\ell} : [0,1] \to \mathbb{R}$ be given by (19). Then, by (ii), $f_{s,\ell}(0) = f_s(0) > c \ge a$ and hence

$$\mathscr{A}_{H_s}(P_s) = f_{s,\ell}(r_s) > f_{s,\ell}(0) > a.$$

If $s < \min\{-1, a - \ell/2\}$, then $f_{s,\ell}(1/2) = s + \ell/2 < a$ and hence

$$\mathscr{A}_{H_s}(P_s') = f_{s,\ell}(r_s') < a.$$

By Theorem 5.2.2, $\mathrm{HF}^{[a,\infty)}(H_s;\alpha) \cong H_*(P_s;\mathbb{Z}_2)$ for $s < \min\{-1, a - \ell/2\}$. By Proposition 4.5.1, the monotone homomorphism $\sigma_{H_{s_1}H_{s_0}} : \mathrm{HF}^{[a,\infty)}(H_{s_0};0) \to \mathrm{HF}^{[a,\infty)}(H_{s_1};0)$ is an isomorphism for $s_1 < s_0 < \min\{-1, a - \ell/2\}$. Step 4 now follows from Lemma 4.7.1(i).

Step 5. If $\ell \le a \le c$, then the homomorphism

$$T_\alpha^{[a,\infty);c} : \underleftarrow{\mathrm{SH}}^{[a,\infty)}(U^*X;\alpha) \to \underrightarrow{\mathrm{SH}}^{[a,\infty);c}(U^*X, X;\alpha)$$

is an isomorphism.

By (ii), $f_s(0) > c \ge a$ for every s. Hence, by step 2, π_s is an isomorphism for every $s \in \mathbb{R}$. Moreover, by step 4, ι_s is an isomorphism for $s \ll -1$. By Proposition 4.8.2, $T_\alpha^{[a,\infty);c} = \iota_s \circ \pi_s$ for every s. Hence $T_\alpha^{[a,\infty);c}$ is an isomorphism.

It remains to prove the statement on $\widehat{C}(U^*X, X; \alpha, a)$. Indeed, it follows from what we proved above that $\mathcal{A}_c(U^*X, X, ; \alpha) = [\ell, c]$ for every $c > 0$. Therefore, for every $a \in \mathbb{R}$, we have

$$\widehat{C}(U^*X, X; \alpha, a) = \inf\left\{c > 0 \,\middle|\, \sup \mathcal{A}_c(U^*X, X; \alpha) > a\right\} = \max\{\ell, a\}.$$

The proof of Theorem 5.1.2 is complete. □

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