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SYMPLECTIC PACKING IN DIMENSION 4

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Abstract

We discuss closed symplectic 4-manifolds which admit full symplectic packings by N equal balls for large N's. We give a homological criterion for recognizing such manifolds. As a corollary we prove that $\mathbb{C}P^2$ can be fully packed by N equal balls for every $N \geq 9$.

1 Introduction

Let (M^4, Ω) be a closed symplectic 4-manifold. We say that (M, Ω) admits a symplectic packing by N balls of radii $\lambda_1, \dots, \lambda_N$ if there exists a symplectic embedding of the disjoint union $\coprod_{q=1}^{N}(B(\lambda_q), \omega_{std})$ into (M, Ω), where $(B(\lambda_q), \omega_{std})$ denotes the standard closed 4-ball of radius λ_q, endowed with the standard symplectic form of $\mathbb{R}^4$, $\omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

We say that (M, Ω) admits a full packing by N equal balls if the supremum of volumes which can be filled by symplectic embeddings of N disjoint equal balls equals to the volume of (M, Ω). Otherwise we say that there is a packing obstruction. Finally, we denote by $v_N(M, \Omega)$ the ratio between the supremum of the fillable volume by packings with N equal balls, and the volume of (M, Ω).

Symplectic packings were studied for the first time by Gromov in [G], and later by McDuff and Polterovich in [MP]. McDuff and Polterovich discovered that for certain manifolds there are packing obstructions. Moreover, they were able to compute v_N of certain manifolds for many values of N. In particular they computed $v_N(\mathbb{C}P^2)$ for any N which is a square and for $1 \leq N \leq 8$. It turned out that for every N which is a square there exists full packing, while for every $1 \leq N \leq 8$ which is not a square there are packing obstructions. We refer the reader to [MP] for more details about the symplectic packing problem.

In this paper we continue the above discussion, concentrating on manifolds which admit full packings by N equal balls for large enough N's. We

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give a homological condition for recognizing such manifolds, and a direct method for computing values of N_0, such that for all $N \geq N_0$ there exists full packings by N equal balls. We then work out several examples including $\mathbb{C}P^2$, which turns out to admit full packings for every $N \geq 9$.

The methods we use are based on the inflation procedure of Lalonde and McDuff and on Taubes theory of Gromov invariants via pseudo-holomorphic curves.

2 Main Results

Our main results are concerned with closed symplectic 4-manifolds of the following types:

- manifolds with $b_2^+ = 1$, $b_1 = 0$.
- ruled manifolds and their blow-ups.

Manifolds of the above types belong to a wider class known as *manifolds which do not have SW-simple type*. We shall denote this class by $\mathcal{C}$, remarking that our main results remain true for manifolds in this class.

We refer the reader to [T1,2], and to [M2,3] for the precise definition of that class. Finally, note that the class $\mathcal{C}$ is closed under the operation of blowing-up.

Given a symplectic manifold (M, Ω), we shall denote by $c_1 = c_1(TM, J)$ the first Chern class of the complex vector bundle (TM, J), where J is any almost complex structure tamed by Ω.

Definition 2.A. Let (M, Ω) be a closed symplectic 4-manifold. Consider the following set

$$\mathcal{D}_\Omega = \left\{ B \in H_2(M; \mathbb{Z}) \mid \Omega(B) > 0,\, c_1(B) \geq 2,\, B \cdot B \geq 0 \right\} .$$

Define

$$d_\Omega = \inf_{B \in \mathcal{D}_\Omega} \frac{\Omega(B)}{c_1(B)} \in [0, \infty] .$$

Here we use the convention that $\inf \emptyset = \infty$.

Before stating our main theorem we mention that given a symplectic manifold (M, Ω), its volume is defined to be $Vol(M, \Omega) = \int_M \frac{1}{2} \Omega \wedge \Omega$.

Theorem 2.B. *Let (M, Ω) be a closed symplectic 4-manifold in the class $\mathcal{C}$. Suppose that $0 < d_\Omega \leq \infty$. Then*

$$v_N(M, \Omega) \geq \min \left\{ 1, \frac{N d_\Omega^2}{2 Vol(M, \Omega)} \right\} .$$

In particular, there exists an integer N_Ω such that for every $N \geq N_\Omega$, (M,Ω) admits a full packing by N equal balls. In fact, N_Ω can be taken to be any integer which satisfies

$$N_\Omega \geq \frac{2Vol(M,\Omega)}{d_\Omega^2} .$$

The proof is given in section 4.1, where a slightly sharper result is stated and proved. See also section 6 for exact computation of v_N.

The above theorem shows that it makes sense to define the following invariant.

Definition 2.C. Let (M,Ω) be a closed symplectic 4-manifold. Define $P_{(M,\Omega)}$, the *packing number* of (M,Ω), as follows:

$$P_{(M,\Omega)} = 1+\max\{N \in \mathbb{N} \mid \text{ there does not exist a full packing by } N \text{ equal balls}\} .$$

Here we use the convention that $\max \emptyset = 0$, while max of an unbounded set is ∞. When there is no risk of confusion we shall denote the packing number of (M,Ω) by P_Ω.

As corollary to Theorem 2.B we prove:

Corollary 2.D. 1) $P_{(\mathbb{C}P^2,\sigma_{std})} = 9$, *where σ_{std} is the standard Kähler form of $\mathbb{C}P^2$.*

2) $2\frac{\beta}{\alpha} \leq P_{(S^2\times S^2,\alpha\sigma\oplus\beta\sigma)} \leq 8\frac{\beta}{\alpha}$, *where σ is the standard symplectic form of S^2 and $0 < \alpha \leq \beta$.*

3) *Let R be an orientable surface of genus $g \geq 1$. Let $\Omega = \beta\sigma_R \oplus \alpha\sigma_{S^2}$ be a symplectic form of $R \times S^2$, where σ_R, σ_{S^2} are area forms of R, S^2 respectively such that $\int_R \sigma_R = \int_{S^2} \sigma_{S^2}$. Then $P_\Omega = \lceil 2\frac{\beta}{\alpha} \rceil$. (Here, $\lceil x \rceil$ denotes the minimal integer which is greater or equal to x).*

The proof of this corollary is given in section 5.

Remark 2.E. 1) Notice that in part 3 of the above corollary, unlike in part 2, we do not assume that $\alpha \leq \beta$.

2) From parts 2 and 3 of the above corollary it follows that P_Ω is not a deformation invariant, since $P_\Omega \to \infty$ as $\alpha \to 0$

Theorem 2.F. *Let (M,Ω) be a closed symplectic minimal 4-manifold in the class $\mathcal{C}$. Suppose that (M,Ω) is not rational or ruled, then $\mathcal{D}_\Omega = \emptyset$. In particular $P_{(M,\Omega)} = 1$.*

The proof is given in section 4.2.

Examples of manifolds satisfying the conditions of the above theorem are: Barlow surfaces, Dolgachev surfaces, hyper-elliptic surfaces and Enriques surface. As explained in [MS1] and [M2] the above manifolds belong to the class $\mathcal{C}$. Since they are minimal and not rational or ruled they satisfy the conditions of Theorem 2.F.

It is worth remarking here that in some cases there exist explicit packing constructions, mainly for rational manifolds. This was done by Karshon in [K] and by Traynor in [Tr].

We conclude this section by mentioning that all symplectic packings of manifolds in the class $\mathcal{C}$ are unique in the sense that given $\lambda_1, \dots, \lambda_N$, any two symplectic embeddings of the disjoint union of balls of radii $\lambda_1, \dots, \lambda_N$ are symplectically isotopic. See [M1] for the case of $\mathbb{C}P^2$ with $N \leq 2$, [B] for the case of $\mathbb{C}P^2$ with $N \leq 6$, and [M2] for the general case.

3 Inflation and Gromov Invariants

In order to prove Theorem 2.B we need a few preliminary lemmas. For completeness we also state some relevant theorems from Taubes theory of Gromov invariants.

Let (M, Ω) be a closed symplectic 4-manifold. An Ω-symplectic exceptional sphere is by definition a symplectically embedded sphere with self intersection number -1. We shall denote by $\mathcal{E}$ the set of all 2-integral homology classes which can be represented by Ω-symplectic exceptional spheres. In what follows we shall use the notation PD for the Poincaré duality.

We first need the following theorem of McDuff which belongs to the framework of Taubes theory of Gromov invariants. For the theory of Gromov invariants see [T1,2], [M3].

Theorem 3.A. (McDuff [M2, Lemma 2.2]) *Let (M, Ω) be a closed symplectic 4-manifold in the class $\mathcal{C}$. Let $A \in H_2(M; \mathbb{Q})$ satisfy $\Omega(A) > 0$ and $A \cdot A > 0$. Then for sufficiently large n, the class nA can be represented by a (possibly disconnected) J-holomorphic curve for generic J. If in addition, for every $E \in \mathcal{E}$, $A \cdot E \geq 0$ then there exists an embedded and connected pseudo-holomorphic curve representing the class nA.*

Combining this theorem with the inflation procedure of Lalonde and McDuff one obtains the following theorem:

Theorem 3.B. (McDuff [M2]) *Let (M, Ω) be a closed symplectic 4-manifold in the class $\mathcal{C}$. Let $A \in H_2(M; \mathbb{Q})$ satisfy $\Omega(A) > 0$ and $A \cdot A > 0$. Assume that for every $E \in \mathcal{E}$ $A \cdot E \geq 0$. Then there exists a closed 2-form ρ, representing $PD(A)$ and such that $\Omega + y\rho$ is symplectic for all $y \geq 0$.*

For more details about this and about the inflation procedure see [M2] and [LM].

REMARK 3.C. An obvious conclusion of the above theorem is:
Under the assumptions of the above theorem, arbitrarily close to $PD(A)$ there exist cohomology classes which represent symplectic forms in the same deformation class as Ω.

We shall also need the following well known lemma (see [M4], [MP]):

LEMMA 3.D. *Let (M,Ω) be a closed symplectic 4-manifold. Denote by $\mathcal{E}$ the set of all homology classes which can be represented by Ω-symplectic exceptional spheres. Then:*

1) *$\mathcal{E}$ depends only on the deformation class of Ω.*
2) *Let $\mathcal{J}(\Omega)$ be the space of all Ω-tamed smooth almost complex structures of M. Then there exists a dense (actually, even residual) subset $\mathcal{J}_{\mathcal{E}} \subseteq \mathcal{J}(\Omega)$ such that for every $J \in \mathcal{J}_{\mathcal{E}}$ all classes in $\mathcal{E}$ admit J-holomorphic representatives which are connected, embedded and of genus 0.*
3) *If E', E'' are distinct classes in $\mathcal{E}$ then $E' \cdot E'' \geq 0$.*

Essential to the symplectic packing problem is the symplectic blow-up operation (see [MP], [MS2]). We shall work in the following setting. Let (M,J) be a 4-dimensional almost complex manifold with J integrable near $x_1,\dots,x_N \in M$. Let $(\overline{M},\overline{J}) \xrightarrow{\Theta} (M,J)$ be the complex blow-up of (M,J) at $x_1,\dots,x_N$. Denote by $\Sigma_q = \Theta^{-1}(x_q)$, $q=1,\dots,N$ the exceptional divisors, and by $E_q \in H_2(\overline{M};\mathbb{Z})$ their homology classes. Finally, we set $e_q = PD(E_q)$.

Recall that a symplectic form taming an almost complex structure J is said to be J-standard near $x \in M$ if the pair (Ω, J) is diffeomorphic to the standard pair (ω_{std}, i) of $\mathbb{R}^4$, near x.

The following lemma is an obvious generalization of Proposition 2.1.C from [MP].

LEMMA 3.E. *Let (M,Ω) be a closed symplectic 4-manifold. Let J be an almost complex structure tamed by Ω, which is integrable near $x_1,\dots,x_N \in M$ and suppose that Ω is J-standard near $x_1,\dots x_N$. Let $\mu_1(0),\dots,\mu_N(0)$ be positive numbers and let*

$$\varphi = \coprod_{q=1}^{N} \varphi_q : \coprod_{q=1}^{N} \big(B(\mu_q(0)), \omega_{std}\big) \to (M,\Omega)$$

be a symplectic embedding which is (i,J)-holomorphic. Denote by $(\overline{M},\overline{\Omega}_0)$

the symplectic blow-up of Ω with respect to φ. Suppose we have a symplectic deformation $\{\overline{\Omega}_t\}_{0\le t\le 1}$ starting with $\overline{\Omega}_0$, and lying in the cohomology class

$$[\overline{\Omega}_t] = [\Theta^*\Omega] - \pi \sum_{q=1}^{N} \mu_q(t)^2 e_q \ .$$

Then (M,Ω) admits a symplectic packing by N balls of radii $\mu_1(1),\ldots,\mu_N(1)$.

The proof of this lemma goes along the same lines as those of Proposition 2.1.C from [MP], only that here one has to adjust the symplectic forms $\overline{\Omega}_t$ on the exceptional divisors so that they become standard. This can be done using the symplectic neighborhood theorem.

4 Proof of the Main Theorems

4.1 Proof of Theorem 2.B. We shall prove a slightly more general version of Theorem 2.B, namely:

Theorem 4.1.A. *Let (M,Ω) be a closed symplectic 4-manifold in the class $\mathcal{C}$. Suppose that $0 < d_\Omega \le \infty$ and let $\lambda_1,\ldots,\lambda_N < \sqrt{d_\Omega}$ be positive numbers which satisfy*

$$\sum_{q=1}^{N} \lambda_q^4 < 2Vol(M,\Omega) \ .$$

Denote by $\overline{M} \overset{\Theta}{\to} M$ a complex blow-up of M at N distinct points. Then the following holds:

1) *The cohomology class*

$$[\Theta^*\Omega] - \sum_{q=1}^{N} \lambda_q^2 e_q$$

admits a symplectic representative.

2) *The manifold $(M,\pi\Omega)$ admits a symplectic packing by N balls of radii $\lambda_1,\ldots,\lambda_N$.*
3) *In particular, if*

$$N \ge \frac{2Vol(M,\Omega)}{d_\Omega^2}$$

then there exists a full packing of (M,Ω) by N equal balls.

Proof. The idea of the proof goes along the following lines. First, endow $\overline{M}$ with some auxiliary symplectic form $\overline{\Omega}$, obtained from blowing-up (M,Ω) with respect to some symplectic embedding of very small balls. Next, consider the homology class $A = PD([\Theta^*\Omega]) - \sum_{q=1}^N \lambda_q^2 E_q$. The idea is to show that for large enough n the class nA represents an embedded, reduced and connected pseudo-holomorphic curve in $\overline{M}$. Once this is proved we can use the inflation procedure to obtain a closed 2-form ρ which lies in the cohomology class $[\Theta^*\Omega] - \sum_{q=1}^N \lambda_q^2 e_q$ such that for every $y \geq 0$ the form $\overline{\Omega} + y\rho$ is symplectic. Dividing by y we see that for every $y > 0$ the form $\frac{1}{y}\overline{\Omega} + \rho$ is symplectic. By taking y to be very large we obtain a symplectic form lying in a cohomology class which is very close to the desired class: $[\Theta^*\Omega] - \sum_{q=1}^N \lambda_q^2 e_q$. It turns out that this approximation is enough for our purposes. We now give the precise details of the proof.

Let J be an almost complex structure tamed by Ω which is integrable near $x_1, \dots, x_N \in M$. Let $(\overline{M}, \overline{J}) \xrightarrow{\Theta} (M, J)$ be the complex blow-up of M at $x_1, \dots, x_N$. Denote by Σ_q the exceptional divisors, by E_q their homology classes, and by e_q the Poincaré dual of E_q. Finally, denote by c_1 the first Chern class of (TM, J) and by $\bar{c}_1$ the first Chern class of $(T\overline{M}, \overline{J})$. Notice that $\bar{c}_1 = c_1 - \sum_{q=1}^N e_q$, under the natural decomposition $H^2(\overline{M}; \mathbb{Z}) = H^2(M; \mathbb{Z}) \oplus \mathbb{Z}e_1 \oplus \dots + \oplus \mathbb{Z}e_N$.

Without loss of generality we may assume that Ω is J-standard near the points $x_1, \dots, x_N$ since Ω is isotopic to such a form (see [MP, Proposition 2.1.A]).

The proof is divided into two steps. In the first we assume that the cohomology class of Ω is rational. The second step is a reduction to the first one.

Step 1: Assume that $[\Omega]$ is a rational class.
Let $N \in \mathbb{N}$, and let λ_q be as in the assumptions of the theorem. Choose $\bar{\lambda}_q \in \mathbb{Q}$ such that $\lambda_q < \bar{\lambda}_q < \sqrt{d_\Omega}$ and such that $\sum_{q=1}^N \bar{\lambda}_q^4 < 2Vol(M, \Omega)$. Set

$$a = [\Theta^*\Omega] - \sum_{q=1}^N \bar{\lambda}_q^2 e_q \in H^2(\overline{M}; \mathbb{Q}) \ .$$

Denote by A the Poincaré dual of a. Clearly $A \cdot A > 0$.

Let $\overline{\Omega}_\epsilon$ be a symplectic form on $\overline{M}$ obtained from symplectic blowing-up with respect to a holomorphic and symplectic embedding of N balls of very small radii. Hence

$$[\overline{\Omega}_\epsilon] = [\Theta^*\Omega] - \epsilon \sum_{q=1}^N e_q \ .$$

By taking ϵ be small enough we may assume that $\overline{\Omega}_\epsilon(A) > 0$. Denote by $\mathcal{E}$ the set of homology classes which can be represented by $\overline{\Omega}_\epsilon$-symplectic exceptional spheres in $\overline{M}$.

We claim that $A \cdot E > 0$ for any $E \in \mathcal{E}$. For this purpose, write

$$E = B - \sum_{q=1}^{N} m_q E_q \,, \text{ where } B \in H_2(M;\mathbb{Z}) \,.$$

Notice first that $\Omega(B) \geq 0$. Indeed, the form $\overline{\Omega}_\epsilon$ can be included in a smooth family of symplectic forms $\{\overline{\Omega}_t\}_{0<t\leq\epsilon}$ such that

$$[\overline{\Omega}_t] = [\Theta^*\Omega] - t\sum_{q=1}^{N} e_q \quad \text{for all } 0 < t \leq \epsilon \,.$$

Since the set $\mathcal{E}$ depends only on the deformation class of $\overline{\Omega}_\epsilon$ we must have $\overline{\Omega}_t(E) > 0$ for all $0 < t \leq \epsilon$ and it easily follows that $\Omega(B) \geq 0$.

Moreover, one can show that:

(i) If $\Omega(B) = 0$ then $B = 0$ and $E = E_q$ for some $1 \leq q \leq N$.

(ii) If $B \neq 0$ then $m_q \geq 0$ for all $1 \leq q \leq N$.

Indeed, suppose that $\Omega(B) = 0$ but $B \neq 0$. Therefore $E \neq E_q$ for all q, so by Lemma 3.D we have $E \cdot E_q \geq 0$, hence $m_q \geq 0$ for all q. It is not possible that all the m_q's are zero, since then $E = B$ and we get $\overline{\Omega}_\epsilon(E) = \Omega(B) = 0$. So there must be at least one positive m_q. But then $\overline{\Omega}_\epsilon(E) = 0 - \epsilon\sum_{q=1}^{N} m_q < 0$, which is impossible. This proves that $B = 0$, and it easily follows that $E = E_q$ for some q. Part (ii) follows immediately from Lemma 3.D.

Now we are ready to show that $A \cdot E > 0$. If $\Omega(B) = 0$ then $E = E_q$ for some q, hence $A \cdot E = \bar{\lambda}_q^2 > 0$. So assume $\Omega(B) > 0$, from which it follows that $m_q \geq 0$ for all q. If all the m_q are zero then clearly $A \cdot E = \Omega(B) = \overline{\Omega}_\epsilon(B) > 0$ and we are done. So assume that at least one of the m_q's is positive, hence

$$\sum_{q=1}^{N} m_q \geq 1 \,.$$

Since $\bar{c}_1(E) = 1$ we must have

$$c_1(B) = 1 + \sum_{q=1}^{N} m_q \geq 2 \,.$$

Furthermore, since $E \cdot E = -1$ we must have

$$B \cdot B - \sum_{q=1}^{N} m_q^2 = -1$$

hence $B \cdot B \geq 0$. This shows that $B \in \mathcal{D}_\Omega$. Set $\bar{\lambda} = \max\{\bar{\lambda}_1, \dots, \bar{\lambda}_N\}$. We have

$$A\cdot E = \Omega(B) - \sum_{q=1}^{N} \bar{\lambda}_q^2 m_q \geq \Omega(B) - \bar{\lambda}^2\big(c_1(B)-1\big) \geq \Omega(B) - d_\Omega c_1(B) + \bar{\lambda}^2 > 0\,.$$

Notice that the case $\mathcal{D}_\Omega = \emptyset$ (which corresponds to $d_\Omega = \infty$), has already been treated since when $\mathcal{D}_\Omega = \emptyset$, the proof of our claim ends at an earlier stage.

By Theorem 3.B there exists a closed 2-form ρ, representing the class $a = PD(A)$ such that $\overline{\Omega}_\epsilon + y\rho$ is symplectic for all $y \geq 0$. As mentioned before the idea now is to consider the symplectic forms $\frac{1}{y}\overline{\Omega}_\epsilon + \rho$. If we take y large enough we obtain in this way a symplectic form which lies in a cohomology class very close to the desired one.

More precisely, consider the symplectic forms

$$\overline{\Omega}_s = \frac{1}{1+s-\epsilon}\left[\overline{\Omega}_\epsilon + (s-\epsilon)\rho\right]\,, \qquad \text{where} \;\; s \geq \epsilon\,. \tag{1}$$

They lie in the cohomology class

$$[\overline{\Omega}_s] = [\Theta^*\Omega] - \frac{1}{1+s-\epsilon}\sum_{q=1}^{N}\big(\epsilon + (s-\epsilon)\bar{\lambda}_q^2\big)e_q\,.$$

Choose $s_0 > \epsilon$ so large that

$$\lambda_q^2 < \frac{\epsilon + (s_0-\epsilon)\bar{\lambda}_q^2}{1+s_0-\epsilon} < \bar{\lambda}_q^2 \quad \text{(we may assume } \epsilon < \bar{\lambda}_q^2 \text{ for all } q)\,.$$

Notice that equation 1 provides a symplectic deformation $\{\overline{\Omega}_s\}_{\epsilon \leq s \leq s_0}$ starting at $\overline{\Omega}_\epsilon$ and ending with $\overline{\Omega}_{s_0}$, and this deformation satisfies the conditions of Lemma 3.E. It follows by this lemma that $(M, \pi\Omega)$ admits a symplectic packing by N balls of radii $\sqrt{\frac{\epsilon+(s_0-\epsilon)\bar{\lambda}_q^2}{1+s_0-\epsilon}} > \lambda_q$ $(q = 1, \dots, N)$. In particular all the conclusions of the theorem hold for $\lambda_1, \dots, \lambda_N$.

Step 2: Consider the general case.
First notice that the proof of Step 1 still holds if we assume that the cohomology class of Ω is a real multiple of a rational class. We need now the following general and simple observation:
Let $a \in \mathbb{R}^n$. Then arbitrarily close to a there exist $a_0, \dots, a_n \in \mathbb{Q}^n \subset \mathbb{R}^n$ and nonnegative real numbers $\alpha_0, \dots, \alpha_n \geq 0$ such that $\sum_{i=0}^n \alpha_i = 1$ and $\sum_{i=0}^n \alpha_i a_i = a$.

Using this, we decompose $a = [\Theta^*\Omega] - \sum_{q=1}^N \bar{\lambda}_q^2 e_q$ into a sum of the form $\sum_{i=0}^n \alpha_i a_i$ where $a_i \in H^2(\overline{M}; \mathbb{Q})$, and $n = b_2(M) + N$. Since the classes a_i can be taken to be arbitrarily close to a we can proceed as we did in Step 1 of the proof. The only difference is that now we have to do $n+1$ inflations, thus obtaining $n+1$ closed 2-forms $\rho_0, \dots, \rho_n$ representing the classes $a_0, \dots, a_n$, and such that $\overline{\Omega}_\epsilon + \sum_{i=0}^N y_i\rho_i$ are symplectic for all $y_i \geq 0$. The proof now proceeds as in Step 1 by taking $y_i = s\alpha_i$, with s very large.

Note that inflation using more than one curve is slightly more complicated than just with one curve. We refer the reader to [M2] for more details on this.

Finally, the third statement of the theorem follows easily from the second. □

4.2 Proof of Theorem 2.F. The proof of Theorem 2.F relies on the following two auxiliary results, which we include here for completeness.

Given a symplectic manifold (M, Ω) we denote by K the canonical class of (M, J), for any almost complex structure J, tamed by Ω.

Theorem 4.2.A. (Liu, Ohta-Ono. See [Li] and [MS1]) *Let M be a minimal symplectic 4-manifold. Then:*

1) *M is rational or ruled if and only if it admits a symplectic structure Ω, with $K \cdot \Omega < 0$.*
2) *If $K^2 < 0$ then M is an irrational ruled surface.*

The second result we need is called the light cone lemma. We shall be using the following set

$$\overline{\mathcal{P}^+} = \left\{a \in H^2(M;\mathbb{R}) \mid a \neq 0, a^2 \geq 0, [\Omega] \cdot a \geq 0\right\}$$

which is commonly called the closure of the forward positive cone.

Lemma 4.2.B. (Light cone lemma. See [M3]) *Let (M, Ω) be a symplectic 4-manifold with $b_2^+ = 1$. If $\alpha, \beta \in \overline{\mathcal{P}^+}$ then $\alpha \cdot \beta \geq 0$, with equality if and only if $\alpha = \lambda\beta$ for some $\lambda > 0$.*

Now we are ready for the

Proof of Theorem 2.F. Denote by $K = -c_1(TM, J)$ the canonical class of (TM, J), where J is any almost complex structure tamed by Ω. Since (M, Ω) is minimal but not rational or ruled, then by Theorem 4.2.A we must have $K^2 \geq 0$ and $K \cdot [\Omega] \geq 0$. We may assume that $K \neq 0$, since otherwise $\mathcal{D}_\Omega = \emptyset$ by definition. Hence K belongs to the closure of the forward positive cone

$$\overline{\mathcal{P}^+} = \left\{a \in H^2(M;\mathbb{R}) \mid a \neq 0, a^2 \geq 0, [\Omega] \cdot a \geq 0\right\} .$$

Assume that $\mathcal{D}_\Omega$ is not empty, and let $B \in \mathcal{D}_\Omega$. Clearly $\mathcal{D}_\Omega \subseteq \overline{\mathcal{P}^+}$. Since manifolds in the class $\mathcal{C}$ have $b_2^+ = 1$, it follows from the light cone lemma that $K \cdot B \geq 0$. But this cannot hold by the definition of the set $\mathcal{D}_\Omega$. $\square$

5 Examples

In this section we work out some examples of computing the packing number.

Minimal rational and ruled manifolds. Consider first minimal rational manifolds, that is $\mathbb{C}P^2$ and $S^2 \times S^2$. We start with $(\mathbb{C}P^2, \sigma_{std})$, where σ_{std} is the standard Kähler form normalized such that $\int_{\mathbb{C}P^1} \sigma_{std} = 1$. Denote by L the homology class of a projective line in $\mathbb{C}P^2$ and by l its Poincaré dual. We have $c_1 = 3l$, $[\sigma_{std}] = l$, hence $d_{\sigma_{std}} = \frac{1}{3}$. Using Theorem 4.1.A we see that given $\lambda_1, \dots, \lambda_N < \frac{1}{\sqrt{3}}$ which satisfy $\sum_{q=1}^N \lambda_q^4 < 1$, there exists a symplectic packing of $(\mathbb{C}P^2, \pi\sigma_{std})$ by N balls of radii $\lambda_1, \dots, \lambda_N$.

In particular, if $N \geq 9$, $\mathbb{C}P^2$ admits a full packing by N equal balls. Note however, that for 8 equal balls there is a packing obstruction (see [MP]). Finally, recall that by a theorem of Taubes any two cohomologous symplectic structures on $\mathbb{C}P^2$ are diffeomorphic. All the above prove:

Corollary 5.A. *For any symplectic form σ on $\mathbb{C}P^2$, $P_\sigma = 9$.*

Let us consider now $S^2 \times S^2$. Again, it is enough to consider the standard split forms, since by theorems of McDuff and Li and Liu (see [LM]), any two cohomologous symplectic forms of $S^2 \times S^2$ are diffeomorphic.

Consider the symplectic form $\Omega = \alpha\sigma \oplus \beta\sigma$, where $\int_{S^2} \sigma = 1$ and $0 < \alpha \leq \beta$. Using Gromov's non-squeezing theorem (see [G]), it is not hard to see that if $B(\lambda)$ embeds symplectically into $(S^2 \times S^2, \Omega)$ then $\pi\lambda^2 < \alpha$, hence we must have

$$P_\Omega \geq 2\frac{\beta}{\alpha} \ .$$

Using Theorem 4.1.A we compute an upper bound for P_Ω as follows: Set $A_1 = [S^2 \times pt]$, $A_2 = [pt \times S^2]$, and $a_i = PD(A_i)$, $i = 1, 2$. We have $c_1 = 2(a_1 + a_2)$ and $Vol(S^2 \times S^2, \Omega) = \alpha\beta$. Let $B \in \mathcal{D}_\Omega$, say $B = n_1 A_1 + n_2 A_2$. It is easy to see that n_1, n_2 are non-negative. Hence

$$\frac{\Omega(B)}{c_1(B)} = \frac{\alpha n_1 + \beta n_2}{2n_1 + 2n_2} \geq \frac{\alpha}{2} > 0 \ .$$

Therefore $d_\Omega \geq \frac{\alpha}{2}$, hence from Theorem 4.1.A we obtain:

COROLLARY 5.B. *Let $0 < \alpha \leq \beta$ and let σ be any area form of S^2. Then $2\frac{\beta}{\alpha} \leq P_\Omega \leq 8\frac{\beta}{\alpha}$.*

Note that in some cases it is possible to give sharper bounds for P_Ω, and even to compute its precise value. It is well known that there exists a diffeomorphism between the blow-up of $S^2 \times S^2$ at N points and the blow-up of $\mathbb{C}P^2$ at $N+1$ points. Furthermore it is not hard to compute that this diffeomorphism can be chosen to induce the following correspondence: packing $(S^2 \times S^2, \pi\Omega)$ by N balls of radii $\lambda_1, \dots, \lambda_N$ corresponds to packing $(\mathbb{C}P^2, \pi(\alpha+\beta-\lambda_1^2)\sigma_{std})$ by $N+1$ balls of radii $\sqrt{\alpha-\lambda_1^2}, \sqrt{\beta-\lambda_1^2}, \lambda_2, \dots, \lambda_N$.

Restricting to $\alpha = \beta$, a straightforward computation shows that there is packing obstruction for 7 balls (one uses the above correspondence and the packing inequalities from [MP]). Hence, if $\alpha = \beta$ then $P_\Omega = 8$.

Let us consider now irrational ruled surfaces.

COROLLARY 5.C. *Let R be an orientable surface of genus $g \geq 1$. Let σ_R, σ_{S^2} be area forms on R,S^2 respectively, such that $\int_R \sigma_R = \int_{S^2} \sigma_{S^2}$. Let α, β be positive numbers, then*

$$P_{(R\times S^2, \beta\sigma_R \oplus \alpha\sigma_{S^2})} = \left\lceil 2\frac{\beta}{\alpha} \right\rceil$$

Proof. Set $\Omega = \beta\sigma_R \oplus \alpha\sigma_{S^2}$. Let J_s be a split complex structure on $R \times S^2$. Denote by $((R\times S^2)_N, \overline{J}_s)$ the complex blow-up of $R \times S^2$ at N generic points, and by $\overline{\Omega}$ a blow-up of Ω. We claim that the set $\mathcal{E}$ of all homology classes which can be represented by $\overline{\Omega}$-symplectic exceptional spheres is:

$$\mathcal{E} = \{E_1, \dots, E_N, S - E_1, \dots, S - E_N\} \ ,$$

where $S = [pt \times S^2]$.

Indeed, let C be an $\overline{\Omega}$-symplectic exceptional sphere in the class E. Choose a generic almost complex structure $\overline{J}$ tamed by $\overline{\Omega}$ for which C is $\overline{J}$-holomorphic.

Consider a generic path $\{\overline{J}_t\}_{0\leq t\leq 1}$ of $\overline{\Omega}$ tamed almost complex structures with $\overline{J}_0 = \overline{J}$ and $\overline{J}_1 = \overline{J}_s$. Since $\overline{J}$ can be chosen to be arbitrarily close to $\overline{J}_s$, we may assume that for all $0 \leq t < 1$ there exist $\overline{J}_t$-holomorphic E-spheres. Using Gromov's compactness theorem we obtain a (possibly cusp) $\overline{J}_s$-holomorphic E curve, $\widetilde{C} = C_1 \cup \ldots \cup C_n$, with $genus(C_i) = 0$.

Denote by $pr_R : R \times S^2 \to R$ the projection and by $\widetilde{pr_R}$ its lifting to $(R\times S^2)_N$. Clearly $\widetilde{pr_R}$ is $\overline{J}_s$-holomorphic. Since $genus(R) \geq 1$ it follows that for every j, $\widetilde{pr_R}(C_j)$ must be a point, say p_j. As $\widetilde{C}$ is connected we see that all the p_j's are equal. Thus $\widetilde{pr_R}(\widetilde{C})$ is a point, and it immediately follows that $E \in \{E_1, \dots, E_N, S - E_1, \dots, S - E_N\}$. Conversely, it is obvious that E_q and $S - E_q$ are $\overline{\Omega}$-symplectic exceptional classes.

Since the exceptional spheres provide all the packing obstructions, it follows that $(R \times S^2, \pi\Omega)$, admits a symplectic packing by N balls of radii $\lambda_1, \dots, \lambda_N$ iff $\lambda_q^2 < \alpha$ and $\sum_{q=1}^N \lambda_q^4 < 2\alpha\beta$. Restricting to equal balls the result easily follows. □

6 Packing Obstructions

In this section we give some generalizations of the main theorems, which enable us to compute the exact values of $v_N(M,\Omega)$ for given N. Recall that $v_N(M,\Omega)$ is defined as the ratio between the supremum of the fillable volume by packing (M,Ω) with N balls and the volume of (M,Ω).

Let (M^4,Ω) be a closed symplectic manifold in the class $\mathcal{C}$. Denote by $\Theta : \widetilde{M_N} \to M$ its blow-up at N points, and write $\mathcal{E}_N$ for the set of 2-homology classes representing exceptional spheres in $\widetilde{M_N}$.

Let

$$pr : H_2(\widetilde{M_N};\mathbb{Z}) \to H_2(M;\mathbb{Z})$$

be the natural projection induced by Θ, and set

$$\mathcal{E}'_N = pr(\mathcal{E}_N) \setminus \{0\} \subset H_2(M;\mathbb{Z})$$
$$d'_N = \inf_{B \in \mathcal{E}'_N} \frac{\Omega(B)}{c_1(B) - 1} \ .$$

Finally, denote by $\lambda_{max}(M,\Omega;N)$ the supremum of the radii λ, for which there exists a symplectic packing of (M,Ω) by N equal balls of radius λ.

Theorem 6.A. *Let (M,Ω) be a closed symplectic 4-manifold in the class $\mathcal{C}$. Then:*

$$\lambda_{max}(M,\Omega;N)^2 = min\left\{\frac{1}{\pi}d'_N, \frac{1}{\pi}\sqrt{\frac{2Vol(M,\Omega)}{N}}\right\}$$
$$v_N(M,\Omega) = min\left\{\frac{N{d'_N}^2}{2Vol(M,\Omega)}, 1\right\} \ .$$

Proof. The proof of the above theorem goes exactly along the same lines as the proof of Theorem 2.B, the main idea being that the only packing obstructions come from exceptional spheres and from volume constrains.

Let $\overline{\Omega}_\epsilon$ be a symplectic form on $\widetilde{M_N}$ obtained from symplectic blowing-up with respect to a holomorphic and symplectic embedding of N balls of very small radii, as in the proof of Theorem 2.B.

Suppose that

$$\lambda^2 < min\left\{\frac{1}{\pi}d'_N, \frac{1}{\pi}\sqrt{\frac{2Vol(M,\Omega)}{N}}\right\} \ , \tag{2}$$

and consider the cohomology class $a = [\Theta^*\Omega] - \pi\lambda^2 \sum_{q=1}^N e_q$. Finally, write A for the Poincaré dual of a.

Exactly as in the proof of Theorem 2.B, we may assume without loss of generality that A is a rational homology class.

Clearly $A \cdot A > 0$, and by taking ϵ to be small enough we also have that $\overline{\Omega}_\epsilon(A) > 0$. Now, denote by $\mathcal{E}_N$ the set of all homology classes which can be represented by $\overline{\Omega}_\epsilon$-symplectic exceptional spheres. We claim that for every $E \in \mathcal{E}_N,\ \ A \cdot E > 0$.

Indeed, let $E = B - \sum_{q=1}^N m_q E_q$, where $B \in H_2(M;\mathbb{Z})$. If $B = 0$ then, as in the proof of Theorem 2.B it is easily seen that $E = E_q$ for some q, hence $A \cdot E > 0$. If $B \neq 0$ then $B \in \mathcal{E}'_N$ by definition. We have $1 = c_1(E) = c_1(B) - \sum_{q=1}^N m_q$, so

$$A \cdot E = \Omega(B) - \pi\lambda^2 \sum_{q=1}^N m_q = \Omega(B) - \pi\lambda^2\big(c_1(B) - 1\big) > 0 \ ,$$

since $\pi\lambda^2 < d'_N$.

It follows from Theorem 3.B that there exists a closed 2-form ρ, representing the class $a = PD(A)$, such that $\frac{1}{y}\overline{\Omega}_\epsilon + \rho$ is symplectic for all $y > 0$. This implies, exactly as in the proof of Theorem 2.B, that (M,Ω) admits a symplectic packing by N equal balls of radius arbitrarily close to λ. Since this is true for every λ which satisfies inequality (2), we have

$$\lambda_{max}(M,\Omega;N)^2 \geq min\left\{\frac{1}{\pi}d'_N, \frac{1}{\pi}\sqrt{\frac{2Vol(M,\Omega)}{N}}\right\} \ .$$

Conversely, if there exists a symplectic packing by N balls of radius λ, then the cohomology class a, must carry a symplectic representative in the same deformation class as $\overline{\Omega}_\epsilon$, hence $a(E) > 0$ for every $E \in \mathcal{E}_N$, and we get as above that $\lambda^2 < \frac{1}{\pi}d'_N$. Finally, $(a \cup a, [M]) > 0$ implies that $\lambda^2 < \frac{1}{\pi}\sqrt{\frac{2Vol(M,\Omega)}{N}}$.

Taking supremum over all such λ's we have

$$\lambda_{max}(M,\Omega;N)^2 \leq min\left\{\frac{1}{\pi}d'_N, \frac{1}{\pi}\sqrt{\frac{2Vol(M,\Omega)}{N}}\right\} \ .$$

The proof is complete. □

6.1 Examples. Since the set $\mathcal{E}'_N$ may be quite complicated it is not too easy to use Theorem 6.A directly. However, we shall consider a larger set, $\mathcal{E}''_N$, which is simpler.

Denote by $\widetilde{\mathcal{E}}_N$ the set of all $E \in H_2(\widetilde{M_N};\mathbb{Z})$ which satisfy the following conditions:

1) $E \cdot E = -1$.
2) $c_1(E) = 1$.
3) $E \cdot E_q \geq 0$ for all $1 \leq q \leq N$.
4) $[\Theta^*\Omega](E) \geq 0$.

Let $\mathcal{E}''_N = pr(\widetilde{\mathcal{E}}_N)\setminus\{0\}$. The set $\mathcal{E}''_N$ is much simpler to calculate than $\mathcal{E}'_N$, because its determination amounts to solving a system of two Diophantine equations. It turns out that in some cases, such as rational manifolds, we can replace $\mathcal{E}'_N$ in Theorem 6.A, by $\mathcal{E}''_N$.

We give now the details of the argument. First consider the manifold $\mathbb{C}P^2$. Let L be the homology class of a line in $\mathbb{C}P^2$. Using L as a basis for $H_2(\mathbb{C}P^2;\mathbb{Z})$, we may identify $\mathcal{E}''_N$ with a set of integers, which we denote by $D_N(\mathbb{C}P^2)$.

More precisely, denote by $D_N(\mathbb{C}P^2)$ the set of all integers d, for which there exists a non-negative solution $d, m_1, \dots, m_N \geq 0$ for the system of Diophantine equations:

$$\begin{cases} d^2 = \sum_{q=1}^N m_q^2 - 1 \\ 3d = \sum_{q=1}^N m_q + 1 \ . \end{cases}$$

Similarly, for $S^2 \times S^2$, by using $[S^2 \times \{pt\}], [\{pt\} \times S^2]$ as a basis for $H_2(S^2 \times S^2;\mathbb{Z})$, we let $D_N(S^2 \times S^2)$ be the set of all pairs of integers (n_1, n_2) for which there exists a non-negative solution $n_1, n_2, m_1, \dots, m_N \geq 0$ for the system of Diophantine equations:

$$\begin{cases} 2n_1 n_2 = \sum_{q=1}^N m_q^2 - 1 \\ 2n_1 + 2n_2 = \sum_{q=1}^N m_q + 1 \ . \end{cases}$$

In each of the two cases the first equation reflects the fact that $E \cdot E = -1$, while the second comes from $c_1(E) = 1$.

Note that for $N \leq 8$, $D_N(\mathbb{C}P^2)$ is a finite set, while $D_N(S^2 \times S^2)$ is finite for $N \leq 7$ (see [D]).

Theorem 6.1.A. 1) *For* $(\mathbb{C}P^2, \sigma_{std})$,

$$v_N = min\left\{1, N \cdot \inf_{d \in D_N(\mathbb{C}P^2)} \left(\frac{d}{3d-1}\right)^2\right\} .$$

2) *Let* σ_{S^2} *be an area form on* S^2. *Then for* $(S^2 \times S^2, \alpha\sigma_{S^2} \oplus \beta\sigma_{s^2})$,

$$v_N = min\left\{1, \frac{N}{2\alpha\beta} \cdot \inf_{(n_1,n_2) \in D_N(S^2 \times S^2)} \left(\frac{\alpha n_1 + \beta n_2}{2n_1 + 2n_2 - 1}\right)^2\right\} .$$

3) *Let* R *be an oriented surface of genus at least* 1, *and let* σ_R *be an area form on* R. *Then for* $(S^2 \times R, \alpha\sigma_{S^2} \oplus \beta\sigma_R)$,

$$v_N = min\left\{1, \frac{N\alpha}{2\beta}\right\} .$$

Proof. In view of Theorem 6.A statement 3 follows immediately from the fact that

$$\mathcal{E}_N = \{E_1, \dots, E_N, S - E_1, \dots, S - E_N\} \ ,$$

already mentioned in the proof of Corollary 5.C. Here, S stands for the homology class $[S^2 \times \{pt\}]$.

We turn now to the proof of statements 1 and 2. The idea of the proof is that although homology classes E, with $E \cdot E = -1$, $c_1(E) = 1$, $E \cdot E_q \geq 0$, $[\Theta^*\Omega](E) \geq 0$ may not represent exceptional spheres they still represent some pseudo-holomorphic curve in $\widetilde{M_N}$, for a generic almost complex structure tamed by $\widetilde{\Omega}$. The argument is based on the the wall crossing formula for Gromov invariants. We refer the reader to [MS1] for more details about Gromov invariants and the wall crossing formula.

Let (M, Ω) be either $(\mathbb{C}P^2, \sigma_{std})$ or $(S^2 \times S^2, \alpha\sigma_{S^2} \oplus \beta\sigma_{s^2})$. Suppose that there exists a symplectic packing of (M, Ω) with N balls of radius λ. It follows that there exists a symplectic form $\widetilde{\Omega}$ on $\widetilde{M_N}$ such that

$$[\widetilde{\Omega}] = [\Theta^*\Omega] - \pi\lambda^2 \sum_{q=1}^{N} e_q \ .$$

Consider the set

$$\mathcal{E}''_N = \left\{ dL \mid d \in D_N(\mathbb{C}P^2) \right\}$$

in case $M = \mathbb{C}P^2$. Here we denote by L the homology class of a line in $\mathbb{C}P^2$.

Similarly, for $M = S^2 \times S^2$ consider the set

$$\mathcal{E}''_N = \left\{ n_1 S_1 + n_2 S_2 \mid (n_1, n_2) \in D_N(S^2 \times S^2) \right\} \ .$$

Here we denote by S_1, S_2 the homology classes, $[S^2 \times \{pt\}], [\{pt\} \times S^2]$ respectively.

Let $B \in \mathcal{E}''_N$. By our assumptions, there exist non-negative integers $m_1, \dots, m_N$ such that the class $E = B - \sum_{q=1}^{N} m_q E_q$ belongs to $\widetilde{\mathcal{E}}_N$.

We claim that $Gr(K - E) = 0$. Indeed, it is easy to see in both cases ($M = \mathbb{C}P^2$ or $M = S^2 \times S^2$) that there exists a symplectic form $\widetilde{\Omega}'$ in the same deformation class as $\widetilde{\Omega}$, such that $\widetilde{\Omega}'(K - E) < 0$.

Since $E \cdot E + c_1(E) = 0$ we can apply the wall crossing formula (see [MS1]) to obtain that $Gr(E) \neq 0$. It follows that $\widetilde{\Omega}(E) > 0$, that is

$$\lambda^2 \leq \frac{1}{\pi} \frac{\Omega(B)}{\sum_{q=1}^{N} m_q} = \frac{1}{\pi} \frac{\Omega(B)}{c_1(B) - 1} \ .$$

This proves that

$$\lambda_{max}(N)^2 \leq \inf_{B \in \mathcal{E}''_N} \frac{1}{\pi} \frac{\Omega(B)}{c_1(B) - 1} \ .$$

On the other hand, since $\mathcal{E}'_N \subset \mathcal{E}''_N$ we must have

$$\inf_{B\in\mathcal{E}''_N} \frac{1}{\pi}\frac{\Omega(B)}{c_1(B)-1} \leq \frac{1}{\pi} d'_N \ .$$

From volume obstruction we have also $\lambda_{max}(N)^2 < \frac{1}{\pi}\sqrt{\frac{2Vol(M,\Omega)}{N}}$.
Now it easily follows from Theorem 6.A that

$$\lambda_{max}(N)^2 = \min\left\{ \inf_{B\in\mathcal{E}''_N} \frac{1}{\pi}\frac{\Omega(B)}{c_1(B)-1}, \frac{1}{\pi}\sqrt{\frac{2Vol(M,\Omega)}{N}} \right\} ,$$

hence

$$v_N = min\left\{1, \frac{N}{2Vol(M,\Omega)} \inf_{B\in\mathcal{E}''} \left(\frac{\Omega(B)}{c_1(B)-1}\right)^2\right\} .$$

This completes the proof of the theorem. □

Solving the relevant Diophantine equations and applying Theorem 6.1.A give the following results:
For $(S^2\times S^2, \sigma_{S^2}\oplus\sigma_{S^2})$, we have
$v_1 = \frac{1}{2}$, $v_2 = 1$, $v_3 = \frac{2}{3}$, $v_4 = \frac{8}{9}$, $v_5 = \frac{9}{10}$, $v_6 = \frac{48}{49}$, $v_7 = \frac{224}{225}$, $v_N = 1$ for any $N \geq 8$.

For $\mathbb{C}P^2$ we get the values already obtained by McDuff and Polterovich in [MP], namely
$v_1 = 1$, $v_2 = \frac{1}{2}$, $v_3 = \frac{3}{4}$, $v_4 = 1$, $v_5 = \frac{20}{25}$, $v_6 = \frac{24}{25}$, $v_7 = \frac{63}{64}$, $v_8 = \frac{288}{289}$, $v_N = 1$ for any $N \geq 9$.

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