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CONSTRUCTING NEW AMPLE DIVISORS OUT OF OLD ONES

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1. Introduction. The main objective of this paper is to propose a method for constructing new ample divisors on rational surfaces by gluing two given ones.

Recall that a divisor D on an algebraic variety X is ample if the corresponding line bundle $\mathcal{O}_X(D)$ is ample, and it is called *nef* (numerically effective) if there exists an ample divisor A such that $A+kD$ is ample for every $k>0$. We refer the reader to [Dem] and [Ha1] for excellent expositions on various aspects of the theory of ample and nef line bundles.

Of fundamental importance is the determination of those classes in $\mathrm{Pic}(X)$ which are ample. Although this problem has a very simple solution for smooth curves, already in dimension 2 the problem becomes much harder. Even for relatively simple surfaces, such as rational, the complete answer is not known. Several conjectures in this direction exist; however, at the present time only estimates on the *ample cone*—the cone generated by the ample classes in $\mathrm{Pic}(X)$—are known. For example, let $d,m>0$ and consider the divisor class

$$D=\pi^*\mathcal{O}_{\mathbb{C}P^2}(d)-m\sum_{j=1}^{N}E_j$$

on the blowup $\pi : V_N \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ at $N\geq 9$ generic points. Nagata conjectured in [Nag] that *D is ample if and only if $D\cdot D>0$*, but was able to prove this only for N*'s* that are squares. In [Xu1] Xu proved that *D is ample provided that $m/d<\sqrt{N-1}/N$*. By making a more detailed analysis of the case $m=1$, Xu proved in [Xu2] that *when $d\geq 3$ the divisor class $D=\pi^*\mathcal{O}_{\mathbb{C}P^2}(d)-\sum_{j=1}^{N}E_j$ is ample if and only if $D\cdot D>0$* (see also [Ku] for a generalization for arbitrary surfaces and [Ang] for an analogous result for $\mathbb{C}P^3$).

A closely related problem is that of computing *Seshadri constants* of ample line bundles, which measure their local positivity. The Seshadri constant $\mathcal{E}(\mathcal{L},p)$ of the line bundle $\mathcal{L}$ at the point $p\in X$ is defined to be *the supremum of all those $\epsilon\geq 0$ for which the $\mathbb{R}$-divisor class $\pi^*\mathcal{L}-\epsilon E$ is nef on the blowup $\pi:\widetilde{X}_p\to X$ of X at the point p with exceptional divisor E.*

Seshadri constants have been much studied by Demailly [Dem]; Ein, Küchle, and Lazarsfeld [EL], [EKL], [Laz]; and Xu [Xu3]. A considerable part of these works is devoted to computations and estimates from below on the values of these constants.

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The present paper is largely motivated by the problem of computing Seshadri constants and the determination of the ample cones of rational surfaces. Our main results provide an algorithmic method for constructing new ample divisors out of the knowledge of ample divisors on simpler rational surfaces. By applying the algorithm recursively we obtain in Section 3 new estimates on Seshadri-like constants and detect new ample divisors. We then propose in Section 4 a conjecture naturally arising from our method, which relates *continued fractions expansions of* $\sqrt{N}$ with the ample cone of $\mathbb{C}P^2$ blown up at N points. Finally, we interpret in Section 7 our main results in the language of symplectic geometry and explain the intuition behind them.

2. Main results. Our main results deal with *simple rational surfaces* S, which by definition are blowups $\Theta : S \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ at n distinct points $p_1, \dots, p_n \in \mathbb{C}P^2$. We denote by $E_i = \Theta^{-1}(p_i)$ the exceptional divisor over the blown-up point p_i and by L a divisor in S, obtained by pulling back via Θ a projective line in $\mathbb{C}P^2$ that does not pass through any of the points $p_1, \dots, p_n$.

A vector $(d; \alpha_1, \dots, \alpha_k) \in \mathbb{Z}_+ \times \mathbb{Z}^k_{\geq 0}$ is called ample (resp., nef) if there exists a simple rational surface S, on which the divisor $dL - \sum_{j=1}^k \alpha_j E_j$ is ample (resp., nef).

The first ingredient of our algorithm is the following gluing theorem.

THEOREM 2.A. *Let* $(d; m_1, \dots, m_n, m)$ *be an ample (resp., nef) vector, and let* $(m; \alpha_1, \dots, \alpha_k) \in \mathbb{Z}^{k+1}_+$ *be a nef vector. Then* $v = (d; m_1, \dots, m_n, \alpha_1, \dots, \alpha_k)$ *is ample (resp., nef). Moreover,* v *can be realized by an ample (resp., nef) divisor on a very general rational surface.*

By a *very general* choice of points $q_1, \dots, q_r$ in an algebraic variety X, we mean that $(q_1, \dots, q_r)$ is allowed to vary in a subset of the configuration space $\mathscr{C}_r(X) = \{(x_1, \dots, x_r) \in X^r \mid x_i \neq x_j\}$ whose complement is contained in a countable union of proper subvarieties of $\mathscr{C}_r(X)$. By a *very general rational surface* we mean one that is obtained by blowing up points $q_1, \dots, q_r \in \mathbb{C}P^2$ which may be chosen to be very general.

The proof of Theorem 2.A together with various generalizations is given in Section 5. Theorem 2.A combined with the action of the Cremona group on the ample cone gives rise to an algorithmic procedure for detecting new ample classes in the Picard group of rational surfaces. The algorithm is explained in Section 3.4.

2.1. Applications to Seshadri constants. Given an ample line bundle $\mathscr{L} \to S$ on a surface S and given a vector $w = (w_1, \dots, w_N)$ of positive numbers, we define the *w-weighted remainder* of $\mathscr{L}$ at the N distinct points $p_1, \dots, p_N \in S$ to be the quantity

$$\mathscr{R}^w(\mathscr{L}, p_1, \dots, p_N) = \frac{1}{\mathscr{L} \cdot \mathscr{L}} \inf_{0 \leq \epsilon \in \mathbb{R}} \left\{ \mathscr{L}_\epsilon \cdot \mathscr{L}_\epsilon \mid \mathscr{L}_\epsilon = \pi^* \mathscr{L} - \epsilon \sum_{j=1}^N w_j E_j \quad \text{is nef} \right\},$$

where $\pi : \widetilde{S} \to S$ is the blowup of S at the points $p_1, \dots, p_N$ with exceptional divisors

$E_i = \pi^{-1}(p_i)$. It is obvious that $0 \le \mathcal{R}^w < 1$. Note that $\mathcal{R}^w$ remains invariant under rescaling of $\mathcal{L}$ and of w, namely, $\mathcal{R}^{aw}(b\mathcal{L}, p_1, \ldots, p_N) = \mathcal{R}^w(\mathcal{L}, p_1, \ldots, p_N)$ for every $a, b > 0$. It is convenient to also define a more global invariant

$$\mathcal{R}_N^w(\mathcal{L}) = \inf\left\{\mathcal{R}^w(\mathcal{L}, p_1, \ldots, p_N) \mid p_1, \ldots, p_N \in S \quad \text{are distinct points}\right\}.$$

Restricting to the case of homogeneous weights, we obtain the *homogeneous remainders*

$$\mathcal{R}(\mathcal{L}, p_1, \ldots, p_N) = \mathcal{R}^{w_h}(\mathcal{L}, p_1, \ldots, p_N), \qquad \mathcal{R}_N(\mathcal{L}) = \mathcal{R}_N^{w_h}(\mathcal{L}),$$

where $w_h = (1, \ldots, 1)$.

The constants $\mathcal{R}^w(\mathcal{L}, p_1, \ldots, p_N)$ are obvious generalizations of the Seshadri constants $\mathcal{E}(\mathcal{L}, p)$ from Section 1 (see also [Xu3] for similar Seshadri-like constants). Several theorems and conjectures related to the ample cone can be neatly formulated using the constants $\mathcal{R}_N$. For example, Nagata's conjecture can be reformulated as "$\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) = 0$ *for every* $N \ge 9$." Similarly, Xu's result from Section 1 states that $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) \le 1/N$. Using our algorithm we prove in Section 3.6 the following asymptotic result.

THEOREM 2.1.A. *Let* $a, l \in \mathbb{N}$. *Then*
(1) *for* $N = a^2l^2 + 2l$, $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) \le 1/(a^2l+1)^2$;
(2) *for* $N = a^2l^2 - 2l$, $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) \le 1/(a^2l-1)^2$;
(3) *for* $N = a^2l^2 + l$ *with* $al \ge 3$, $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) \le 1/(2a^2l+1)^2$.

In Section 4 we view this result in a more general context by proposing a conjecture that bounds $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1))$ in terms of continued fractions approximations of $\sqrt{N}$.

Our methods also yield, as a corollary, the following generalization of a theorem of Xu [Xu2] and Küchle [Ku].

COROLLARY 2.1.B. *Let* $d \in \mathbb{N}$. *The divisor* $D = \pi^*\mathcal{O}_{\mathbb{C}P^2}(d) - 2\sum_{j=1}^N E_j$ *on the blowup of* $\mathbb{C}P^2$ *at* N *very general points is nef if and only if* $D \cdot D \ge 0$.

The proof appears in Section 3.6. Let us conclude this section with the following, somewhat amusing, corollary of Theorem 2.A.

COROLLARY 2.1.C. *If Nagata's conjecture holds for* N_1 *and* N_2, *then it holds also for* N_1N_2.

The proof is given in Section 3.7.

3. Asymptotics on the remainders of $\mathcal{O}_{\mathbb{C}P^2}(1)$. In order to obtain estimates on $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1))$, we successively use Theorem 2.A in combination with the *Cremona action*. The point is that the Cremona group acts on the set of ample (resp., nef) vectors. Let us briefly summarize the needed facts about the Cremona action. We refer the reader to [DoOr] for more details.

3.1. The Cremona action on the ample cone. For $k \geq 3$, denote by $(H_k, \langle\, ,\rangle)$ the hyperbolic lattice $H_k = \mathbb{Z}l \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k$ with the bilinear form $\langle\, ,\rangle$ defined by

$$\langle l, l\rangle = 1, \qquad \langle l, e_j\rangle = 0, \qquad \langle e_i, e_j\rangle = -\delta_{ij}.$$

Consider the subgroup $Cr_k \subset \mathrm{Aut}(H_k, \langle\, ,\rangle)$, generated by

(1) the symmetric group $S_k \hookrightarrow \mathrm{Aut}(H_k, \langle\, ,\rangle)$ acting on the last k components by permutations;

(2) the reflection $R_{123} : (H_k, \langle\, ,\rangle) \to (H_k, \langle\, ,\rangle)$ defined by $R_{123}(\eta) = \eta + \langle \eta, r_{123}\rangle r_{123}$, where $r_{123} = l - e_1 - e_2 - e_3$.

The group Cr_k is called the *Cremona group*.

Given a simple rational surface obtained by blowing up $\Theta : V \to \mathbb{C}P^2$ of $p_1, \ldots, p_n \in \mathbb{C}P^2$, there is an isomorphism of lattices $m_\Theta : (\mathrm{Pic}(V), \cdot) \to (H_k, \langle\, ,\rangle)$, where $\cdot$ stands for the intersection form on $\mathrm{Pic}(V)$. The isomorphism m_Θ sends L to l and E_i to e_i.

The main (classical) feature of the Cremona group we use is the following.

PROPOSITION 3.1.A. *For $k \geq 3$, the set of ample (resp., nef) vectors (viewed as a subset of H_k) remains invariant under the action of Cr_k.*

The proof of this well-known fact is sketched in Section 6.3 below.

3.2. Three useful Cremona transformations. In applying our algorithm, it is convenient to use the following transformations based on the Cremona action.

Reflections. It can be easily seen that the reflections $R_{ijk}(\eta) = \eta + \langle \eta, r_{ijk}\rangle r_{ijk}$, where $r_{ijk} = l - e_i - e_j - e_k$, belong to Cr_k.

Sorting. This transformation is denoted by $SR : H_k \to H_k$. By definition, SR takes a vector $v = (d; m_1, \ldots, m_k) \in H_k$ and sorts its last k components. In other words, $SR(v) = (d; m_{\tau(1)}, \ldots, m_{\tau(k)})$, where τ is a permutation of $\{1, \ldots, k\}$ for which $m_{\tau(1)} \geq \cdots \geq m_{\tau(k)}$. It is obvious that for every vector $v \in H_k$, there exists $\sigma \in Cr_k$ such that $SR(v) = \sigma(v)$. Thus, $SR(v)$ is ample (resp., nef) if and only if v is.

Minimization of degree. This transformation is denoted by $\Pi : H_k \to H_k$. Its description is the following. Let $v = (d; m_1, \ldots, m_k) \in H_k$. Put $v_0 = v$, and for every $n \geq 0$ define

$$v_{n+1} = R_{123}\big(SR(v_n)\big).$$

Next, let $d_n = \deg(v_n)$ be the degree of v_n, which by definition is the first component of the vector, namely, $\deg(v_n) = \langle v, l\rangle$. Let $n_0 \geq 0$ be the minimal nonnegative integer for which $d_{n_0+1} \geq d_{n_0} \geq 0$. If no such n_0 exists[1] we set $n_0 = 0$ by default. Now define $\Pi(v) = v_{n_0}$. Intuitively, one can view the transformation $v \to \Pi(v)$ as finding an element $\Pi(v)$ that lies in the orbit $Cr_k v$ and, locally in the orbit, has minimal degree. As before, it is obvious that $\Pi(v)$ is ample (resp., nef) if and only if v is.

[1]This can happen only if $d = d_0 < 0$, in which case v cannot be nef anyway.

3.3. The ample and nef cones. Before describing the algorithmic procedure for detection of ample classes, we remark that the set of ample (resp., nef) vectors is closed under addition and under multiplication by positive (resp., nonnegative) integers. This follows easily from the fact that there exists a (very general) simple rational surface on which all the ample (resp., nef) vectors can simultaneously be realized by ample (resp., nef) divisors. This can be proved using Lemma 6.1.A below, for example.

Henceforth we denote by $\mathcal{K}_k \subset H_k \otimes \mathbb{R}$ (resp., $\overline{\mathcal{K}_k}$) the cone generated by all ample (resp., nef) vectors.

3.4. An algorithmic procedure for detecting ample classes. Given two vectors $v_1 = (d; m_1, \dots, m_n) \in H_n$ and $v_2 = (\delta; \alpha_1, \dots, \alpha_k) \in H_k$ with $\delta = m_i$ for some $1 \le i \le n$, define a new vector $v_1 \#_i v_2 \in H_{n+k-1}$ by setting

$$v_1 \#_i v_2 = (d; m_1, \dots, m_{i-1}, \alpha_1, \dots, \alpha_k, m_{i+1}, \dots, m_n).$$

Theorem 2.A states that if v_1 is ample (resp., nef) and v_2 is nef, then $v_1 \#_i v_2$ is ample (resp., nef).

Given a vector $v_0 \in H_N$, the ampleness of which we want to prove, we try to find a decomposition $v_0 = v_1 \#_{i_1} u_1$, where $u_1 \in H_{k_1}$ is known to be nef and $v_1 \in H_{n_1}$, $(k_1 + n_1 - 1 = N)$. If v_1 turns to be ample, then we are done in view of Theorem 2.A. To check the ampleness of v_1, we first "simplify" it by applying to it Cremona transformations. For example, we may try, using Cremona transformations to reduce the degree of v_1, by applying to it the transformation Π defined above. Let v_1' be a simpler vector in the same orbit of v_1 under the action of Cr_{n_1} (e.g., v_1' having minimal degree in the orbit or having some other convenient feature). By Proposition 3.1.A, v_1 is ample if and only if v_1' is. Now we apply the whole process to v_1', and so on. In this way we obtain a sequence of vectors $v_1, u_1, v_1', \dots, v_r, u_r, v_r'$, where v_j' is a Cremona simplification of $v_j \in H_{n_j}$, $u_j \in H_{k_j}$ is a nef vector, and $v_j' = v_{j+1} \#_{i_{j+1}} u_{j+1}$ for some i_{j+1}.

Note that at each stage the number of blown-up points decreases, namely, $n_{j+1} < n_j$, provided that $k_j > 1$. The process ends successfully as soon as we are able to prove that v_r is ample for some r. We remark that if one of the v_j turns out not to be ample then the process fails to give any information because the converse of Theorem 2.A is not true. However, we may attempt to find other decomposition sequences $v_1, u_1, v_1', \dots, v_r, u_r, v_r'$.

The same procedure can be applied for proving nefness of a vector v_0 by requiring that v_r is nef instead of ample. In the next subsection we apply this process in order to prove Theorem 2.1.A and Corollary 2.1.B.

In order to make the preceding procedure applicable, we must first endow ourselves with a large enough initial collection of ample and nef vectors that play the role of the u_j's and of v_r. To simplify notation let us agree that $(d; \alpha_1^{\times r_1}, \dots, \alpha_k^{\times r_k})$ stands for

$$(d; \underbrace{\alpha_1, \ldots, \alpha_1}_{r_1 \text{ times}}, \ldots\ldots, \underbrace{\alpha_k, \ldots, \alpha_k}_{r_k \text{ times}}) \in H_N, \quad \text{where } N = \sum_{j=1}^{k} r_j.$$

The next lemma provides a modest initial collection of ample and nef vectors, which is sufficient for our purposes.

LEMMA 3.4.A. *The following vectors are nef (resp., ample) on a very general rational surface:*
(1) $(d; 1^{\times r})$*, where* $d^2 \geq r$ *(resp.,* $d^2 > r$*);*
(2) $(d; m_1, m_2, 1^{\times r})$*, where* $d \geq m_1 + m_2$ *and* $d^2 \geq m_1^2 + m_2^2 + r$*.*

The proof is given in Section 6.2. The "ample" case of statement (1) above has been proved by Xu in [Xu2] and by Küchle in [Ku]. Below, however, we present an alternative proof pointed out to us by Ilya Tyomkin.

Remark. Note that the cones $\mathcal{K}_n$ (resp., $\overline{\mathcal{K}}_n$) can be explicitly computed when $n < 9$ (see [Dmz], [FM]) and so can be joined to the initial collection of ample and nef vectors to be applied in the framework of the process mentioned above.

3.5. A simple example. The following is a simple numerical illustration of how to practically apply our algorithm.

Let $N = 14$ and consider the vector $v = (15; 4^{\times 14}) \in H_{14}$. We claim that the vector v is nef. Note that $\langle v, v \rangle = 1$; hence, nefness of v implies that $\mathcal{R}_{14}(\mathcal{O}_{\mathbb{C}P^2}(1)) \leq 1/15^2 = 1/225$.

We start with the decomposition

$$\big(15; 4^{\times 14}\big) = \big(15; 4^{\times 10}, 8\big) \# \big(8; 4^{\times 4}\big).$$

By Lemma 3.4.A, the vector $(8; 4^{\times 4}) = 4(2; 1^{\times 4})$ is nef; hence, we are reduced to proving nefness of $v_1 = (15; 4^{\times 10}, 8)$. Applying the transformation Π to v_1 (see Section 3.2), we obtain by a straightforward calculation that $v_1' = \Pi(v_1) = (10; 3^{\times 11})$.

Next, decompose v_1' as

$$v_1' = \big(10; 3^{\times 11}\big) = \big(10; 3^{\times 7}, 6\big) \# \big(6; 3^{\times 4}\big).$$

Again by Lemma 3.4.A, the vector $(6; 3^{\times 4})$ is nef, so the problem is reduced to nefness of $v_2 = (10; 3^{\times 7}, 6)$. Now apply Π to v_2 to obtain $\Pi(v_2) = (1; 0^{\times 8})$. This vector is obviously nef; therefore our original vector $v = (15; 4^{\times 14})$ is also nef.

As shown below, similar application of our algorithm leads to a proof of Theorem 2.1.A.

3.6. Proofs of Theorem 2.1.A and Corollary 2.1.B

Proof of Theorem 2.1.A. (1) Let $N = a^2 l^2 + 2l$ and $v_0 = (a^2 l + 1; a^{\times N})$. As $\langle v_0, v_0 \rangle = 1$, nefness of v_0 gives the needed estimate for $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1))$.

The decomposition $N = (al-1)^2 + n$, where $n = 2al + 2l - 1$, leads us to $v_0 = v_1 \#_1 u_1$, where

$$v_1 = \big(a^2 l + 1; a(al-1), a^{\times n}\big) \in H_{n+1}, \qquad u_1 = a\Big(al-1; 1^{\times (al-1)^2}\Big) \in H_{(al-1)^2}.$$

By Lemma 3.4.A, u_1 is nef; hence, in view of Theorem 2.A we are reduced to proving that v_1 is nef. This turns out to be easy by using Cremona transformations. Indeed, let

$$v_1' = R_{1,n-1,n} \circ R_{1,n-3,n-2} \circ \cdots \circ R_{123}(v_1),$$

where $R_{ijk} \in Cr_{n+1}$ are defined in Section 3.2. A straightforward computation shows that $v_1' = (a+l; l-1, 1^{\times n-1}, a)$. This vector is nef by Lemma 3.4.A, and, therefore, v_1 is too.

(2) Let $N = a^2 l^2 - 2l$ and $v_0 = (a^2 l - 1; a^{\times N})$. Again, $\langle v_0, v_0 \rangle = 1$; hence, in order to prove the needed estimate on $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1))$, we have to prove that v_0 is nef. Using the decomposition $N = (al-2)^2 + n_1$, where $n_1 = 4al - 2l - 4$, we note that $v_0 = v_1 \#_1 u_1$, where

$$v_1 = \Big(a^2 l - 1; a(al-2), a^{\times n_1}\Big) \in H_{n_1+1}, \qquad u_1 = a\Big(al-2; 1^{\times (al-2)^2}\Big) \in H_{(al-2)^2}.$$

By Lemma 3.4.A, u_1 is nef. We are thus reduced to proving nefness of v_1. By applying similar Cremona transformations as in (1), we obtain that $v_1' = (a^2 l - 2al + l + 1; (al - l - 2)(a-1), (a-1)^{\times n_1})$ lies in the same orbit as v_1.

Next we apply the same algorithm again on v_1'. For this, consider the decomposition $v_1' = v_2 \#_1 u_2$, where

$$v_2 = \Big(a^2 l - 2al + l + 1; (al-l-1)(a-1), (a-1)^{\times 2al-1}\Big) \in H_{2al},$$

$$u_2 = (a-1)\Big(al-l-1; al-l-2, 1^{\times 2al-2l-3}\Big).$$

By Lemma 3.4.A, u_2 is nef; thus, we are reduced to proving that v_2 is nef. Using Cremona transformations similar to those in (1), we obtain that $v_2' = \big(a+l-1; l-1, 1^{\times 2al-2}, a-1\big)$ lies in the same orbit as v_2. But by Lemma 3.4.A, v_2' is nef.

(3) The proof of the case $N = a^2 l^2 + l$ requires a more general gluing procedure than the one described in Theorem 2.A. We therefore postpone the proof to Section 5.3. □

Proof of Corollary 2.1.B. Let $D = \pi^* \mathcal{O}_{\mathbb{C}P^2}(d) - 2\sum_{j=1}^{N} E_j$, and suppose that $D \cdot D \geq 0$.

Step 1. Consider first the case $N = k^2 + k$ for some k. The case $k = 1$ can be easily proved directly, so assume that $k > 1$. By the third statement of Theorem 2.1.A, we have that

$$\mathcal{R}_N\big(\mathcal{O}_{\mathbb{C}P^2}(1)\big) \leq \frac{1}{(2k+1)^2}.$$

This implies that $\pi^*\mathbb{O}_{\mathbb{C}P^2}(2k+1) - 2\sum_{j=1}^{N} E_j$ is nef. Since $D \cdot D \geq 0$ and d is an integer, we have that $d \geq 2k+1$, which immediately implies nefness of D.

Step 2. Consider the general case. The condition $D \cdot D \geq 0$ reads $d^2 \geq 4N$. We may assume that d is odd, for the case of even d is precisely the contents of Xu's theorem from Section 1 (see [Xu2]). Writing $d = 2k+1$, the condition $d^2 \geq 4N$ implies that $d^2 \geq 4N+1$; hence, $k^2 + k \geq N$. By step 1, $\pi^*\mathbb{O}_{\mathbb{C}P^2}(d) - 2\sum_{j=1}^{k^2+k} E_j$ is nef, hence $\pi^*\mathbb{O}_{\mathbb{C}P^2}(d) - 2\sum_{j=1}^{N} E_j$ is nef as well. □

Remark. More careful considerations, in the spirit of the proof of Theorem 2.1.A, actually show that *when* $d > 5$, *the divisor* $D = \pi^*\mathbb{O}_{\mathbb{C}P^2}(d) - 2\sum_{j=1}^{N} E_j$ *is ample if and only if* $D \cdot D > 0$.

To prove this, one first has to sharpen the second statement of Lemma 3.4.A and prove that $(d; m_1, m_2, 1^{\times r})$ is ample when $d > m_1 + m_2$ and $d^2 > m_1^2 + m_2^2 + r$. This can be done by arguments similar to, though more delicate than those used to prove nefness of these vectors. Then, using the "ample + nef ⇒ ample" case of Theorem 2.A, one deduces as in the proof of Theorem 2.1.A that the divisor D is ample for $N = k^2 + k$, when $k > 2$. The case of general N can be easily reduced to $N = k^2 + k$ as in the preceding proof.

3.7. Proof of Corollary 2.1.C. Let $N = N_1 N_2$. Nagata's conjecture for N is equivalent to the nefness of vector $v = (d; m^{\times N})$ for every $d, m > 0$ that satisfy $d^2 - Nm^2 > 0$.

Let d, m be two such numbers. Choose a positive rational number x such that $d^2 > x^2 N_2 > m^2 N$. The assumption that Nagata's conjecture is true for N_1 and N_2 implies that the vectors $u = (x; m^{N_1}) \in \mathbb{Q}^{N_1+1}$ and $w = (d; x^{\times N_2}) \in \mathbb{Q}^{N_2+1}$ are nef.

We have $v = \big(\cdots((w\#_{N_2} u)\#_{N_2-1} u)\cdots\big)\#_1 u$. Observing that Theorem 2.A also remains valid for vectors of rational numbers, we conclude that v is also nef. □

4. A conjecture relating continued fractions and remainders of a line bundle. The goal of this section is to propose a conjecture concerning estimates on the values of the homogeneous remainders of $\mathbb{O}_{\mathbb{C}P^2}(1)$, defined in Section 2.1. It turns out that all the cases appearing in the statement of Theorem 2.1.A are particular cases of this conjecture.

First we recall some relevant facts from classical number theory. Given a nonsquare natural number N, consider the following Diophantine equation in the unknowns d, m:

$$d^2 - Nm^2 = 1.$$

This equation was attached to the name *Pell's equation* in the ancient literature and has been extensively studied by many mathematicians in the seventeenth and eighteenth centuries, including Leonard Euler (see [NZ], [IrRo], [VndP]). The classical result about this equation is that all of its solutions come from *continued-fractions expansions* of $\sqrt{N}$. We write $\langle a_0, a_1, \ldots, a_n \rangle$ for the continued fractions expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

Similarly, we denote by $\langle a_0, a_1, \dots\rangle$ an infinite continued-fractions expansion. It is not hard to see that the continued-fractions expansion of $\sqrt{N}$ must be of the special periodic form

$$\sqrt{N} = \langle a_0, a_1, \dots, a_{n-1}, 2a_0, a_1, \dots, a_{n-1}, 2a_0, \dots\rangle;$$

hence, we write from now on $\sqrt{N} = \langle a_0, \overline{a_1, \dots, a_{n-1}, 2a_0}\rangle$, where the bar denotes the periodic part. Moreover, it turns out that $a_i = a_{n-i}$ for every $1 \le i \le n-1$ (i.e., $(a_1, \dots, a_{n-1})$ is a palindrome). Define a rational number d/m as follows: if n is even put

$$\frac{d}{m} = \langle a_0, a_1, \dots, a_{n-1}\rangle,$$

while for n odd,

$$\frac{d}{m} = \langle a_0, a_1, \dots, a_{n-1}, 2a_0, a_1, \dots, a_{n-1}\rangle.$$

It is well known that (d, m) provides the minimal solution of Pell's equation, called the *fundamental solution*. Moreover, any other solution of Pell's equation is obtained in a similar manner—by truncating the infinite continued fraction of $\sqrt{N}$ one term before the end of one of its periods. More precisely, (d, m) solves Pell's equation if and only if

$$\frac{d}{m} = \Big\langle a_0, \overline{a_1, \dots, a_{n-1}, 2a_0}^{\times r}, a_1, \dots, a_{n-1}\Big\rangle, \quad \text{where } r \text{ is odd if } n \text{ is odd.}$$

This formula means that the periodic part $a_1, \dots, a_{n-1}, 2a_0$ should be taken r times and then once more without the last member $2a_0$. The number r is allowed to be any nonnegative integer in the case when n is even, and r must be odd if n is odd. Our conjecture follows.

Conjecture 4.A. *Let $N > 9$ be a nonsquare integer and let d/m be the fundamental solution of the corresponding Pell's equation. Then*
(1) *the vector $(d; m^{\times N})$ is nef;*
(2) $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) \le 1/d^2$.

Our conjecture is much weaker than Nagata's; on the other hand it seems more accessible. Indeed, our methods provide a proof for the conjecture in the following cases.

(1) *Consider the case that $\sqrt{N}$ has a 2-periodic, continued-fractions expansion* $\sqrt{N} = \langle a_0, \overline{a_1, 2a_0}\rangle$. It is easy to see that this is the case if and only if $a_1 \mid 2a_0$ and $N = a_0^2 + 2(a_0/a_1)$. The fundamental solution of Pell's equation is $d = a_0a_1 + 1, m = a_1$.

(A) *Suppose that* $a_1 \mid a_0$. Putting $a = a_1$ and $l = a_0/a_1$, we get $N = a^2l^2 + 2l$ and $d = a^2l + 1$. By Theorem 2.1.A, our conjecture holds in this case.

(B) *Suppose that* $a_1 \nmid a_0$. Note that since $a_1 \mid 2a_0$, a_1 must be even. Putting $a = a_1/2$ and $l = 2a_0/a_1$, we obtain $N = a^2l^2 + l$ and $d = 2a^2l + 1$. By Theorem 2.1.A, our conjecture holds.

(2) *Consider N's of the form* $N = a^2l^2 - 2l$. It is not hard to see that $d = a^2l - 1, m = a$ satisfy Pell's equation $d^2 - Nm^2 = 1$. By Theorem 2.1.A, we have $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) \leq 1/d^2$. Therefore, if (d', m') is the fundamental solution of Pell's equation, then $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1)) \leq 1/d^2 \leq 1/d'^2$, and so the conjecture holds. Note that in this case the period of expansion of $\sqrt{N}$ is usually longer than 2 (for example, $\sqrt{14} = \langle 3, \overline{1, 2, 1, 6} \rangle$).[2]

(3) Here are two other examples which do not fall into the above categories.

(A) $N = 19$. In this case $\sqrt{19} = \langle 4, \overline{2, 1, 3, 1, 2, 8} \rangle$. The fundamental solution is $d = 170, m = 39$. Thus, our conjecture suggests that $\mathcal{R}_{19}(\mathcal{O}_{\mathbb{C}P^2}(1)) \leq 1/170^2$.

(B) $N = 22$. In this case $\sqrt{22} = \langle 4, \overline{1, 2, 4, 2, 1, 8} \rangle$. The fundamental solution is $d = 197, m = 42$. Thus, our conjecture suggests that $\mathcal{R}_{22}(\mathcal{O}_{\mathbb{C}P^2}(1)) \leq 1/197^2$.

Let us prove that the conjecture indeed holds in these two cases. We start with $N = 19$. By Lemma 3.4.A the vector $u = 39(2; 1^{\times 4}) = (78; 39^{\times 4})$ is nef. We have

$$(170; 39^{\times 19}) = \Big(\big((170; 78^{\times 3}, 39^{\times 7}) \#_3 u \big) \#_2 u \Big) \#_1 u.$$

Thus, we are reduced to proving that $v = (170; 78^{\times 3}, 39^{\times 7})$ is nef. To do this we apply the transformation Π to v (see Section 3.2). One can easily compute that $\Pi(v) = (1; 0^{\times 10})$, which is nef.

The case $N = 22$ is similar. Here we use the decomposition

$$(197; 42^{\times 22}) = \Big(\big(\big((197; 84^{\times 4}, 42^{\times 6}) \#_4 u \big) \#_3 u \big) \#_2 u \Big) \#_1 u,$$

with $u = 42(2; 1^{\times 4}) = (84; 42^{\times 4})$. Applying Π to $(197; 84^{\times 4}, 42^{\times 6})$, we again obtain the vector $(1; 0^{\times 10})$, which is nef.

5. Gluing ample divisors. We derive Theorem 2.A as a corollary of a more general gluing theorem, which we now present. First we need to introduce some new notation.

Let X be a smooth projective variety and $\mathcal{L}$ a line bundle over X. Given rational numbers $m_1, \dots, m_k$, we say that $(X, \mathcal{L}; m_1, \dots, m_k)$ is ample (resp., nef) if the $\mathbb{Q}$-divisor class

$$\pi^* \mathcal{L} - \sum_{j=1}^{k} m_j E_j$$

[2] It is not hard to see that if $N = a^2l^2 - 2l$, then $\sqrt{N}$ has 2-periodic expansion with minus signs.

is ample (resp., nef) on the blowup $\pi : \widetilde{X} \to X$ of X at some k distinct points $p_1, \dots, p_k$. Here, E_j is the exceptional divisor over p_j. In the case $X = \mathbb{C}P^n$, we say that $(d; m_1, \dots, m_k)$ is ample (resp., nef) if $(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}(d); m_1, \dots, m_k)$ is ample (resp., nef).

Given a smooth subvariety $V \subset X$, we define a new variety, denoted $S_{V/X}$, to be the total space of the fiber bundle

$$S_{V/X} = \mathbb{P}(N_{V/X} \oplus \mathcal{O}_V) \xrightarrow{p_V} V,$$

where $N_{V/X}$ is the normal bundle of V in X. Next, put $Z_{V/X}$ to be the divisor

$$Z_{V/X} = \mathbb{P}(N_{V/X} \oplus 0) \subset S_{V/X}.$$

Finally, write $\Theta : \widetilde{X}_V \to X$ for the blowup of X along V and write $E_V \subset \widetilde{X}_V$ for the exceptional divisor of this blowup.

THEOREM 5.A. *Let $\mathcal{L}$ be a line bundle on X, and let $V \subset X$ be a smooth subvariety. Suppose that*

(1) *the $\mathbb{Q}$-divisor class $\Theta^*\mathcal{L} - mE_V$ is ample (resp., nef) on $\widetilde{X}_V$ for some $m \in \mathbb{Q}$;*
(2) *$(S_{V/X}, p_V^*(\mathcal{L}|_V) + mZ_{V/X}; \alpha_1, \dots, \alpha_k)$ is ample (resp., nef) for some $\alpha_1, \dots, \alpha_k \in \mathbb{Q}$.*

Then $(X, \mathcal{L}; \alpha_1, \dots, \alpha_k)$ is ample (resp., nef).

The proof is presented in Section 5.2 below. Here is a simple corollary of this theorem.

COROLLARY 5.B. *Let X be a smooth, n-dimensional, projective variety. Suppose that $(X, \mathcal{L}; m)$ is ample (resp., nef) and that $(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}(m); \alpha_1, \dots, \alpha_k)$ is nef, where $\alpha_1, \dots, \alpha_k > 0$. Then $(X, \mathcal{L}; \alpha_1, \dots, \alpha_k)$ is ample (resp., nef).*

Proof. Take V to be a point in X; then we have $S_{V/X} = \mathbb{C}P^n$ and $Z_{V/X}$ is just a hyperplane. Obviously we have $p_V^*(\mathcal{L}|_V) = \mathcal{O}_{\mathbb{C}P^n}$. Substituting all this in Theorem 5.A the proof readily follows for the case ample + ample $\Rightarrow$ ample.

To prove the ample + nef $\Rightarrow$ ample case observe that since $\pi^*\mathcal{L} - mE$ is ample it follows from Seshadri's criterion for ampleness that there exists an $\epsilon > 0$ such that $\pi^*\mathcal{L} - (m + \epsilon)E$ is still ample. Since $(m; \alpha_1, \dots, \alpha_k)$ is nef it easily follows from Kleiman's criterion for ampleness that $(m + \epsilon; \alpha_1, \dots, \alpha_k)$ is ample (note that we assume that all the α_j's are strictly positive). By the ample + ample $\Rightarrow$ ample case we see that $(X, \mathcal{L}; \alpha_1, \dots, \alpha_k)$ is ample. □

Notice that for $n = 2$, and $\Theta : X \to \mathbb{C}P^2$ a simple rational surface with an ample line bundle $\mathcal{L} = \Theta^*\mathcal{O}_{\mathbb{C}P^2}(d) - \sum_{j=1}^n m_j E_j$, Corollary 5.B becomes precisely Theorem 2.A.

5.1. Different approaches to gluing. Theorem 2.A was originally conjectured by the author in view of (very simple) arguments coming from symplectic geometry (see the discussion in Section 7 below). Our original proof of this theorem made use of an extension, due to Shustin, of the Viro method for gluing algebraic curves while preserving singularities.

This method works in the following setting. Suppose that $C_1, \dots, C_n$ are plane curves with Newton polygons $\Delta_1, \dots, \Delta_n$ that have mutually disjoint interiors and match together to a bigger polygon $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$. Shustin's method allows us, under some transversality conditions on the equisingular strata corresponding to $C_1, \dots, C_n$, to construct a new curve C with Newton polygon Δ and with singular points "inherited" from $C_1, \dots, C_n$. We refer the reader to [Sh] for a detailed presentation of the general method and for interesting applications.

In our case, we have two ample vectors $(d; m_1, \dots, m_n, m)$, and $(m; \alpha_1, \dots, \alpha_k)$. After multiplication by a large enough number, we may assume that there exist irreducible smooth curves in the linear system corresponding to these vectors. The condition that the last components of the first vector equals the degree of the second means that the Newton polygons of our curves match as required by the Shustin-Viro method. Then this method yields a new irreducible curve, obtained by "gluing" the two given ones which lies in the linear system of the vector $(d; m_1, \dots, m_n, \alpha_1, \dots, \alpha_k)$.

Although this approach to Theorem 2.A is quite constructive and explicit, the proof is much longer and complicated than the rather standard approach we take in Section 5.2 below, which is based on a degeneration argument. This type of argument also leads to the more general version stated in Corollary 5.B. The whole idea of using degenerations was kindly suggested to me by Robert Lazarsfeld, whom I would like to thank.

5.2. Proof of Theorem 5.A. This proof is based on a degeneration argument (compare [CiMi], [R]). Assume first that both $\Theta^* \mathcal{L} - m E_V$ and $(S_{V/X}, p_V^*(\mathcal{L}|_V) + m Z_{V/X};\ \alpha_1, \dots, \alpha_k)$ are ample. The case when both of them are nef follows immediately by passing to limits, since the nef cone is the closure of the ample cone. Also note that it is enough to prove the theorem under the assumption that m and the α_j's are all integers by multiplying them and $\mathcal{L}$ by a suitable integer N, for which $Nm, N\alpha_j \in \mathbb{Z}$.

Let $\Delta \subset \mathbb{C}$ be a disc centered at the origin. Denote by $\sigma : Y \to X \times \Delta$ the blowup of $X \times \Delta$ along the subvariety $V \times \{0\}$, and let $\widehat{E}_V = \sigma^{-1}(V \times \{0\}) \subset Y$ be the exceptional divisor. For every $t \in \Delta$, we denote by X_t the total transform of $X \times \{t\}$ in Y with respect to σ. When $t \neq 0$ we have $X_t = X$, while for $t = 0$ we have $X_0 = \overline{X} \cup \widehat{E}_V$, where $\overline{X}$ is the strict transform of $X \times \{0\}$ in Y.

Restricting σ to $\overline{X}$, it is easy to see that $\sigma : \overline{X} \to X$ is just the blowup $\Theta : \widetilde{X}_V \to X$ of X along V. In the same manner, we have that $\widehat{E}_V$ is just $S_{V/X}$. These two last identifications match together to give

$$X_0 = \widetilde{X}_V \cup_{E_V = Z_{V/X}} S_{V/X}.$$

Consider now the line bundle on Y

$$\widehat{\mathcal{L}} = q^* \mathcal{L} \otimes \mathcal{O}_Y\big(-m \widehat{E}_V\big),$$

where $q : Y \to X$ is the composition of σ with the projection $X \times \Delta \to X$. A simple calculation shows that

$$\widehat{\mathscr{L}}|_{\widehat{E}_V} = p_V^*(\mathscr{L}|_V) \otimes \mathbb{O}_{S_{V/X}}(mZ_{V/X}),$$

and that

$$\widehat{\mathscr{L}}|_{\overline{X}} = \Theta^* \mathscr{L} \otimes \mathbb{O}_{\widetilde{X}_V}(-mE_V).$$

Now, let $x_1, \dots, x_k \in \widehat{E}_V = S_{V/X}$ be very general points for which the divisor class

$$\pi^*\big(\widehat{\mathscr{L}}|_{\widehat{E}_V}\big) - \sum_{j=1}^{k} \alpha_j E_j = \pi^*\big(p_V^*(\mathscr{L}|_V) + mZ_{V/X}\big) - \sum_{j=1}^{k} \alpha_j E_j$$

is ample on the blowup $\pi : \mathrm{Bl}(\widehat{E}_V) \to \widehat{E}_V$ at $x_1, \dots, x_k$.

Consider a deformation of points $x_1(t), \dots, x_k(t) \in X$ depending holomorphically on $t \in \Delta$ with $x_i(0) = x_i$, and such that $x_i(t) \neq x_j(t)$ for every $i \neq j$ and $t \in \Delta$. Denote by $\Sigma_i \subset Y$ the strict transform of the subvarieties $\{(x_i(t), t) \mid t \in \Delta\} \subset X \times \Delta$ and let $\widetilde{\pi} : \widetilde{Y} \to Y$ be the blowup of Y along $\cup_{i=1}^k \Sigma_i$ (note that the Σ_i's are disjoint).

Let $\mathcal{N}$ be the line bundle corresponding to the divisor class

$$\mathcal{N} = \widetilde{\pi}^* \widehat{\mathscr{L}} - \sum_{j=1}^{k} \alpha_j \widetilde{\Sigma}_j,$$

where $\widetilde{\Sigma}_j$ is the exceptional divisor over Σ_j. The variety $\widetilde{Y}$ is a flat family over Δ with fibers $\widetilde{Y}_t$. When $t \neq 0$, $\widetilde{Y}_t$ is just the blowup of X at $x_1(t), \dots, x_k(t)$. The fiber over $t = 0$, namely, $\widetilde{Y}_0$, is the blowup of X_0 at $x_1, \dots, x_k \in \widehat{E}_V$. Thus, $\widetilde{Y}_0$ consists of two components

$$\widetilde{Y}_0 = \overline{X} \cup \mathrm{Bl}\big(\widehat{E}_V\big),$$

and we have that

$$\mathcal{N}\big|_{\overline{X}} = \widehat{\mathscr{L}}\big|_{\overline{X}} \qquad \text{and} \qquad \mathcal{N}|_{\mathrm{Bl}(\widehat{E}_V)} = \pi^*\big(\widehat{\mathscr{L}}|_{\widehat{E}_V}\big) - \sum_{j=1}^{k} \alpha_j E_j.$$

It follows that $\mathcal{N}|_{\widetilde{Y}_0}$ is ample, because its restriction to each irreducible component of $\widetilde{Y}_0$ is. Since in flat families ampleness is an open condition in the base, we see that $\mathcal{N}|_{\widetilde{Y}_t}$ is ample for t sufficiently close to zero, and, consequently, for such t's

$$\pi^* \mathscr{L} - \sum_{j=1}^{k} \alpha_j E_j$$

ample on the blowup $\pi : \widetilde{X} \to X$ of X at $x_1(t), \dots, x_k(t)$. □

5.3. More asymptotics on remainders. In order to prove the third statement of Theorem 2.1.A, we use Theorem 5.A with V being a curve.

Let X be a smooth surface and $V \subset X$ a smooth curve. Clearly, the blowup $\widetilde{X}_V$ of

X along V coincides with X itself, and $S = S_{V/X} \xrightarrow{p_V} V$ is a ruled surface over V. It is easy to see that S has the form $S = \mathbb{P}(\mathcal{O}_V(V) \oplus \mathcal{O}_V)$, where $\mathcal{O}_V(V)$ is the restriction of $\mathcal{O}_X(V)$ to V. Note that $\deg \mathcal{O}_V(V) = V \cdot V$. Denote by F the numerical equivalence class of a fiber of the ruled surface $p_V : S \to V$. It is easy to check that the divisor $Z = Z_{V/X} \subset S$ is just a section of p_V with self-intersection number $Z \cdot Z = -V \cdot V$. Next, we have that $p_V^*(\mathcal{L}|_V)$ is numerically equivalent to $(\mathcal{L} \cdot V)F$. Substituting all this in Theorem 5.A we obtain the following result.

COROLLARY 5.3.A. *Let $\mathcal{L}$ be a line bundle over a smooth surface X and $V \subset X$ a smooth curve. Suppose that*

(1) *$\mathcal{L} - mV$ is ample (resp., nef);*

(2) *$(S, (\mathcal{L} \cdot V)F + mZ; \alpha_1, \dots, \alpha_k)$ is ample (resp., nef), where $\alpha_1, \dots, \alpha_k > 0$.*

Then $(X, \mathcal{L}; \alpha_1, \dots, \alpha_k)$ is ample (resp., nef).

As in Lemma 3.4.A, we need an initial collection of nef vectors to begin with. Consider a smooth surface X and an *irrational* smooth curve $V \subset X$. Suppose that

- $V \cdot V = 0$, and
- $N_{V/X} \neq \mathcal{O}_V$, where $N_{V/X}$ is the normal bundle of V in X.

Note that in particular these two conditions imply that $N_{V/X}$ has no global nonzero sections.

Under such assumptions we have the next result.

LEMMA 5.3.B. *Let $V \subset X$ be as above. Then for every $n \geq 0$, $(S_{V/X}, nF + 2Z_{V/X}; 2^{\times n})$ is nef.*

The proof is given in the end of this section. We turn now to the proof of the third statement of Theorem 2.1.A.

Proof of Theorem 2.1.A (3). In order to prove the needed estimate for $\mathcal{R}_N(\mathcal{O}_{\mathbb{C}P^2}(1))$, we must prove that the vector $(2a^2 l + 1; 2a^{\times a^2 l^2 + l})$ is nef.

Consider a smooth curve $V' \subset \mathbb{C}P^2$, of degree al, and let $p_1, \dots, p_{a^2 l^2} \in V'$ be distinct points such that no other curve of degree al passes through them (recall that $al \geq 3$). Let $\pi : X \to \mathbb{C}P^2$ be the blowup of $\mathbb{C}P^2$ at these points, and let $V \subset X$ be the proper transform of V' in X. Finally, put

$$\mathcal{L} = \pi^* \mathcal{O}_{\mathbb{C}P^2}\big(2a^2 l + 1\big) - 2a \sum_{j=1}^{a^2 l^2} E_j.$$

Since $\mathcal{L} - 2aV \equiv \pi^* \mathcal{O}_{\mathbb{C}P^2}(1)$ is nef, it is enough, by Corollary 5.3.A, to prove that

$$D = \big(S, (\mathcal{L} \cdot V)F + 2aZ; 2a^{\times l}\big) = a\big(S, lF + 2Z; 2^{\times l}\big) \qquad (*)$$

is nef, where $S = S_{V/X}$ and $Z = Z_{V/X}$. Due to our choices of $p_1, \dots, p_{a^2 l^2} \in V' \subset \mathbb{C}P^2$, it is not hard to see that the conditions of Lemma5.3.B are satisfied. Thus, the vector $(*)$ is nef. □

Proof of Lemma 5.3.B. The statement is trivial in the case $n = 0$; hence, we assume that $n \geq 1$. We argue by induction on n.

Step A. [3] Let $n = 1$. From our assumptions on $N_{V/X}$ it follows that $S_{V/X} \to V$ has exactly two sections that are numerically equivalent to $Z_{V/X}$, namely, $Z_\infty = Z_{V/X}$ itself (the section at infinity) and $Z_0 = \mathbb{P}(0 \oplus \mathbb{O}_V)$ (the zero section). We denote their numerical equivalence class by Z.

Choose a point $p \in S_{V/X} \setminus (Z_\infty \cup Z_0)$ and denote by $\pi : \widetilde{S}_{V/X} \to S_{V/X}$ the blowup of $S_{V/X}$ at p. With this notations put $D = \pi^*(F + 2Z) - 2E$.

Since $D \cdot D = 0$, to prove nefness of D it is enough to show that for every irreducible curve $G \subset \widetilde{S}_{V/X}$, we have $D \cdot G \geq 0$. Indeed, suppose that $G \equiv \pi^*(aF + bZ) - mE$. If $G \equiv \pi^*(F - E)$, then $D \cdot G = 0$, so assume that $G \not\equiv \pi^*(F - E)$. By the same argument, we may also assume that $G \not\equiv \pi^* F, \pi^* Z, -mE$. Thus, we obtain that $0 \leq G \cdot (\pi^* F - E) = b - m$, $0 \leq G \cdot \pi^* Z = a$ and $0 \leq G \cdot E = m$. Summarizing all this, we have that

$$b \geq m \geq 0 \qquad \text{and} \qquad a \geq 0.$$

Let $\overline{G}$ be the normalization of the curve G. Composing the projection $\overline{G} \to G$ with the projection $S_{V/X} \to V$ we obtain a branched covering $\overline{G} \to V$ (recall that we assume that $G \not\equiv \pi^* F, \pi^* F - E$). The degree of this branched covering is at least $G \cdot \pi^* F = b$.

Let $g = \text{genus}(V)$. Using the adjunction and Riemann-Hurwitz formulas, we obtain the inequality

$$1 + \frac{G \cdot G + K \cdot G}{2} \geq b(g-1) + 1, \tag{1}$$

where K is the canonical class of $\widetilde{S}_{V/X}$. Let us substitute into this the numerical equivalence class of G; this and a straightforward simplification transform inequality (1) to

$$2a(b-1) \geq m^2 - m. \tag{2}$$

If $m = 0$ then it is easy to see that $D \cdot G \geq 0$, and if $m = 1$ then $D \cdot G = 2a + b - 2 \geq 0$, because if $a = 0$ then we must have $b \geq 2$, for otherwise $G \equiv \pi^* Z - E$, which is impossible by our choice of the point p and our assumptions on $N_{V/X}$. It remains to prove that $D \cdot G \geq 0$ for the case $m \geq 2$. Under this assumption, we have (in view of (2)) that

$$D \cdot G = b + 2a - 2m \geq b + \frac{m^2 - m}{b - 1} - 2m = \frac{(b - m)\big(b - (m+1)\big)}{b - 1}.$$

It easily follows now that $D \cdot G \geq 0$.

Step B. Let $n \geq 2$, and assume that the statement is true for every $n' < n$. Assuming this we now prove the statement for n.

[3] I would like to thank Aaron Bertram for suggesting to me the proof presented here.

Consider the blowup of $\pi : \widetilde{S}_{V/X} \to S_{V/X}$ at $n-1$ points not lying on $Z_\infty \cup Z_0$, and put

$$\mathscr{L} = \pi^*\big(nF + 2Z_\infty\big) - 2\sum_{j=1}^{n-1} E_j,$$

where the E_j's are the exceptional divisors over the blown-up points. Consider a fiber of $S_{V/X} \to V$ which passes through the last of the $n-1$ blown-up points, and let C be its proper transform in $\widetilde{S}_{V/X}$. We have $C \equiv \pi^*F - E_{n-1}$.

By the induction hypothesis $\mathscr{L} - 2C$ is nef, and so by Corollary 5.3.A, in order to prove nefness of $(S_{V/X}, nF + 2Z; 2^{\times n})$ it is enough to show that the divisor

$$D = \pi'^*\big((\mathscr{L}\cdot C)F' + 2Z'\big) - 2E = 2\big(\pi'^* Z' - E\big)$$

is nef on the blowup $\pi' : \widetilde{S}' \to S'$ of the (rational ruled) surface

$$S' = \mathbb{P}\big(N_{C/S_{V/X}} \oplus \mathcal{O}_C\big)$$

at some point $p \in S'$. Here we denote by Z' the divisor $Z' = Z_{C/S_{V/X}}$. Note that we are free to choose the point $p \in S'$ as we wish, so let us choose it to lie on Z'. This implies that the divisor D is equivalent to an effective irreducible divisor, and since $Z' \cdot Z' = -C \cdot C = 1$, we have that $D \cdot D = 0$. Nefness of D is now obvious. □

6. Proof of miscellaneous lemmas and a proposition

6.1. Passing from specific points to very general. In order to prove Lemma 3.4.A, we choose the k blown-up points $q_1, \dots, q_k \in \mathbb{C}P^2$ to lie in a very specific position is not generic. This is justified by the following lemma.

LEMMA 6.1.A. *Let F be a divisor on a smooth projective variety S, and let $q_1^{(0)}, \dots, q_k^{(0)} \in S$ be distinct points. Let $\pi_0 : \widetilde{S}_0 \to S$ be the blowup of S at $q_1^0, \dots, q_k^0$ with exceptional divisors $E_i^0 = \pi_0^{-1}(q_i^0), i = 1, \dots, k$. Suppose that for some $f_1, \dots, f_k \geq 0$, the divisor $\pi_0^* F - \sum_{j=1}^k f_j E_j^0$ is ample (resp., nef). Then, for a very general choice of points $q_1, \dots, q_k \in S$, the divisor*

$$\pi^* F - \sum_{j=1}^k f_j E_j$$

is ample (resp., nef) on the blowup $\pi : \widetilde{S} \to S$ of S at $q_1, \dots, q_k$ with exceptional divisors $E_j = \pi^{-1}(q_j), \quad j = 1, \dots, k$.

Proof. This lemma is well known and rather standard. The idea of its proof is as follows. Denote by $\mathscr{C}_k$ be the k-configuration space of S. One can easily construct a flat family $p : Y \to \mathscr{C}_k$ whose fiber over $\underline{x} = (x_1, \dots, x_k) \in \mathscr{C}_k$ is the blowup

$\pi_{\underline{x}} : Y_{\underline{x}} \to S$ of S at $x_1, \dots, x_k$. Furthermore, there exists a line bundle $\mathcal{L}$ on Y whose restriction to each fiber $Y_{\underline{x}}$ is in the class of the divisor

$$D_{\underline{x}} = \pi_{\underline{x}}^* F - \sum_{j=1}^{k} E_j^{\underline{x}}.$$

Here, we denote by $E_j^{\underline{x}} \subset Y_{\underline{x}}$ the exceptional divisor over $x_j \in S$.

Now, in flat families ampleness is an open condition on the base. Therefore, since $D_{\underline{x}}$ is ample for $\underline{x}_0 = (q_1^0, \dots, q_k^0)$, the same holds also for every $\underline{x}$ in a Zariski open neighborhood of $\underline{x}_0 \in \mathcal{C}_k$.

The proof of the "nef" statement is similar. □

6.2. Proof of Lemma 3.4.A. Notice first that in view of Lemma 6.1.A it is enough to prove that our vectors are nef (resp., ample) on a specific simple rational surface.

(1) Consider first the case $d^2 > r$. In [Nag] (consult also [ShTy]) Nagata proved that if N is a square, then for generic points $p_1, \dots, p_N \in \mathbb{C}P^2$, and for every irreducible curve $C \subset \mathbb{C}P^2$, the *strict* inequality

$$\deg(C) > \frac{\sum_{j=1}^{N} \mathrm{mult}_{p_j}(C)}{\sqrt{N}} \tag{1}$$

holds. Let V_r be the blowup of $\mathbb{C}P^2$ at r generic points, and denote by $\Theta : \widetilde{V}_r \to V_r$ the blowup of V_r at $d^2 - r$ generic points. Thus, $\widetilde{V}_r$ is the same as the blowup of $\mathbb{C}P^2$ at $N = d^2$ generic points, and it follows from inequality (1) that the divisor $\widetilde{D} = \Theta^*(dL - \sum_{j=1}^{r} E_j) - \sum_{j=r+1}^{d^2} E_j$ intersects every curve positively. This immediately implies that $D = dL - \sum_{j=1}^{r} E_j$ intersects any curve in V_r positively. As $D \cdot D > 0$, the statement follows from the Nakai-Moishezon criterion (see [Ha1]).

The proof for the nef case ($d^2 \geq r$) is much easier. Indeed, let $C \subset \mathbb{C}P^2$ be an irreducible smooth curve of degree d, and let $p_1, \dots, p_r$ be distinct points on C. Let V be the blowup of $\mathbb{C}P^2$ at $p_1, \dots, p_r$, and let D be the proper transform of C in V, $D \in |dL - \sum_{j=1}^{r} E_j|$. As D is an irreducible curve of nonnegative self-intersection, the vector $(d; 1^{\times r})$ corresponding to the divisor class of D is nef on V.

(2) Set $D = dL - m_1 E_1 - m_2 E_2$. Consider the linear system $|D|$ on V_2—the blowup of $\mathbb{C}P^2$ at two points. As $D = m_1(L - E_1) + m_2(L - E_2) + (d - m_1 - m_2)L$ it is easy to see that $|D|$ is not empty and has no base-points; hence, by Bertini's theorem, there exists an irreducible (smooth) curve $C \in |D|$. Choose r distinct points $p_1, \dots, p_r \in C \setminus (E_1 \cup E_2)$, and let $\widetilde{V}$ be the blowup of V at $p_1, \dots, p_r$. Finally, denote by $\widetilde{C}$ be the proper transform of C in $\widetilde{V}$.

We have $\widetilde{C} \in |dL - m_1 E_1 - m_2 E_2 - \sum_{j=3}^{r+2} E_j|$. As $\widetilde{C}$ is irreducible and $\widetilde{C} \cdot \widetilde{C} \geq 0$, the vector $(d; m_1, m_2, 1^{\times r})$ is nef. □

6.3. Proof of Proposition 3.1.A. The fact that the Cremona action on H_k preserves the ample and nef cones follows easily from Lemma 6.1.A and the following lemma, which essentially appears in [DoOr]. We use the notation introduced in Section 3.1.

LEMMA 6.3.A. *Let V be a simple rational surface obtained by blowing up $\Theta : V \to \mathbb{C}P^2$ points $p_1, \dots, p_n \in \mathbb{C}P^2$ in general position. Then for every $\sigma \in Cr_k$, there exists a simple rational surface V_σ obtained by blowing up $\Theta_\sigma : V_\sigma \to \mathbb{C}P^2$ points $q_1, \dots, q_k$ in general position and a biholomorphism $f_\sigma : V_\sigma \to V$ making the commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}(V) & \xrightarrow{f_\sigma^*} & \mathrm{Pic}(V_\sigma) \\ {\scriptstyle m_\Theta}\downarrow & & \downarrow{\scriptstyle m_{\Theta_\sigma}} \\ H_k & \xrightarrow{\sigma} & H_k. \end{array}$$

7. Symplectic interpretations. The purpose of this section is to explain the intuition that gave rise to the gluing Theorem 2.A and to Corollaries 5.B and 5.3.A. Interestingly enough, this comes from symplectic geometry.

Symplectic geometry is the branch of geometry dealing with the structure of symplectic manifolds that are by definition pairs (M, Ω) consisting of a smooth manifold M and a nondegenerate, closed, differential 2-form Ω. The reader is referred to [AG] and [MS] for the foundations.

Due to developments in this field of research in the last decade, many analogies have been discovered between symplectic and complex manifolds. These become especially striking in dimension 4, where symplectic 4-manifolds play the role of complex surfaces. In several cases it turns out that algebro-geometric considerations remain true when properly translated into the symplectic category and so give rise to new theorems in the symplectic framework. This principle is reflected very well in Donaldson's symplectic submanifolds theory, in the classification of rational and ruled symplectic manifolds of Lalonde-McDuff, in the symplectic packing theorems of McDuff-Polterovich, in Ruan's symplectization of the extremal rays theory, and in the work of others.

In this paper we have, in some sense, reversed this direction of reasoning. Our main theorem is in fact an algebro-geometric translation of a very simple symplectic fact arising from the theory of symplectic packing. We refer the reader to [MP] for an excellent exposition on the symplectic packing problem.

Recall from [MP] that a *symplectic packing* of (M, Ω) by N balls of radii $\lambda_1, \dots, \lambda_N$ is a symplectic embedding of a disjoint union of N balls into (M, Ω)

$$\varphi = \coprod_{j=1}^{N} \varphi_j : \coprod_{j=1}^{N} B(\lambda_j) \to (M, \Omega),$$

where $B(\lambda_j)$ stands for the standard Euclidean closed ball of radius λ_j of the same dimension as M, endowed with its standard symplectic structure $\omega_{std} = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$.

It was discovered by McDuff [M] that every symplectic packing gives rise to a symplectic form $\widetilde{\Omega}$ on the blowup $\Theta : \widetilde{M} \to M$ of M at the points $p_1 = \varphi_1(0), \dots, p_N =$

$\varphi_N(0)$. This form lies in the cohomology class

$$[\widetilde{\Omega}] = [\Theta^*\Omega] - \pi \sum_{j=1}^{N} \lambda_j^2 e_j, \qquad (1)$$

where e_j denotes the Poincaré dual to the homology class of the exceptional divisor E_j of the blowup. This procedure is called *symplectic blowing up*.

Conversely, given a symplectic form $\widetilde{\Omega}$ on $\widetilde{M}$ which is nondegenerate on the exceptional divisors E_j and with cohomology class as in (1) above, one can perform *symplectic blowing down* at the exceptional divisors and obtain a symplectic form Ω on M and a symplectic packing $\varphi = \coprod_{j=1}^{N} \varphi_j : \coprod_{j=1}^{N} B(\lambda_j) \to (M, \Omega)$.

Consider the symplectic manifold $(\mathbb{C}P^2, \sigma)$, where σ is the Fubini-Studi Kähler form, normalized such that the area of a projective line is π. Its cohomology class is πl, where $l \in H^2(\mathbb{C}P^2, \mathbb{Z})$ is the standard positive generator. Call a vector of positive numbers $(d; m_1, \dots, m_k)$ *symplectic* if the cohomology class

$$d\Theta_V^* l - \sum_{j=1}^{k} m_j e_j$$

can be represented by a symplectic form $\widetilde{\omega}$ on the blowup $\Theta_V : V \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ at some k distinct points.

Now, let M be a complex surface and $\Theta_p : \widetilde{M}_p \to M$ its blowup at a point $p \in M$ with exceptional divisor E. Denote by e the Poincaré dual to the homology class of E.

PROPOSITION 7.A. *Let $a \in H^2(M)$, and suppose that there exists a positive number m such that the cohomology class $\Theta_p^* a - me \in H^2(\widetilde{M}_p)$ can be represented by a symplectic form whose restriction to E is nondegenerate.*[4] *Then, for every symplectic vector $(m; \alpha_1, \dots, \alpha_k)$, the cohomology class $\Theta^* a - \sum_{j=1}^{k} \alpha_j e_j$ on the blowup $\Theta : \widetilde{M} \to M$ at some k points can be represented by a symplectic form.*

The proof is based on the following very simple observation. If $\Theta_p^* a - me$ has a symplectic representative $\widetilde{\Omega}$, then by symplectic blowing down, one obtains a symplectic form Ω on M and an embedding φ of a standard 4-dimensional ball of radius $\sqrt{m/\pi}$ into (M, Ω). The same argument with slight modifications, applied to the vector $(m; \alpha_1, \dots, \alpha_k)$, implies that the standard ball of radius $\sqrt{m/\pi}$ admits a symplectic packing, say ϕ, by k balls of radii $\sqrt{\alpha_1/\pi}, \dots, \sqrt{\alpha_k/\pi}$. Composing these two embeddings we conclude that (M, Ω) admits a symplectic packing $\varphi \circ \phi$ of k balls of radii $\sqrt{\alpha_1/\pi}, \dots, \sqrt{\alpha_k/\pi}$. The proposition follows now from symplectic blowing up. For completeness, here are the precise arguments of the proof.

Proof. Let $\widetilde{\Omega}$ be a symplectic form on $\widetilde{M}_p$ lying in the cohomology class $\Theta_p^* a -$

[4] This means that E is a symplectic submanifold with respect to this form.

me, and suppose that the restriction of $\widetilde{\Omega}$ to E is nondegenerate. Applying symplectic blowing down to $\widetilde{\Omega}$, we obtain a symplectic form Ω on M lying in the cohomology class a and a symplectic embedding $\varphi : B(\sqrt{m/\pi}) \to (M, \Omega)$.

Let $\widetilde{\omega}$ be a symplectic form on the blowup $\Theta_V : V \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ lying in the cohomology class $m\Theta_V^* l - \sum_{j=1}^k \alpha_j e_j$. Since the Gromov invariants of the homology classes of each of the exceptional divisors are not zero we may assume that $\widetilde{\omega}$ is nondegenerate on the exceptional divisors E_j. Blowing down symplectically, we obtain a symplectic form ω on $\mathbb{C}P^2$ lying in the cohomology class ml and a symplectic packing $\psi : \coprod_{j=1}^k B(\sqrt{\alpha_j/\pi}) \to (\mathbb{C}P^2, \omega)$. Since any two cohomologous symplectic forms on $\mathbb{C}P^2$ are symplectomorphic, we may assume that $\omega = (m/\pi)\sigma$. It can be proved by the methods of [MP] that there exists a symplectic submanifold (with respect to ω) $L \subset M$, isotopic to a projective line, which is disjoint from Image ψ. It is well known that $(\mathbb{C}P^2 \setminus L, (m/\pi)\sigma)$ is symplectomorphic to $B(\sqrt{m/\pi})$; hence, we obtain a symplectic packing $\phi : \coprod_{j=1}^k B(\sqrt{\alpha_j/\pi}) \to B(\sqrt{m/\pi})$.

The composition $\varphi \circ \phi$ is a symplectic packing of (M, Ω) by k balls of radii $\sqrt{\alpha_1/\pi}, \ldots, \sqrt{\alpha_k/\pi}$. Blowing up symplectically with respect to this embedding yields a symplectic form on the blowup $\Theta : \widetilde{M} \to M$ of M at k points, which lies in the cohomology class $\Theta^* a - \sum_{j=1}^k \alpha_j e_j$. □

Let us try to translate Proposition 7.A to the language of algebraic geometry. Keeping in mind that in the symplectic category the role of Kähler forms is played by symplectic forms and the role of complex submanifolds by symplectic submanifolds, the Kählerian translation should read: "*If the cohomology class $\Theta_p^* a - me$ has a Kähler representative then for every Kähler vector $(m; \alpha_1, \ldots, \alpha_k)$ the cohomology class $\Theta^* a - \sum_{j=1}^k \alpha_j e_j$ has a Kähler representative, too.*"

Here, we call a vector $(m; \alpha_1, \ldots, \alpha_k)$ Kähler if the cohomology class

$$m\Theta_V^* l - \sum_{j=1}^k \alpha_j e_j$$

can be represented by a Kähler form $\widetilde{\omega}$ on some simple rational surface $\Theta_V : V \to \mathbb{C}P^2$.

Due to the Lefschetz theorem on $(1,1)$ classes and Kodaira's embedding theorem, it follows that on a complex manifold there is a bijection—via Poincaré duality—between the set of homology classes of ample $\mathbb{Q}$-divisors and the set of rational cohomology classes which can be represented by Kähler forms. Poincaré dualizing the "Kählerian translation," we are naturally led to the following: "*Let D be a divisor on M such that $\Theta_p^* D - mE$ is ample. Then for every ample vector $(m; \alpha_1, \ldots, \alpha_k)$ the divisor $\Theta^* D - \sum_{j=1}^k \alpha_j E_j$ is ample, too*". This is precisely the contents of Theorem 2.A and Corollary 5.B. It is worth mentioning here that Corollary 5.3.A has a symplectic interpretation as well. The symplectic analogue can be formulated in terms of Gompf's fiber sum construction [Go].

7.1. Symplectic meaning of the remainders $\mathcal{R}_N(\mathcal{L})$. In Section 2.1 we defined the homogeneous remainders $\mathcal{R}_N(\mathcal{L})$ of an ample line bundle $\mathcal{L}$ over a surface. The definition naturally extends to n-dimensional smooth varieties X in the following obvious way. Given $p_1, \dots, p_N \in X$, set

$$\mathcal{R}(\mathcal{L}, p_1, \dots, p_N) = \frac{1}{\mathcal{L}^n} \inf_{0 \le \epsilon \in \mathbb{R}} \left\{ \mathcal{L}_\epsilon^n \mid \mathcal{L}_\epsilon = \pi^*\mathcal{L} - \epsilon \sum_{j=1}^N E_j \quad \text{is nef} \right\},$$

where $\pi : \widetilde{X} \to X$ is the blowup of X at the points $p_1, \dots, p_N$ with exceptional divisors $E_i = \pi^{-1}(p_i)$. To get a more global invariant, define

$$\mathcal{R}_N(\mathcal{L}) = \inf \{\mathcal{R}(\mathcal{L}, p_1, \dots, p_N) \mid p_1, \dots, p_N \in X \quad \text{are distinct points}\}.$$

Let us explain the symplectic meaning of these constants. Let (M, Ω) be a symplectic manifold. Following McDuff and Polterovich [MP], consider the quantity

$$v_N(M, \Omega) = \sup_{\varphi, \lambda} \frac{\text{Vol(Image } \varphi)}{\text{Vol}(M, \Omega)},$$

where (φ, λ) passes over all the possible symplectic packings φ of (M, Ω) with N equal balls of (varying) radius λ. The volume of the manifold is defined to be $\text{Vol}(M, \Omega) = \int_M (1/n!)\Omega^{\wedge n}$.

The constants $v_N(M, \Omega)$ admit values between 0 and 1 and measure the maximal part of the volume of (M, Ω) that can be filled by symplectic packing with N equal balls. When $v_N = 1$ we say that there exists a *full packing* of (M, Ω) by N equal balls, while in the case $v_N < 1$ we say that there exists a *packing obstruction*.

In view of the preceding discussion it is easy to see that the homogeneous remainders $\mathcal{R}_N(\mathcal{L})$ of an ample line bundle over a complex manifold M, play the algebro-geometric role of the quantity $1 - v_N(M, \Omega)$, where Ω is a Kähler form representing the first Chern class of $\mathcal{L}$. In fact, it is easy to prove that the inequality

$$1 - v_N(M, \Omega) \le \mathcal{R}_N(\mathcal{L}) \tag{2}$$

holds.

Note that there are cases in which one always has equality in (2). For example, it follows from the work of McDuff and Polterovich [MP] that this is the case for $\mathbb{C}P^2$ when $N < 9$. The point is that the symplectic cone and the Kähler cone of del Pezzo surfaces coincide. Note that in (real) dimension 4, more is known about the constants v_N than about $\mathcal{R}_N$ (see [Bi1], [Bi2]). It would be interesting to know whether there exist cases in which a strict inequality occurs in (2).

We conclude by pointing out another interesting approach to bounding Seshadri constants via symplectic packing, due to Lazarsfeld [Laz]. The idea is that given a Kähler form Ω on a complex manifold and a symplectic packing φ that is also holomorphic, the symplectic blowup of Ω associated to φ is Kähler. This situation happens when the associated Kähler metric on the image of φ is flat. Applying this to the case of a principally polarized abelian variety, Lazarsfeld obtains nontrivial estimates on Seshadri constants of the corresponding ample divisor.

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