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LAGRANGIAN NON-INTERSECTIONS

P. Biran

1 Introduction and Main Results

The present paper is devoted to Lagrangian submanifolds of symplectic manifolds and their intersection patterns.

One of the cornerstones of symplectic topology is the rigidity of symplectic structures reflected in the behavior of Lagrangian submanifolds and their intersections. The first non-trivial restrictions on Lagrangian embeddings go back to Gromov's work of 1985 [G]. One of the many results of that paper is the following fundamental theorem: *In $\mathbb{R}^{2n}$ there are no closed exact Lagrangian submanifolds, in particular there are no closed Lagrangian submanifolds $L \subset \mathbb{R}^{2n}$ with $H^1(L;\mathbb{R}) = 0$.*

Since then, Gromov's techniques have been extended and, in combination with other methods, have led to more restrictions on the topology of Lagrangian submanifolds, most of the results being for Lagrangians submanifolds of $\mathbb{R}^{2n}$ and of cotangent bundles (see e.g. [AuLP], [Bu], [E3], [LS], [Mo1,2], [O2,3], [P], [S1], [V1,2] for a partial list of older and newer results).

Only relatively recently first results on the topology of Lagrangians in closed manifolds have been obtained by Seidel [S3] and later on by Biran and Cieliebak [BC2]. Note that when studying Lagrangians in an arbitrary manifold one encounters all Lagrangian submanifolds of $\mathbb{R}^{2n}$. This is due to the fact that every symplectic manifold M^{2n} is locally modeled on $\mathbb{R}^{2n}$, hence every Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ can also be Lagrangianly embedded into M^{2n}. Thus, Lagrangian embeddings into $\mathbb{R}^{2n}$ should, in a sense, be regarded as the local case. However, our understanding of Lagrangian submanifolds of $\mathbb{R}^{2n}$ is still quite limited, in particular also that of "local" Lagrangian submanifolds in any symplectic manifold M^{2n}. (Thus, in this case, "local" turns out to be difficult.)

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In this paper we concentrate on "global" Lagrangian submanifolds. One way to "mod out" the local Lagrangians is to assume for example that the first homology of the Lagrangians is either zero or torsion. In view of the preceding theorem of Gromov such Lagrangian submanifolds must be global in the sense that they cannot lie entirely in a Darboux chart.

One of the phenomena arising from our results below is that in certain symplectic manifolds the topology of Lagrangians with small first homology is extremely restricted. It turns out that in some cases (e.g. $M = \mathbb{C}P^n$) certain assumptions on $H_1(L;\mathbb{Z})$ of a Lagrangian L completely determine the entire homology of L. This phenomenon is illustrated in Theorems A–C of section 1.1 below. Note that very recently examples of this phenomenon have been discovered also for cotangent bundles of spheres by Buhovski [Bu] and by Seidel [S1].

The second phenomenon presented in this paper belongs to the framework of Lagrangian intersections. Our results show that certain symplectic manifolds M contain some kind of "Lagrangian core" $\Lambda \subset M$ which dominates intersections, in the sense that many global Lagrangians $L \subset M$ must intersect Λ, the intersection points being irremovable by symplectic diffeomorphisms. Results in this direction are presented in section 1.3.

1.1 Homological uniqueness of Lagrangian submanifolds. Here and in what follows all Lagrangian submanifolds are assumed to be compact and without boundary, unless otherwise explicitly stated.

Lagrangian submanifolds of $\mathbb{C}P^n$. Let $\mathbb{C}P^n$ be the complex projective space, endowed with its standard Kähler structure. It is well known that $\mathbb{C}P^n$ has no Lagrangian submanifolds L with $H_1(L;\mathbb{Z}) = 0$ (see [S3], see also [BC2]). Note however that there do exist Lagrangians $L \subset \mathbb{C}P^n$ with $H_1(L;\mathbb{Z})$ torsion. For example, the real projective space

$$\mathbb{R}P^n \approx \big\{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \bigm| z_i \in \mathbb{R}\ \forall i\,\big\} \subset \mathbb{C}P^n$$

is such a Lagrangian submanifold (for $n \geq 2$). In fact Seidel proved in [S3] that *every Lagrangian submanifold $L \subset \mathbb{C}P^n$ with $H^1(L;\mathbb{Z}_{2n+2}) = \mathbb{Z}_2$ must satisfy $H^*(L;\mathbb{Z}_2) \cong H^*(\mathbb{R}P^n;\mathbb{Z}_2)$ as graded vector spaces.* Below we shall prove a stronger statement which gives information also on the cohomology *ring* of L. Henceforth we say that an abelian group H is q-torsion if for every $\alpha \in H$ we have $q\alpha = 0$. (This, by our conventions, *includes* the case when H is the trivial group.) Our first result is:

Theorem A. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $H_1(L;\mathbb{Z})$ 2-torsion. Then:*