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Connectedness of Spaces of Symplectic Embeddings

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1 Main result

The present note is devoted to the symplectic packing problem in dimension 4 (see [MP] for the exposition). Our main result is the following.

Theorem 1.1. The space of symplectic packings of $\mathbb{C}P^2$ by k equal balls of a given radius is connected, for $3 \leq k \leq 6$. ☐

In view of [M2], Theorem 1.1 is equivalent to the following.

Theorem 1.2. There exists a unique symplectic blowup of $\mathbb{C}P^2$ at k points with equal weights, for $3 \leq k \leq 6$. ☐

The proof is outlined in Section 3 below (we refer the reader to [Bi1] for the complete proofs).

For $k = 1$ and 2 balls, the theorems above were proved by McDuff in [M1], [M2]. Connectedness of spaces of symplectic packings of ruled surfaces was studied by Lalonde [La] and Gatien [Ga]. After the expanded version of this note [Bi1] was written, McDuff obtained a generalization of our results to an arbitrary number of balls and more general symplectic 4-manifolds [M3]. Notice that she uses Taubes-Seiberg-Witten theory, while we remain in the more familiar framework of Gromov-Witten invariants.

2 Fundamental lemma

We start with the following definitions.

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Let J be an almost complex structure on a manifold M. A closed 2-form ω is said to *tame* J if $\omega(X, JX) > 0$ for all nonzero $X \in TM$. Similarly, we say that a closed 2-form τ *weakly tames* J if $\tau(X, JX) \geq 0$ for all $X \in TM$. Notice that a weakly taming form may be degenerated, while a taming form must be symplectic.

We say that an almost complex structure J is *positive* if there are no J-holomorphic spheres having nonpositive first Chern number. Note that for symplectic 4-manifolds, the set of positive almost complex structures is dense in the space of all almost complex structures tamed by the symplectic structure (see [MS]).

Lemma 2.1. Let (M^4, ω_0) be a symplectic closed 4-manifold. Let $S_0 \subset M$ be an ω_0-symplectically embedded 2-sphere with $S_0 \cdot S_0 = 0$. Suppose that there does not exist any $B \in H_2(M; \mathbb{Z})$ such that $[S_0] = 2B$. Let J_0 be a positive almost complex structure tamed by ω_0, such that S_0 is J_0-holomorphic. Let $\{\omega_t\}_{t\in[0,1]}$ be a deformation of symplectic forms starting at ω_0. Then there exists a path of ω_t-tamed almost complex structures $\{J_t\}_{t\in[0,1]}$ starting with J_0, and a smooth family of closed 2-forms τ_t weakly taming J_t, such that $[\tau_t]$ is Poincaré dual to $[S_0]$ for all t. Moreover, the above remains true if we have a finite set $S_0^1, \ldots, S_0^r$ of embedded J_0-holomorphic spheres with self-intersection numbers zero, that is, there exists a smooth family of closed 2-forms $\tau_t^1, \ldots, \tau_t^r$ weakly taming J_t, such that $[\tau_t^i]$ is Poincaré dual to $[S_0^i]$ for all $1 \leq i \leq r$. □

The idea of the proof is to use Gromov-Witten invariants in order to show that there exists a path of almost complex structures $\{J_t\}_{t\in[0,1]}$ and a smooth family of embedded J_t-holomorphic spheres $\{S_t\}_{t\in[0,1]}$ starting with S_0 and representing the homology class $[S_0]$ (see [MS, Chapter 7]). The homological condition that $[S_0]$ is not divisible by 2 is imposed in order to control the behavior of the cusp-curves which may appear during the deformation, so that the Gromov-Witten invariants are well defined.

In order to construct the forms τ_t, notice that the spheres S_t have trivial normal bundle, and hence a small enough tubular neighborhood $\mathcal{U}_t$ of S_t looks like a product of a 2-disc and a 2-sphere. Clearly, it is enough to take τ_t to be a suitable split form supported in $\mathcal{U}_t$. For more details, see [Bi1, Section 4].

3 The inflation procedure

Following Lalonde and McDuff [LM], we use the so-called inflation procedure to correct symplectic deformations into isotopies.

Proof of Theorem 1.2 for $k = 4$. Let $\mathbb{C}P^2_4$ be the complex blowup of $\mathbb{C}P^2$ at four generic points, and $\bar{J}_0$ be the standard complex structure on $\mathbb{C}P^2_4$. Let $L \in H_2(\mathbb{C}P^2_4; \mathbb{Z})$ be the homology class of a projective line which does not pass through the exceptional divisors, and denote by E_q, $q = 1, \dots, 4$ the homology classes of the exceptional divisors. Finally, we write l, e_q for the Poincaré duals of the classes L, E_q.

Let $\overline{\Omega}_{-1}, \overline{\Omega}_1$ be two symplectic forms on $\mathbb{C}P^2_4$, obtained by symplectic blowing-up, with equal weights of σ, the standard Kähler form on $\mathbb{C}P^2$. Here σ is normalized such that $\int_{\mathbb{C}P^1} \sigma = \pi$.

Assume that the cohomology class of the forms $\overline{\Omega}_{-1}$, $\overline{\Omega}_1$ is $\pi(l - \lambda^2 \sum_{q=1}^4 e_q)$. It follows from [MP] (see also [Bi1]) that there exists a symplectic deformation $\{\overline{\Omega}_t\}_{t\in[-1,1]}$ connecting $\overline{\Omega}_{-1}$ and $\overline{\Omega}_1$ such that $\overline{\Omega}_0$ tames $\overline{J}_0$ and such that

$$[\overline{\Omega}_t] = \pi\left(l - \alpha(t)^2 \sum_{q=1}^{4} e_q\right)$$

where $0 < \alpha(t) \leq \lambda$ for all t, and $\alpha(-1) = \alpha(1) = \lambda$.

The main idea of the proof is to perform inflation along curves which belong to "pencils" of pseudoholomorphic conics passing through the four blown-up points. The existence of these pencils follows from Lemma 2.1 above. More precisely, consider a $\overline{J}_0$-holomorphic sphere S_0 in the homology class $2L - \sum_{q=1}^4 E_q$ (that is, the proper transform of a conic through the four blown-up points). By Lemma 2.1, there exists a path of almost complex structures $\{\overline{J}_t\}_{-1\leq t\leq 1}$, tamed by $\overline{\Omega}_t$, and a smooth family of closed 2-forms τ_t, weakly taming $\overline{J}_t$, such that

$$[\tau_t] = 2l - \sum_{q=1}^{4} e_q.$$

We find now smooth functions $x(t) > 0$ and $y(t) \geq 0$, such that $x(-1) = x(1) = 1$, $y(-1) = y(1) = 0$, and such that the form $\overline{\varpi}_t = x(t)\overline{\Omega}_t + y(t)\tau_t$ lies in the cohomology class $\pi(l - \lambda^2 \sum_{i=1}^4 e_i)$ for all t. The forms $\overline{\varpi}_t$ are symplectic because τ_t weakly tames $\overline{J}_t$, and hence the family $\overline{\varpi}_t$ is an isotopy connecting $\overline{\Omega}_{-1}$ with $\overline{\Omega}_1$.

It remains to prove the existence of the coefficients $x(t), y(t)$. They must satisfy

$$x(t)[\overline{\Omega}_t] + y(t)[\tau_t] = \pi\left(l - \lambda^2 \sum_{q=1}^{4} e_q\right).$$

This is equivalent to the system of two linear equations

$$\begin{cases} \pi x(t) + 2y(t) = \pi \\ \pi\alpha(t)^2 x(t) + y(t) = \pi\lambda^2 \end{cases}$$

which has the solution

$$x(t) = \frac{1 - 2\lambda^2}{1 - 2\alpha(t)^2}, \qquad y(t) = \pi\frac{\lambda^2 - \alpha(t)^2}{1 - 2\alpha(t)^2}.$$

Indeed, $x(t) > 0$, $y(t) \geq 0$ and $x(-1) = x(1) = 1$, $y(-1) = y(1) = 0$. (Note that $\lambda^2 < 1/2$, since otherwise the space of symplectic packings of $\mathbb{C}P^2$ by four equal balls of radius λ would have been empty, because of volume obstruction.) ■

The proof of Theorem 1.2 for $k = 3, 5, 6$ is essentially the same as for $k = 4$. The main difference is that in each case we have to chose different types of curves to inflate along. For example, for $k = 5$ one has to inflate along five conics, each passing through four of the blown-up points. For $k = 6$, one first blows up $\mathbb{C}P^2_6$ at an auxiliary point, and then inflates along curves on $\mathbb{C}P^2_7$. The suitable curves here are in the homology class $A = 5L - 2\sum_{q=1}^{6} E_q - E_7$, that is, proper transforms of quintics with six double points. Finally, after inflating, one has to blow down the inflated deformation at the seventh exceptional divisor in order to get the desired isotopy. The reader is referred to [Bi1] for the complete proofs.

4 Further results

Theorem 1.1 can be applied to extensions of symplectic packings. One can easily prove the following (see [Bi1]).

Corollary 4.1. Let $1 \leq k \leq 6$, and let $\mu > 0$ be such that the space of symplectic packings of $\mathbb{C}P^2$ by k equal balls of radius μ is nonempty.

(1) Let $\lambda < \mu$; then every symplectic embedding $\coprod_1^k B^4(\lambda) \to (\mathbb{C}P^2, \sigma)$ can be extended to a symplectic embedding $\coprod_1^k B^4(\mu) \to (\mathbb{C}P^2, \sigma)$.

(2) Let $r < k$; then every symplectic embedding $\coprod_1^r B^4(\mu) \to (\mathbb{C}P^2, \sigma)$ can be extended to a symplectic embedding $\coprod_1^k B^4(\mu) \to (\mathbb{C}P^2, \sigma)$. □

The inflation procedure can also be used to provide some information on homotopy groups of some spaces of unparameterized symplectic packings.

Theorem 4.2. The space of symplectic packings of $\mathbb{C}P^2$ by k equal balls of a given radius is simply connected, for $k = 1, 2$. □

The proof of this theorem is based on 2-parametric inflation along suitable curves. In contrast to the case of 1-parametric deformations, here it is harder to make sure that the curves needed for the inflation persist along the deformation. The reason is that in two parametric families, curves of zero first Chern number may bubble off. However, for $k = 1, 2$, one can show that no such curves occur. This is done in [Bi2].

Finally, we mention that the limits of the above method depend on the curves one uses—the “better” the curves, the sharper the results obtained by the inflation procedure. McDuff’s generalization of our results for arbitrary number of balls is based on inflation along curves of very high genus. The existence of these curves follows from some new results in Taubes-Seiberg-Witten theory.

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