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CALABI QUASIMORPHISMS FOR THE SYMPLECTIC BALL

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We prove that the group of compactly supported symplectomorphisms of the standard symplectic ball admits a continuum of linearly independent real-valued homogeneous quasimorphisms. In addition these quasimorphisms are Lipschitz in the Hofer metric and have the following property: the value of each such quasimorphism on any symplectomorphism supported in any "sufficiently small" open subset of the ball equals the Calabi invariant of the symplectomorphism. By a "sufficiently small" open subset we mean that it can be displaced from itself by a symplectomorphism of the ball. As a byproduct we show that the (Lagrangian) Clifford torus in the complex projective space cannot be displaced from itself by a Hamiltonian isotopy.

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1. Introduction and Results

A quasimorphism on a group G is a function $\mu : G \to \mathbb{R}$ which satisfies the homomorphism equation up to a bounded error: there exists $R > 0$ such that

$$|\mu(fg) - \mu(f) - \mu(g)| \leq R\,,$$

for all f, $g \in G$ (see [1] for preliminaries on quasimorphisms). A quasimorphism μ is called *homogeneous* if $\mu(g^m) = m\mu(g)$ for all $g \in G$ and $m \in \mathbb{Z}$. It is easy to see that a homogeneous quasimorphism on an abelian group is always a genuine homomorphism.

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In this paper we focus on the case when $G = \mathrm{Symp}_0(B^{2n})$ is the identity component of the group of all compactly supported symplectic diffeomorphisms of the $2n$-dimensional ball

$$B^{2n} = \{|p|^2 + |q|^2 < 1\} \subset \mathbb{R}^{2n}$$

equipped with the symplectic form $dp \wedge dq$, where $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$ are coordinates on $\mathbb{R}^{2n}$. A celebrated result due to Banyaga [2] states that the real vector space of homomorphisms $G \to \mathbb{R}$ is one-dimensional: all of them are proportional to the classical Calabi homomorphism (see Sec. 2 below). Our main result shows that in contrast to this G carries a lot of homogeneous quasimorphisms.

Theorem 1.1. *The real vector space of homogeneous quasimorphisms on* $\mathrm{Symp}_0(B^{2n})$ *is infinite-dimensional.*

Corollary 1.2. *The second bounded cohomology of* $\mathrm{Symp}_0(B^{2n})$ *is an infinite-dimensional vector space over* $\mathbb{R}$.

It is unknown whether analogues of these results are valid for smooth and/or volume-preserving diffeomorphisms. The property featured in the theorem above is known for Gromov-hyperbolic groups [4, 7]. We refer the reader to [8] for a historical review of the "hunt" for quasimorphisms. Let us mention that the first genuine homogeneous quasimorphism (i.e. not a homomorphism) on $\mathrm{Symp}_0(B^{2n})$ was discovered by Barge and Ghys in [3].

The case $n = 2$ was settled by Entov–Polterovich in [8] and by Gambaudo–Ghys in [9]. Though the methods used in these papers are quite different, both of them appeal in a crucial way to 2-dimensional topology. To prove Theorem 1.1 we extend the symplecto-topological approach developed in [8].

The quasimorphisms guaranteed by Theorem 1.1 have a number of interesting additional features which will be described below: they are Lipschitz with respect to the Hofer metric [10, 12] and enjoy the so-called Calabi property.

As a byproduct we prove a result on Lagrangian intersection. Consider the complex projective space $\mathbb{C}P^n$ endowed with its standard symplectic structure. Then the so called *Clifford torus*

$$T^n_{\mathrm{clif}} := \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \,|\, |z_0| = \cdots = |z_n|\} \subset \mathbb{C}P^n\,,$$

is a Lagrangian torus (see Sec. 3). Note that under the usual symplectic identification of $\mathbb{C}P^n \backslash \mathbb{C}P^{n-1}$ with the open unit ball $\mathrm{Int}\, B^{2n}(1)$, the torus T^n_{clif} corresponds to the "split" torus $\{(w_1, \ldots, w_n) \in \mathbb{C}^n \,|\, |w_i|^2 = \frac{1}{n+1}, i = 1, \ldots, n\}$ (see Sec. 4).

Theorem 1.3. *The Lagrangian torus* $T^n_{\mathrm{clif}} \subset \mathbb{C}P^n$ *cannot be displaced from itself by a Hamiltonian isotopy.*

This result admits a generalization to certain products of complex projective spaces (see [Sec. 7, Example 7.1]).

2. Hamiltonian Diffeomorphisms and Calabi Quasimorphisms

We will work in the following general setting. Let (M, ω) be a connected symplectic manifold without boundary and let $G = \mathrm{Ham}(M, \omega)$ denote the group of compactly supported Hamiltonian symplectomorphisms of (M, ω). For simply connected symplectic manifolds M (in particular, balls) G coincides with the identity component of the group of all compactly supported symplectomorphisms of M. For preliminaries on G see e.g. [11, 12].

Given a Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ set $H_t := H(t, \cdot) : M \to \mathbb{R}$ and denote by $\{\phi_H^t\}_{0 \le t \le 1}$, $\phi_H^0 = \mathbf{1}$, the Hamiltonian flow of H. Denote the time-1 map of this flow by $\phi_H \in G$.

The group G has a natural class of subgroups associated to non-empty open subsets $U \subseteq M$. Namely, consider Hamiltonian functions $H : [0, 1] \times M \to \mathbb{R}$ such that for each t the support of H_t lies inside U. The subgroup $G_U \subset G$ is formed by all elements ϕ_H generated by such Hamiltonian functions H. When the symplectic form ω is exact on U the formula

$$\phi_H \mapsto \int_0^1 dt \int_M H_t \omega^n \,. \tag{2.1}$$

gives rise to a well-defined homomorphism

$$\mathrm{Cal}_U : G_U \to \mathbb{R}\,,$$

called the *Calabi homomorphism* [2, 6]. Note that $G_U \subset G_V$ for $U \subset V$ and in this case $\mathrm{Cal}_U = \mathrm{Cal}_V$ on G_U.

In what follows we deal with the class $\mathcal{D}_{ex}$ of all non-empty open subsets U which can be *displaced* by a Hamiltonian diffeomorphism:

$$hU \cap \text{ Closure } (U) = \emptyset \quad \text{for some } h \in G\,,$$

and such that ω is exact on U. The latter condition is automatically fulfilled when M is a subset of the standard symplectic $\mathbb{R}^{2n}$.

Definition 2.1. A quasimorphism on G coinciding with the Calabi homomorphism $\{\mathrm{Cal}_U : G_U \to \mathbb{R}\}$ on every $U \in \mathcal{D}_{ex}$ is called *a Calabi quasimorphism.*

Definition 2.2. We say that a quasimorphism μ is *Lipschitz with respect to the Hofer metric* if there exists a constant $K > 0$ so that

$$|\mu(\phi_F) - \mu(\phi_H)| \le K \cdot \|F - H\|_{C^0}$$

for all Hamiltonians F, $H : [0; 1] \times M \to \mathbb{R}$. (For the relation of $\|F - H\|_{C^0}$ to the Hofer distance between ϕ_F and ϕ_H see [10, 12], also see [8].)

With this language we can refine Theorem 1.1 as follows.

Theorem 2.3. *The real affine space of homogeneous Calabi quasimorphisms on* $\mathrm{Symp}_0(B^{2n})$ *which are Lipschitz with respect to the Hofer metric is infinite-dimensional.*

Remark 2.4. A straightforward modification of our arguments below yields the same result when one replaces the ball B^{2n} by the unit ball bundle over the n-torus, $T^n \times B^n \subset T^*T^n$, equipped with the standard symplectic structure.

The key ingredient in our proof of Theorem 2.3 is the following fact established in [8]:

Theorem 2.5. *The group of Hamiltonian diffeomorphisms of the complex projective space $\mathbb{C}P^n$ endowed with the Fubini–Study symplectic form carries a homogeneous Calabi quasimorphism which is Lipschitz with respect to the Hofer metric.*

The paper [8] contains an explicit construction of such a quasimorphism. However it is unknown whether it is unique.

3. A Calabi Quasimorphism on $\mathbb{C}P^n$ and the Clifford Torus

Consider $M = \mathbb{C}P^n$ equipped with the Fubini–Study symplectic structure ω normalized so that the integral of ω over the complex projective line is 1 and hence $\mathrm{Vol}\,(\mathbb{C}P^n) := \int_{\mathbb{C}P^n} \omega^n = 1$. The torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ acts on $\mathbb{C}P^n$ in a Hamiltonian way as follows:

$$(s_1, \ldots, s_n) : [z_0 : \cdots : z_n] \mapsto [z_0 : e^{2\pi s_1} z_1 : \cdots : e^{2\pi s_n} z_n]\,.$$

A moment map $\Phi : \mathbb{C}P^n \to \mathbb{R}^n$ for the action is given by the formula:

$$\Phi : [z_0 : \cdots : z_n] \mapsto \left(\frac{|z_1|^2}{|z_0|^2 + \cdots + |z_n|^2}, \ldots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2} \right).$$

The image of Φ is the following closed convex polytope:

$$\Delta = \{ p \in \mathbb{R}^n | p_1 + \cdots + p_n \le 1; p_i \ge 0, i = 1, \ldots, n \}\,,$$

where $p = (p_1, \ldots, p_n)$ are coordinates on $\mathbb{R}^n$. The inverse image of a point $p \in \Delta$ under Φ is an isotropic torus which is an orbit of the T^n-action and whose dimension is equal to i if and only if p lies inside some i-dimensional face of Δ (i.e. it belongs to that i-dimensional face but not to any face of smaller dimension). Let

$$p_{\mathrm{clif}} := \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1} \right)$$

denote the barycenter of Δ. The Lagrangian torus $T^n_{\mathrm{clif}} := \Phi^{-1}(p_{\mathrm{clif}})$ is called the *Clifford torus.*

We say that a function on the simplex Δ is smooth if it extends to a smooth function in a neighborhood of Δ in $\mathbb{R}^n$. The proof of our main results is based on the following calculation.

Theorem 3.1. *Let $\mu : G \to \mathbb{R}$ be any Calabi quasimorphism which is continuous with respect to the Hofer metric. Then for every autonomous Hamiltonian $F : \mathbb{C}P^n \to \mathbb{R}$ of the form $\Phi^* \bar{F}$, where $\bar{F} : \Delta \to \mathbb{R}$ is a smooth function on Δ,*

$$\mu(\phi_F) = \int_{\mathbb{C}P^n} F\omega^n - \bar{F}(p_{\mathrm{clif}})\,.$$

The proof of Theorem 3.1 is postponed till Sec. 5.

4. Proof of Theorem 2.3

Let $B^{2n}(\frac{1}{\sqrt{\pi}}) \subset \mathbb{C}^n$ be the closed symplectic ball of radius $\frac{1}{\sqrt{\pi}}$. Define an embedding $\vartheta_\delta : \operatorname{Int} B^{2n}(\frac{1}{\sqrt{\pi}}) \to \mathbb{C}P^n$, $0 < \delta \le 1$, as:

$$\vartheta_\delta : (w_1, \ldots, w_n) \mapsto \left[\left(\frac{1}{\pi} - \sum_{i=1}^{n} \delta |w_i|^2 \right)^{\frac{1}{2}} : \sqrt{\delta} w_1 : \cdots : \sqrt{\delta} w_n \right],$$

where $w_1, \ldots, w_n$ are the standard complex coordinates on $\mathbb{C}^n$. Each embedding ϑ_δ is conformally symplectic: it maps the symplectic form ω on $\mathbb{C}P^n$ to $\delta\omega_B$, where ω_B is the standard symplectic form on the ball. In particular, $\vartheta := \vartheta_1$ is a genuine symplectic embedding, its image is the complement of a projective hyperplane in $\mathbb{C}P^n$. The ball $\vartheta(\operatorname{Int} B^{2n}(\frac{1}{\sqrt{\pi}})) \subset \mathbb{C}P^n$ is invariant under the T^n-action on $\mathbb{C}P^n$ — the corresponding T^n-action on $\operatorname{Int} B^{2n}(\frac{1}{\sqrt{\pi}})$ is described by the formula:

$$(s_1, \ldots, s_n) : (w_1, \ldots, w_n) \mapsto (e^{2\pi i s_1} w_1, \ldots, e^{2\pi i s_n} w_n).$$

The orbits of the action lying inside the ball are the tori

$$T(r_1, \ldots, r_n) := \{(w_1, \ldots, w_n) \in \mathbb{C}^n |\, |w_i|^2 = r_i, i = 1, \ldots, n\},$$

where $r_1, \ldots, r_n$ is a sequence of non-negative numbers satisfying $r_1 + \cdots + r_n \le 1/\pi$. In fact ϑ_δ maps each $T(r_1, \ldots, r_n) \subset \operatorname{Int} B^{2n}(\frac{1}{\sqrt{\pi}})$ into the torus $\Phi^{-1}(\pi\delta r_1, \ldots, \pi\delta r_n)$. In particular,

$$\vartheta_\delta \left(T \left(\frac{1}{\delta\pi(n+1)}, \ldots, \frac{1}{\delta\pi(n+1)} \right) \right) = \Phi^{-1}(p_{\mathrm{clif}}) \tag{4.1}$$

as long as δ is sufficiently close to 1.

Put $M = B^{2n}(\frac{1}{\sqrt{\pi}})$, $G = \mathrm{Ham}(M)$. The conformally symplectic embeddings $\vartheta_\delta : M \to \mathbb{C}P^n$, $0 < \delta \le 1$, induce monomorphisms $\vartheta_{\delta,*} : G \to \mathrm{Ham}(\mathbb{C}P^n)$. Let $\mu : \mathrm{Ham}(\mathbb{C}P^n) \to \mathbb{R}$ be a Calabi quasimorphism on $\mathrm{Ham}(\mathbb{C}P^n)$ which is continuous with respect to the Hofer metric (see Theorem 2.5). Then each $\mu_\delta = \delta^{-n-1} \cdot \mu \circ \vartheta_{\delta,*}$ is a Calabi quasimorphism on G.

Fix $\delta_0 < 1$ sufficiently close to 1. We will show now that every finite collection of quasimorphisms of the form μ_δ, where $\delta \in [\delta_0, 1]$, is linearly independent over $\mathbb{R}$. Indeed, take any compactly supported Hamiltonian of the form $F = \tilde{F}(\pi|w|^2)$ on M, where $\tilde{F} : \mathbb{R} \to \mathbb{R}$ is a smooth function supported inside $[-1, 1]$ and $w = (w_1, \ldots, w_n)$ are the complex coordinates as above. It follows from Theorem 3.1 combined with formula (4.1) that for any $\delta \in [\delta_0, 1]$

$$\mu_\delta(\phi_F) = \int_{B^{2n}(\frac{1}{\sqrt{\pi}})} F\omega_B^n - \delta^{-n} \tilde{F} \left(\frac{\delta^{-1} n}{n+1} \right), \tag{4.2}$$

where ω_B is the symplectic form on $B^{2n}(\frac{1}{\sqrt{\pi}})$. This immediately yields the linear independence and proves the theorem for $M = B^{2n}(\frac{1}{\sqrt{\pi}})$. Finally observe that in the standard symplectic $\mathbb{R}^{2n}$ balls of different radii are conformally symplectomorphic and thus their groups of Hamiltonian diffeomorphisms are isomorphic. This completes the proof.

5. Proof of Theorem 3.1

We start with the following elementary lemma.

Lemma 5.1. *If $p \in \Delta$, $p \neq p_{\rm clif}$, then the subset $\Phi^{-1}(p) \subset \mathbb{C}P^n$ can be displaced from itself by a Hamiltonian diffeomorphism of $\mathbb{C}P^n$.*

Together with Theorem 1.3, which will be proven below, the lemma shows that the Clifford torus is the only non-displaceable torus among all the Lagrangian and isotropic tori $\Phi^{-1}(p) \subset \mathbb{C}P^n$, $p \in \Delta$.

Proof of Lemma 5.1. The action of T^n on $\mathbb{C}P^n$ extends to an action of $U(n+1) \supset T^n$ on $\mathbb{C}P^n$ — the latter is induced by the natural action of $U(n+1)$ on $\mathbb{C}^{n+1}$. A permutation of homogeneous coordinates on $\mathbb{C}P^n$ can be viewed as the action on $\mathbb{C}P^n$ of some element from $U(n+1)$ and thus it represents a Hamiltonian diffeomorphism of $\mathbb{C}P^n$. On the other hand, such permutations map each orbit of the T^n-action on $\mathbb{C}P^n$ to an orbit of the same action. Therefore each permutation of homogeneous coordinates on $\mathbb{C}P^n$ induces an affine transformation of the simplex Δ which permutes the corresponding vertices. The barycenter $p_{\rm clif} \in \Delta$ is the only fixed point of the group of these permutations, hence the lemma is proved. □

Now we return to the proof of Theorem 3.1. Let $\bar{F}$ be a smooth function defined in a neighborhood V of Δ. Fix $\epsilon > 0$. Perturb $\bar{F}$ to a function $\bar{F}'$ so that

- $\|\bar{F} - \bar{F}'\|_{C^0} < \epsilon$
- $\bar{F}'$ is constant in a γ-neighborhood of $p_{\rm clif}$, namely $\bar{F}' \equiv \bar{F}'(p_{\rm clif})$ in that neighborhood. (Here $\gamma > 0$ is a parameter.)

Let $U_0, \ldots, U_m \subset V$ be a finite cover of Δ with Euclidean balls of radius $\gamma/2$ such that

(i) $p_{\rm clif} \in U_0$;
(ii) $p_{\rm clif} \notin {\rm Closure}(U_j)$ for any $j \geq 1$.

Using a partition of unity write

$$\bar{F}' - \bar{F}'(p_{\rm clif}) = \bar{F}_1 + \cdots + \bar{F}_m \,,$$

where ${\rm supp}\, \bar{F}_j \subset U_j$ for all $j \geq 1$.

Put $F' = \Phi^* \bar{F}'$, $F_j = \Phi^* \bar{F}_j$. Note that the Poisson brackets of all these functions vanish since all fibers of Φ are Lagrangian or isotropic submanifolds of $\mathbb{C}P^n$. Therefore the diffeomorphisms ϕ_{F_j} pairwise commute, hence

$$\phi_{F'} = \phi_{F_1} \circ \cdots \circ \phi_{F_m} \,.$$

Since the restriction of any homogeneous quasimorphism to an abelian subgroup is a homomorphism we conclude that

$$\mu(\phi_{F'}) = \mu(\phi_{F_1}) + \cdots + \mu(\phi_{F_m})\,. \tag{5.1}$$

If γ is small enough, condition (ii) above and Lemma 5.1 guarantee that the sets $\Phi^{-1}(U_j)$, $j \geq 1$, are displaceable — they are neighborhoods of displaceable sets. Therefore

$$\mu(\phi_{F_j}) = \int_{\mathbb{C}P^n} F_j \omega^n, j \geq 1\,,$$

because μ is a Calabi quasimorphism. Substituting this into Eq. (5.1) we obtain that

$$\mu(\phi_{F'}) = \sum_{j=1}^{m} \int_{\mathbb{C}P^n} F_j \omega^n = \int_{\mathbb{C}P^n} F' \omega^n - \bar{F}'(p_{\mathrm{clif}})\,.$$

Since this is true for any $\epsilon > 0$ and $\mu(\phi_F)$ depends continuously on F (recall that μ is Lipschitz with respect to the Hofer metric) this proves Theorem 3.1.

6. Proof of Theorem 1.3

If the Clifford torus is displaceable by a Hamiltonian diffeomorphism then so is a sufficiently small open neighborhood U of it. Thus the restriction of the Calabi quasimorphism μ to the group G_U of Hamiltonian diffeomorphisms supported inside U is the classical Calabi homomorphism. On the other hand, Theorem 3.1 states that if $\bar{F}(p_{\mathrm{clif}}) \neq 0$ then the values of μ and of the Calabi homomorphism on the Hamiltonian diffeomorphism generated by $F = \Phi^* \bar{F}$ do *not* coincide. Hence the Clifford torus is not displaceable. This proves Theorem 1.3.

7. An Outlook

It would be interesting to find a more general setting where one can derive Lagrangian intersection results from the pure existence of a Calabi quasimorphism which is continuous with respect to the Hofer metric. Our proof of the fact that the Clifford torus is non-displaceable (Theorem 1.3) gives an example of this kind. Below we present some speculations in this direction.

Let (M^{2n}, ω) be a symplectic manifold whose group of Hamiltonian diffeomorphisms carries such a quasimorphism, say μ. Assume that for a simplicial complex Δ there exists a map $\Phi : M \to \Delta$ with the following property: the preimage $\Phi^{-1}(p)$ of any $p \in \Delta$ is an isotropic submanifold of M. Let $\mathcal{F}$ be a class of "smooth" functions $\bar{F} : \Delta \to \mathbb{R}$ so that the pull-back $F = \Phi^* \bar{F}$ is smooth. We assume that $\mathcal{F}$ is "sufficiently ample", in particular there exist partitions of unity associated to arbitrarily small coverings.

We say that a point $p \in \Delta$ is displaceable if the preimage $\Phi^{-1}(p)$ can be displaced by a Hamiltonian isotopy and non-displaceable otherwise. Arguing exactly as in the proof of Theorem 3.1 we get the following.

A. Assume that there is a point $p_* \in \Delta$ such that the set $\Delta \backslash \{p_*\}$ consists of displaceable points. Then for every smooth function $\bar{F} \in \mathcal{F}$

$$\mu(\phi_F) = \int_M F \cdot \omega^n - \bar{F}(p_*) \,. \tag{7.1}$$

This immediately yields (cf. the proof of Theorem 1.3) that p_* is non-displaceable. As a logical corollary we get that non-displaceable points do exist!

B. The correspondence

$$\bar{F} \mapsto \mu(\phi_F)$$

is a *linear functional* on the space of smooth functions $\mathcal{F}$. Indeed, all Poisson brackets $\{F, H\}$ with $F = \Phi^* \bar{F}$, $H = \Phi^* \bar{H}$ vanish, and hence the Hamiltonian diffeomorphisms ϕ_F and ϕ_H commute. The desired linearity follows from the fact that μ restricted to an abelian subgroup is a homomorphism.

C. Denote by $\Sigma \subset \Delta$ the subset of all non-displaceable points. Assume first that a function $\bar{F} \in \mathcal{F}$ vanishes on Σ. Using partitions of unity and the Calabi property of μ one readily shows that

$$\mu(\phi_F) = \int_M F \cdot \omega^n \,.$$

This in turn yields the following generalization of formula (7.1): there exists a measure, say, σ on Σ so that

$$\mu(\phi_F) = \int_M F \cdot \omega^n - \int_\Sigma \bar{F} \cdot d\sigma \tag{7.2}$$

for every $\bar{F} \in \mathcal{F}$. For instance in the setting of the previous sections, σ is the Dirac measure concentrated at the value of the moment map corresponding to the Clifford torus.

Let us list some specific examples where in our opinion it would be interesting to test these suggestions. Note that the existence of a Calabi quasimorphism is meanwhile established for complex projective spaces, complex Grassmannians and more generally for spherically monotone symplectic manifolds with semi-simple quantum homology algebra — see [8]. An extra complication is due to the fact that in general a Calabi quasimorphism is defined not on the group $\mathrm{Ham}(M, \omega)$ itself but on its universal cover.

Example 7.1. Let $\Phi : M \to \Delta$ be the moment map associated to a Hamiltonian action of a half-dimensional torus. Is it true that the measure σ from formula (7.2) above is again the Dirac measure at the barycenter of Δ?

As an illustration consider the direct product $M = \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_l}$ equipped with the monotone symplectic structure $\omega = \omega_1 \oplus \cdots \oplus \omega_l$. Monotone means that $[\omega] \in H^2(M)$ is a positive multiple of the first Chern class c_1 of the (complex) tangent bundle of M. More explicitly, if we denote by $h_i \in H^2(\mathbb{C}P^{n_i}; \mathbb{Z})$ the positive generator, monotonicity means that $[\omega] = \lambda((n_1+1)h_1 \oplus \cdots \oplus (n_l+1)h_l)$ for some $\lambda > 0$. Consider the torus action on M defined as the direct product of the standard torus actions on the factors. Denote by T_{clif} the Lagrangian torus in M which is the product of the Clifford tori in $\mathbb{C}P^{n_i}$, $i = 1, \ldots, l$. A simple modification of our arguments in the previous sections shows that the measure σ is indeed the Dirac measure concentrated at the value of the moment map corresponding to the Clifford torus. As a consequence, T_{clif} cannot be displaced from itself by a Hamiltonian isotopy.

Example 7.2. Let (M, ω) be a projective algebraic manifold endowed with the Fubini–Study form. It follows from the theory of Lagrangian skeletons [5] that M carries *many* singular foliations by Lagrangian tori and thus provides a natural playground for our speculation. Note that in this situation a careful description of the space of leaves Δ is already a non-trivial problem. A potential outcome would be the existence of a non-displaceable Lagrangian torus in a large class of symplectic manifolds.

Example 7.3. Take the 2-sphere S^2 equipped with an area form. Every Morse function, say ψ, on S^2 defines a tree T_ψ, called the Reeb graph, whose points correspond to connected components of the level sets of ψ (see [8]). One has a natural map $\Psi : S^2 \to T_\psi$ whose preimages define a singular foliation by circles of S^2. It turns out that the corresponding measure σ is the Dirac measure concentrated at a special point called the median of the tree (see [8]).

Given any triple (M, Φ, Δ) as in the examples above, define its *stabilization* by

$$(S^2 \times M, \Psi \times \Phi, T_\psi \times \Delta)\,.$$

This gives rise to new interesting examples.

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