inventory would vary from 2μ, at the moment a shipment is received, down to zero, immediately prior to the next shipment. Average long-run inventory cost would be $h^+\mu$ per week.

According to equation (1), the inventory cost associated with the retailer would be $2h^+\mu$: a cost of $h^+\mu$ would be counted for each of the retailer's two tours ($f_l = \frac{1}{2}$). It would be as though both vehicles delivered their shipments at the same time, every other week. That is, two shipments of size 2μ would arrive one week, none the second, and two shipments again the following week.

The objective function, as formulated, underestimates the benefit of assigning a retailer to multiple tours. In an optimal solution, retailers facing high demands should be assigned to multiple tours, and visited frequently, as a way of reducing inventory costs. Retailers facing low demands should not be assigned to multiple tours. Hence, they would be visited infrequently. The frequency at which any given tour is covered can vary enormously, depending on the combination of retailers assigned, and the relative frequencies at which the retailers are visited. Hence, the algorithm presented in the paper negates a major benefit of assigning retailers to multiple tours.

Reference

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REJOINDER TO "COMMENTS ON ONE-WAREHOUSE MULTIPLE RETAILER SYSTEMS WITH VEHICLE ROUTING COSTS"

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In Anily and Federgruen (1990) we consider distribution systems with a depot and many geographically dispersed retailers each of which faces external demands occurring at constant, deterministic but retailer specific rates. All stock enters the system through the depot from where it is distributed to the retailers by a fleet of capacitated vehicles combining deliveries into efficient routes.

Optimal replenishment strategies (i.e., inventory rules and routing patterns) minimizing infinite horizon long-run average transportation and inventory costs can be very complex. As pointed out in the paper, the complexity of the structure of optimal policies makes them difficult, if not impossible, to implement even if they could be computed efficiently.

The above paper therefore restricts itself to a class Φ of replenishment strategies with the following properties: a replenishment strategy specifies a collection of *regions* (subsets of outlets) covering all outlets: if an outlet belongs to several regions a specific fraction of its sales/operations is assigned to each of these regions. Each time one of the outlets in a given region receives a delivery, this delivery is made by a vehicle who visits all other outlets in the region as well (in an efficient sequence or *route*). This implies that under strategies in Φ deliveries are never coordinated among different tours and that outlets that are assigned to *different* regions are never served in a common route even though in an *optimal* strategy any given outlet may be served in *varying* rather than *constant* combinations of other outlets. This is illustrated by the example in Hall (1991).

As is the case for all restriction approaches, our partitioning method results in some loss of optimality. (The costs of our proposed heuristics are shown to be close to a *lower bound* for the minimum cost among all strategies in Φ but it is as of yet unknown how close they are to the unrestricted minimum.) The following observations reflect our current understanding of the loss of optimality incurred by restriction approaches in joint replenishment models.

As pointed out in the paper the partitioning restriction approach to the class Φ is similar to that applied in many other joint replenishment problems with considerably simpler joint cost structures see, e.g., Chakravarty et al. (1982, 1985), Barnes et al. (1989), Burns et al. (1985), and the literature review in Federgruen and Zheng (1988). In fact all existing solution approaches for joint replenishment models appear to be based on one of the following two types of restrictions or hybrid combinations thereof: (i) partitioning approaches; (ii) integer-ratio or power-of-two policies: all items or locations are replenished at constant intervals which are integer or power-of-two multiples of some base planning period. (Under these policies, an outlet may be served in varying combinations of other outlets.) Examples of this second approach are Jackson et al. (1985, 1988), Roundy (1985, 1986) and Federgruen and Zheng (1988).

For certain settings, it is well known that the second (power-of-two) restriction approach results in policies whose cost is guaranteed to come within 2% of optimality (see the papers mentioned above). All of these settings assume however that *unlimited* quantities can be delivered for a given fixed cost and that the joint cost structure exhibits economies of scale in the sense of submodularity: the incremental cost due to the addition of a new location to a given collection of locations is no larger than if the same location were added to a subset of this collection. For such settings Federgruen and Zheng (1988) describe an example in which the cost of the best partitioning strategy exceeds that of the best power-of-two strategy by 22.4%. Zheng (1987) conjectures that this represents the worst-case optimality gap. Queyranne and Sun (1989) *prove* that the worst case optimality gap between the two approaches is less than 44.8% for uncapacitated models with submodular joint cost structures.

As shown in Jackson et al. (1988), restriction to integer-ratio or power-of-two strategies may result in considerably larger optimality gaps in *capacitated* models where only a limited amount may be delivered for a given fixed cost (even when the cost structure is submodular). As explained in the Appendix of our paper, the joint cost structure fails to be submodular and replenishment opportunities are capacitated in the model considered ibid. Herer (1990, Theorem 2) shows that no (upper bound) submodular approximation for routing costs needs to exist, whose approximation error is uniformly bounded in the number of delivery points n. (The approximation error is $O(\log n)$, see Theorem 6 ibid.)

Finally, the attractiveness of a given restricted class of strategies cannot exclusively be assessed on the basis of the associated worst-case or average optimality gap (compared to the minimum achievable cost). Other important aspects include the ease of implementation and monitoring as well as other managerial advantages. We refer to the bottom two paragraphs of p. 93 in our paper for a discussion of the latter. In particular, even if a considerably cheaper strategy could be found outside of Φ, it is likely to involve highly intricate coordination and scheduling problems among all outlets and routes or regions.

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