"Technical Appendix to: Keynesian Economics, Monetary Policy and the Business Cycle - New and Old"

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This appendix provides and proves some technical results underlying subsections 5.3 and 5.5 in Cukierman (2005) article and should be read in conjunction with that paper.

1 The case $u(C) + v(l) = \frac{C^{1-\gamma}}{1-\gamma} + ql$

The discussion in this section underlies subsection 5.3 in Cukierman (2005). Here $u(C)$ is a CRRA in consumption and γ and q are positive parameters. This is a convenient case since its equilibrium can be obtained explicitly from the first order conditions for an internal maximization of utility by the representative individual. It is shown in subsection 5.2 of Cukierman (2005) that the equilibrium values of consumption and leisure under flexible (FPE) and under sticky prices and wages (SPE) are given respectively by

$$C_f = \left(\frac{a}{q\mu^*}\right)^{\frac{1}{\gamma}}, \quad l_f = 1 - \left(\frac{1}{q\mu^*}\right)^{\frac{1}{\gamma}} a^{\frac{1-\gamma}{\gamma}} \tag{1}$$

$$C_s = \left(\frac{w_s}{q}\right)^{\frac{1}{\gamma}}, \quad l_s = 1 - \left(\frac{w_s}{q}\right)^{\frac{1}{\gamma}} \frac{1}{a} \tag{2}$$

where

$$w_s = \frac{1}{\mu^* E\frac{1}{a}}. \tag{3}$$

Proposition 1 *For the family of utility functions, $u(c)+v(l) = \frac{c^{1-\gamma}}{1-\gamma}+ql$, expected welfare under* *flexible prices and wages is larger than expected welfare under sticky prices and wages for all possible markups and distributions of productivity shocks.*

Proof. *Substituting the optimal values of consumption and of leisure from equations (1) and (2) into the utility function we obtain the indirect utility functions for the equilibria under flexible prices and wages (FPE) and under the sticky prices and wages (SPE) as functions of the realization of the productivity shock, a. They are given respectively by*

$$\Lambda_f(a) \equiv q + \left[\frac{1}{1-\gamma}\left(\frac{1}{q\mu^*}\right)^{\frac{1-\gamma}{\gamma}} - \left(\frac{q^{\gamma-1}}{\mu^*}\right)^{\frac{1}{\gamma}}\right]\left(\frac{1}{a}\right)^{\frac{\gamma-1}{\gamma}} \tag{4}$$

$$\Lambda_s(a) \equiv q + \frac{1}{1-\gamma}\left(\frac{w_s}{q}\right)^{\frac{1-\gamma}{\gamma}} - \left(q^{\gamma-1}w_s\right)^{\frac{1}{\gamma}}\frac{1}{a} \tag{5}$$

Taking expected values of those expressions after substitution of w_s from equation (3) into the expression for $\Lambda_s(a)$ we obtain after some algebra that $E\Lambda_s(a) < E\Lambda_f(a)$ if and only if

$$K\left(Eb\right)^{\frac{\gamma-1}{\gamma}} < KE\left(b\right)^{\frac{\gamma-1}{\gamma}} \tag{6}$$

where

$$b \equiv \frac{1}{a} \text{ and } K \equiv \frac{1}{1-\gamma}\left(\mu^*\right)^{\frac{\gamma-1}{\gamma}} - \left(\frac{1}{\mu^*}\right)^{\frac{1}{\gamma}} \tag{7}$$

Further algebra establishes that $K > 0$ is equivalent to $\mu^* > 1-\gamma$ for $\gamma < 1$ and that, when $\gamma > 1$, $K < 0$ is equivalent to $\mu^* > 1-\gamma$. Since $\mu^* > 1$ and $\gamma > 0$ the inequality $\mu^* > 1-\gamma$ is always satisfied implying that

$$\begin{aligned} K &> 0 \; for \; \gamma < 1 \\ K &< 0 \; for \; \gamma > 1. \end{aligned} \tag{8}$$

Let $f(b) = b^{\frac{\gamma-1}{\gamma}}$. Then

$$f''(b) = -\frac{\gamma-1}{\gamma^2}b^{-\left(1+\frac{1}{\gamma}\right)} \tag{9}$$

implying that $f(b)$ is a concave, linear or convex function of b depending on whether γ is larger than, equal to, or smaller than one.

(i) The case $\gamma > 1$***:*** *Since $f(b)$ is concave Jensen's inequality in conjunction with the fact that K is negative in this case imply that condition (6) is always satisfied so that expected welfare under a FPE is higher than under a SPE.*

(ii) The case $\gamma < 1$***:*** *Since $f(b)$ is convex Jensen's inequality in conjunction with the fact that K is positive imply that condition (6) is still satisfied so that expected welfare under a FPE is, again, higher than under a SPE.*

(iii) The case $\gamma = 1$***:*** *In this case utility from consumption is logarithmic implying that, in the FPE, labor input is constant across states of nature and*

$$\Lambda_f(a) \equiv \ln \frac{a}{q\mu^*} + q\left(1 - \frac{1}{q\mu^*}\right) \tag{10}$$

$$\Lambda_s(a) \equiv \ln \frac{w_s}{q} + q\left(1 - \frac{w_s}{qa}\right) \tag{11}$$

Using (3) in (11), taking expected values of the resulting expression and of (10), implies after some algebra that

$$E\Lambda_f(a) > E\Lambda_s(a) \ \ < - - - - > \ E\ln\frac{1}{a} < \ln E\frac{1}{a}$$

Since the ln function is concave the second inequality must hold by Jensen's inequality so that, again, the FPE dominates the SPE. ■

2 Derivation of a necessary and sufficient condition for the welfare dominance of the SPE

The discussion in this section underlies subsection 5.5 in Cukierman (2005). Condition (24) for the determination of a_0 in Cukierman (2005) may be rewritten as

$$\sum_i p_i Q(a_i, a_0) = 0 \tag{12}$$

where

$$Q(a_i, a_0) \equiv n(a_i, \frac{a_0}{\mu^*})(a_i - a_0) \tag{13}$$

and p_i is the probability that shock a_i realizes. Since a_0 belongs to the interior of the support of a, the value of $Q(a, a_0)$ is negative for all $a < a_0$ and positive for all $a > a_0$. Let

$$\psi(a, a_0) \equiv \frac{\Delta(a)}{Q(a, a_0)} \tag{14}$$

where $\Delta(a)$ is the difference between welfare under a SPE and a FPE given the value of a.[1] Let $H(a)$ be the subset of all $a's$ in the range $a > a_0$ such that $\psi(a, a_0) > k$ for some arbitrary k and let $L(a)$ be the subset of all $a's$ in the range $a < a_0$ such that $\psi(a, a_0) \leq k$. This section establishes the following (proposition 4 in subsection 5.5 of Cukierman (2005) article).

Proposition 2 *There exist probability distributions of productivity shocks such that expected welfare under the SPE is higher than under the FPE ($E\Delta(a) > 0$) if and only if there exists a real number k such that each of the sets $H(a)$ and $L(a)$ is non empty.*

Proof. *(i) Sufficiency: The definitions of the sets $H(a)$ and $L(a)$ imply*[2]

$$\begin{aligned} \psi(a_i, a_0) &> k \textit{ for all } a_i \text{ that belong to } H(a) \\ \psi(a_i, a_0) &\leq k \textit{ for all } a_i \text{ that belong to } L(a). \end{aligned} \tag{15}$$

Furthermore

$$\begin{aligned} Q(a_i, a_0) &> 0 \textit{ for all } a_i > a_0 \\ Q(a_i, a_0) &< 0 \textit{ for all } a_i < a_0. \end{aligned} \tag{16}$$

Since $H(a)$ is a subset of the set $\{a_i > a_0\}$ and $L(a)$ is a subset of the set $\{a_i < a_0\}$ equations (15) and (16) and the definition of $\psi(a_i, a_0)$ in (14) imply

$$\begin{aligned} \Delta(a_i) &> kQ(a_i, a_0) \textit{ for all } a_i \text{ that belong to } H(a) \\ \Delta(a_i) &\geq kQ(a_i, a_0) \textit{ for all } a_i \text{ that belong to } L(a). \end{aligned} \tag{17}$$

[1] $\Delta(a)$ indirectly depends on a_0. We subsume this dependence into the functional form to economize on notation.

[2] *Note that, in anticipation of a summation, (below) over discrete values of a indexed by "i" this counter is attached as a subscript to the productivity shock, a.*

Allocate all the probability mass to the sets $H(a)$ and $L(a)$ and consider

$$E\Delta(a) = \sum_{a_i \in H(a)} p_i \Delta(a_i) + \sum_{a_i \in L(a)} p_i \Delta(a_i) > k \left[\sum_{a_i \in H(a)} p_i Q(a_i, a_0) + \sum_{a_i \in L(a)} p_i Q(a_i, a_0) \right] = 0 \quad (18)$$

where the inequality follows from (17). The equality to zero follows from the fact that all the probability mass is allocated to the sets $H(a)$ and $L(a)$ in conjunction with equation (12).

(ii) Necessity: The proof is established by showing that if at least one of $H(a)$ and $L(a)$ is empty for all possible values of k, then $E\Delta(a) \leq 0$ for all possible probability distributions.[3] *Let ψ_L be the minimal value of $\psi(a_i, a_0)$ in the range $a_i < a_0$ and let ψ_H be the maximal value of $\psi(a_i, a_0)$ in the range $a_i > a_0$. Since, for every k, at least one of $H(a)$ and $L(a)$ is empty, $\psi_L \geq \psi_H$ (for otherwise there would exist a real number, k such that both inequalities in equation (15) are satisfied). This implies that all the values of $\psi(a_i, a_0)$ in the range $a_i < a_0$ have to be larger than, or equal to, all the values of $\psi(a_i, a_0)$ in the range $a_i > a_0$. Hence there exists another scalar, k_1, such that*

$$\begin{aligned} \psi(a_i, a_0) &\geq k_1 \text{ for all } a_i < a_0 \\ \psi(a_i, a_0) &\leq k_1 \text{ for all } a_i > a_0 \end{aligned} \quad (19)$$

which implies

$$\begin{aligned} \Delta(a_i) &\leq k_1 Q(a_i, a_0) \text{ for all } a_i < a_0 \\ \Delta(a_i) &\leq k_1 Q(a_i, a_0) \text{ for all } a_i > a_0. \end{aligned} \quad (20)$$

Consider $E\Delta(a)$ for any arbitrary distribution, p_i. By definition, and since $\Delta(a_0) = 0$ and $Q(a_0, a_0) = 0$,

$$E\Delta(a) = \sum_{a_i > a_0} p_i \Delta(a_i) + \sum_{a_i < a_0} p_i \Delta(a_i) \leq k_1 \left[\sum_{a_i > a_0} p_i Q(a_i, a_0) + \sum_{a_i < a_0} p_i Q(a_i, a_0) \right] = 0 \quad (21)$$

[3] *This suffices to establish that the non emptiness of both $H(a)$ and $L(a)$ is necessary for $E\Delta(a) > 0$. For suppose, by way of negation, that this is not the case so that $E\Delta(a) > 0$ and at least one of those sets is empty. Then the statement proven in the text implies that $E\Delta(a) \leq 0$ producing a contradiction to the supposition $E\Delta(a) > 0$.*

where the inequality follows from equation (20) and the equality to zero from the definition of a_0 *in equation (12). Hence* $E\Delta(a) \leq 0$ *for all possible probability distributions.* ■

3 References

Cukierman A., (2005), "Keynesian Economics, Monetary Policy and the Business Cycle - New and Old", September. Available on the web at: http://www.tau.ac.il/~alexcuk/pdf/keynesian-econ-new-old-7.pdf