August 03, 2008 (Correction)

Monetary Theory and Policy under Sticky Prices and Wages

2008B

Solution to homework 1

Exercise 1

Derivation of the linearized inter-temporal Euler equation:

We begin with rewriting the inter-temporal Euler condition, equation (7) in chapter 2:

$$1 = E_t\left\{\frac{\beta}{Q_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right\}$$

$$= E_t\left[\exp^{\ln\left\{\frac{\beta}{Q_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+1}}\right\}}\right]$$

$$= E_t\left[\exp^{i_t-\sigma\Delta c_{t+1}-\pi_{t+1}-\rho}\right] \quad , \tag{1}$$

where we use the definitions $x \equiv \ln x$, $\rho \equiv -\ln\beta$, $\pi_t \equiv \ln P_t - \ln P_{t-1}$, $Q_t \equiv \frac{1}{1+i_t}$ and the log-linear approximation $ln(1+i) \approx ln(1) + \frac{\partial ln(x)}{\partial x}_{|x=1} \cdot i = 0 + i = i$.

Let π and γ be, respectively, constant steady-state rates of inflation and consumption growth. It follows that in the steady state, (1) reduces to

$$\begin{aligned} &i = \rho + \pi + \sigma\gamma \\ &\Rightarrow \\ &\rho = i - \pi - \sigma\gamma \quad . \end{aligned} \tag{2}$$

First order approximation, around the steady state, of the expression inside the exponent of (1) is:

$$\begin{aligned}\exp(i_t-\sigma\Delta c_{t+1}-\pi_{t+1}-\rho)&\approx 1+(i_t-i)-\sigma(\Delta c_{t+1}-\gamma)-(\pi_{t+1}-\pi)\\&=1+i_t-\sigma\Delta c_{t+1}-\pi_{t+1}-\rho\quad .\end{aligned}\tag{3}$$

Note that the last equality is obtained by utilizing equation (2).

Taking expected values of the RHS in (3) and substituting in (1) we get

$$1\approx 1+i_t-\sigma E_t\Delta c_{t+1}-E_t\pi_{t+1}-\rho\,.$$

It follows that

$$c_t\approx E_tc_{t+1}-\frac{1}{\sigma}\left[i_t-E_t\pi_{t+1}-\rho\right]\ .$$

Q.E.D.

Note that one convenience, of the log-linear approximation, is the decomposition of interacting terms inside expectation operators, even when they are not independent. This is exactly the case we have here, as C_{t+1} and P_{t+1} are stochastic and not independent; treating them as independent terms, is part of the linear approximation.

Exercise 2

Given the Dixit-Stiglitz consumption composite

$$C_t\equiv\left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\cdot di\right)^{\frac{\varepsilon}{\varepsilon-1}}\ ,\tag{4}$$

show that the efficient allocation across goods satisfies

$$C_t(i)=\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}C_t\,;\ \ \forall i\in(0,1)\ .\tag{5}$$

Proof:

Step 1. Consider the following Lagrangean formulation of the above problem:

$$L=\left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}di\right]^{\frac{\varepsilon}{\varepsilon-1}}-\lambda\left(\int_0^1 P_t(i)\cdot C_t(i)\cdot di-\text{Total expenditure}\right)\ .$$

Differentiating WRT $C_t(i), \forall i\in(0,1)$ and combining the FOCs for any two goods, say goods a and b, we get the following conditions:

$$\lambda=\frac{\partial C_t}{\partial C_t(a)}\cdot\frac{1}{P_t(a)}=\frac{\partial C_t}{\partial C_t(b)}\cdot\frac{1}{P_t(b)}\quad , \tag{6}$$

where C_t is defined by (4).

The conditions in (6) have the standard economic interpretation: in the optimum, the shadow price, λ, is the same across all differentiated goods. That is, the marginal Shekel would contribute to C_t the same amount, λ, regardless the marginal differentiated good it buys.

Substituting the explicit derivatives and rearranging, we get:

$$\frac{C_t(a)}{C_t(b)}=\left[\frac{P_t(a)}{P_t(b)}\right]^{-\varepsilon}\quad . \tag{7}$$

Here we see that ε is the constant elasticity of substitution (CES) between any two differentiated goods, regardless the quantities consumed. Note how in the limiting case of $\varepsilon\to 1$ the optimal allocation in (7) is reduced to the private case of Cobb-Douglas consumption composite, in which all goods has the same exponent.

Step 2. Rewrite (7) as

$$P_t(i)=\left[\frac{C_t(i)}{C_t(a)}\right]^{\frac{-1}{\varepsilon}}P_t(a)\ . \tag{8}$$

Define a new variable, P_t, which is the minimal expenditure for a single unit of the consumption composite. That is:

$$\text{Minimal expenditure} \equiv Min\{P_t C_t\} = Min\left\{\int_0^1 P_t(i)\cdot C_t(i)\cdot di\right\} , \tag{9}$$

or

$$\mathrm{P_t} \equiv ArgMin\left\{\frac{\text{Total expenditure}}{\mathrm{C_t}}\right\} = ArgMin\left\{\frac{\int_0^1 P_t(i)\cdot C_t(i)\cdot di}{C_t}\right\} . \tag{9'}$$

From (9') we see that the price aggregate, P_t, is defined in such a way that the total expenditure ends up being $P_t C_t$.

Note that deriving the optimal price aggregate, P_t, is the dual problem of deriving the optimal consumption composite:

$$C_t(i) = ArgMax\left\{\frac{C_t}{\int_0^1 P_t(i)\cdot C_t(i)\cdot di}\right\};\ \forall\, i \in (0,1) .$$

Now, substitute (8) for every differentiated good, $i \in [0,1]$, in the total expenditure, $P_t C_t = \int_0^1 P_t(i)\cdot C_t(i)\cdot di$, to get:

$$\begin{aligned}
P_t C_t &= P_t(a)\cdot C_t(a)^{\frac{1}{\varepsilon}} \cdot \int_0^1 C_t(i)^{\frac{-1}{\varepsilon}+1} \cdot di \\
&= P_t(a)\cdot C_t(a)^{\frac{1}{\varepsilon}} \cdot \int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \cdot di \\
&= P_t(a)\cdot C_t(a)^{\frac{1}{\varepsilon}} \cdot \left\{\left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \cdot di\right]^{\frac{\varepsilon}{\varepsilon-1}}\right\}^{\frac{\varepsilon-1}{\varepsilon}} \\
&= P_t(a)\cdot C_t(a)^{\frac{1}{\varepsilon}} \cdot C_t^{\frac{\varepsilon-1}{\varepsilon}} .
\end{aligned} \tag{10}$$

The last equality utilizes the definition in (4). Rearranging (10), we get (5).

Q.E.D.

Some intuition:

Note that if instead of $C_t(b)$ and $P_t(b)$ in equation (7), we substitute C_t and P_t, respectively, we could get (5) immediately. The intuition is that in the optimum, the shadow price has to be the same across any two differentiated goods, and therefore, also between any good and the entire basket. However, C_t —although homogenous of degree one—is not a basket in the full sense, so we need some algebraic tricks in order to get (5) from (7).

Exercise 3

Given the definition of P_t, equation (9') above, show that

$$P_t = \left[\int_0^1 P_t(i)^{(1-\varepsilon)} \cdot di\right]^{\frac{1}{1-\varepsilon}} \quad . \tag{11}$$

Note that this is the dual problem of the one from the previous exercise we just solved. That is, the optimal $C_t(i)$ as expressed by equation (5), is the one that minimizes the expenditure for a given indirect utility level.

This price aggregate has a very important property—it is homogenous of degree one. It means that like a standard price index, if all prices are doubled, the price index is doubled as well. However, unlike what the Central Bureau for Statistics (CBS) measures, this is the optimal (and 'true') price index, with the weights being continuously updated. By contrast, in the CBS price index, the weights remain the same from one period to another.

Proof:

Use (5) to substitute for $C_t(i)$ in (9), to get :

$$P_t C_t = P_t^{\varepsilon} \cdot C_t \left[\int_0^1 P_t(i)^{1-\varepsilon} \cdot di \right] . \tag{12}$$

C_t is canceled out from both sides, and after rearrangement we get (11).

Q.E.D.

Note that although not a price index in the full sense, yet P_t has a very important price-index property—it is homogenous of degree one, just like the consumption composite. Therefore, after linearization, both P_t and C_t look like price and consumption indices.

Exercise 4

Given that $(1-\theta) \in (0,1)$ is the probability that Calvo will come to visit, show that the expected time interval between any two visits is:

$$E[D] = \frac{1}{1-\theta} \quad . \tag{13}$$

Proof:

The probability of price update signal is independent of the signals' history. Hence, the probability to receive the signal every g periods is given by $\Pr(freq = g) = \theta^{g-1}(1-\theta)$. The left multiplier in the R.H.S. is the probability that the price is not optimized during the first $g-1$ periods; the right multiplier is the probability of optimization signal in the $g'th$ period. The expected frequency of price optimization signal is hence the average of frequencies, weighted by the corresponding probabilities: $\sum_{g=1}^{\infty} \{ g \cdot \Pr(freq = g) \}$. Substituting for every $\Pr(freq = g)$ and using some straightforward algebra, we get:

$$\sum_{g=1}^{\infty}\left\{g\cdot\theta^{g-1}\left(1-\theta\right)\right\}=\left(1-\theta\right)\left\{1+2\cdot\theta^{1}+3\cdot\theta^{2}+\ldots+t\cdot\theta^{t-1}+\ldots+\infty\cdot\alpha^{\infty}\right\}=$$

$$\left(1-\theta\right)\left\{1+2\cdot\left(\theta^{1}+\theta^{2}+\ldots+\theta^{\infty}\right)+\left(\theta^{2}+\theta^{3}+\ldots+\theta^{\infty}\right)+\ldots+\infty\cdot\left(\theta^{\infty}\right)\right\}=$$

$$\left(1-\theta\right)\left\{1+\frac{2\cdot\theta}{1-\theta}+\frac{\theta^{2}}{1-\theta}+\frac{\theta^{3}}{1-\theta}+\ldots+\frac{\theta^{\infty}}{1-\theta}\right\}=$$

$$\left(1-\theta\right)+2\cdot\theta+\theta^{2}+\theta^{3}+\ldots+\theta^{\infty}=$$

$$1+\theta+\theta^{2}+\theta^{3}+\ldots+\theta^{\infty}=$$

$$\sum_{i=0}^{\infty}\theta^{i}=$$

$$\frac{1}{1-\theta}\quad.$$

In the last equality we used the formula for the sum of converging geometric series.

Q.E.D.

Note that this result is very intuitive: if any given period, a fraction of $1-\theta$ optimizes, than the optimization frequency is simply the inverse: $(1-\theta)^{-1}$ (Just think of numerical example).

Exercise 5

We saw that the price-setting first order condition is:

$$E_t\sum_{k=0}^{\infty}\theta^{k}\left\{Q_{t,t+k}\,Y_{t+k|t}\,\frac{P_t^{*}}{P_{t-1}}\right\}=E_t\sum_{k=0}^{\infty}\theta^{k}\left\{Q_{t,t+k}\,Y_{t+k|t}\mathcal{M}\,MC_{t+k|t}\,\Pi_{t-1,t+k}\right\}. \tag{14}$$

Show that after log linearization, it becomes:

$$p_t^{*}-p_{t-1}=\left(1-\beta\theta\right)E_t\sum_{k=0}^{\infty}\left(\beta\theta\right)^{k}\left\{\hat{mc}_{t+k|t}+\left(p_{t+k}-p_{t-1}\right)\right\}\quad. \tag{15}$$

Proof:

Note that, using the standard IS condition, we have that

$$E_t Q_{t,t+k} \equiv E_t \prod_{j=1}^{k} \frac{1}{1+i_{t+j}} = E_t \prod_{j=1}^{k} \left[\beta \left(\frac{C_{t+j}}{C_{t+j-1}} \right)^{-\sigma} \frac{P_{t+j-1}}{P_{t+j}} \right] = \beta^k E_t \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \quad . \qquad (16)$$

In the zero-inflation deterministic steady-state we have that:

$$Y_{t+k|t} = \overline{Y} \quad ; \quad P_t^* / P_{t-1} = 1 \quad ; \quad Q_{t,t+k} = \beta^k \quad ; \quad MC_{t+k|t} = \overline{MC} = 1/\mathcal{M} \quad ; \quad \Pi_{t-1,t+k} \quad .$$

Substituting (16) into (14), we can see that in the steady state, (14) reduces to:

$$\sum_{k=0}^{\infty} \theta^k \beta^k \, \overline{Y} = \sum_{k=0}^{\infty} \theta^k \beta^k \, \overline{Y} \quad . \qquad (17)$$

First order Taylor expansion of (14), around this steady state, yields:

$$\sum_{k=0}^{\infty} \theta^k \beta^k \, \overline{Y} + \Xi + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \cdot \left(Y_{t+k} - \overline{Y} \right) + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \overline{Y} \left[\frac{\left(P_t^* - \overline{P} \right) - \left(P_{t-1} - \overline{P} \right)}{\overline{P}} \right] \approx$$

$$\sum_{k=0}^{\infty} \theta^k \beta^k \, \overline{Y} + \Xi + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \cdot \left(Y_{t+k} - \overline{Y} \right) + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \overline{Y} \, \mathcal{M} \Delta MC_{t+k|t} + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \overline{Y} \left[\frac{\left(P_{t+k} - \overline{P} \right) - \left(P_{t-1} - \overline{P} \right)}{\overline{P}} \right],$$

where $\Delta MC_{t+k|t} \equiv \left(MC_{t+k|t} - \overline{MC} \right)$, $\overline{P}$ is the constant steady-state price level and Ξ collects the terms that include the variables in $Q_{t,t+k}$.

Dividing through by $\overline{Y}$, canceling out terms that appear in both sides, and approximating $\frac{P_x - \overline{P}}{\overline{P}} \approx \ln \frac{P_x}{\overline{P}} \equiv p_x$,we get:

$$\left(p_t^* - p_{t-1} \right) E_t \sum_{k=0}^{\infty} \theta^k \beta^k \approx E_t \sum_{k=0}^{\infty} \theta^k \beta^k \, \mathcal{M} \Delta MC_{t+k|t} + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \left(p_{t+k} - p_{t-1} \right).$$

Using the formula for sum of converging geometric series, we have that:

$$\left(p_t^* - p_{t-1}\right)\frac{1}{(1-\beta\theta)} \approx E_t \sum_{k=0}^{\infty} \theta^k \beta^k \; \mathcal{M}\Delta MC_{t+k|t} + E_t \sum_{k=0}^{\infty} \theta^k \beta^k \; \left(p_{t+k} - p_{t-1}\right). \tag{18}$$

Rearranging, we get:

$$\left(p_t^* - p_{t-1}\right) \approx (1-\beta\theta)\left\{ \mathrm{E}_{\mathrm{t}} \sum_{k=0}^{\infty} (\beta\theta)^k \left[\mathcal{M}\, \Delta MC_{t+k|t} + \left(p_{t+k} - p_{t-1}\right)\right]\right\}. \tag{19}$$

But,

$$\mathcal{M} = \frac{1}{\overline{MC}}$$

$$\Rightarrow$$

$$\mathcal{M}\, \Delta MC_{t+k|t} = \frac{\Delta MC_{t+k|t}}{\overline{MC}} \approx \ln MC_{t+k|t} - \ln \overline{MC} \equiv \hat{mc}_{t+k|t} \quad .$$

Substituting in (19) we get (15).

Q.E.D.

Exercise 6

Given that

$$mc_{t+k|t} = mc_{t+k} + \frac{\alpha}{1-\alpha}\left(y_{t+k|t} - y_{t+k}\right) \;, \tag{20}$$

Show that

$$mc_{t+k|t} = mc_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}\left(p_t^* - p_{t+k}\right) \;. \tag{21}$$

Proof:

We already know the Dixit-Stiglitz demand for the firm output as a function of its relative price and of the aggregate demand (= aggregate output):

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k} \;.$$

Applying logs on both sides and rearranging, we get:

$$y_{t+k|t} - y_{t+k} = -\varepsilon\left(p_t^* - p_{t+k}\right) . \qquad (22)$$

Substituting in (10) we get (21)—the firm MC as a function of the aggregate MC and of the firm relative-price.

Q.E.D.

Note that the more competitive is the market—that is, the higher is the constant-elasticity-of-substitution parameter (CES), ε—the greater is the firm MC elasticity WRT to its relative price. Higher CES leads to a more responsive demand-determined output, which leads to a more responsive MC with respect to relative prices.

Exercise 7

Given (21), derive the New-Keynesian Phillips curve.

Proof:

Substitute (21) into (15), to get the optimal price setting as function of future aggregate MC:

$$p_t^* - p_{t-1} = (1-\beta\theta)E_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\hat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}\left(p_t^* - p_{t+k}\right) + \left(p_{t+k} - p_{t-1}\right)\right\} .$$

Rearrange, to get:

$$p_t^* - p_{t-1} = (1-\beta\theta)E_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\hat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}p_t^* + \frac{1-\alpha+\alpha\varepsilon}{1-\alpha}p_{t+k} + \left(\overbrace{\frac{\alpha\varepsilon}{1-\alpha} - \frac{1-\alpha+\alpha\varepsilon}{1-\alpha}}^{=-1}\right)p_{t-1}\right\}$$

$$= (1-\beta\theta)E_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\hat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}\left(p_t^* - p_{t-1}\right) + \frac{1-\alpha+\alpha\varepsilon}{1-\alpha}\left(p_{t+k} - p_{t-1}\right)\right\} .$$

Move the term $\frac{\alpha\varepsilon}{1-\alpha}\left(p_t^* - p_{t-1}\right)$ to the LHS and multiply the resulted equation by $\frac{(1-\alpha)}{(1-\alpha+\alpha\varepsilon)}$, to get:

$$\left(p_t^* - p_{t-1}\right) = (1-\beta\theta)E_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\frac{(1-\alpha)}{(1-\alpha+\alpha\varepsilon)}\hat{mc}_{t+k} + \left(p_{t+k} - p_{t-1}\right)\right\} .$$

Note that $E_t\left(p_{t+k} - p_{t-1}\right) = E_t\sum_{s=0}^{k}\pi_{t+s}$. Substitute it in the last equation to get:

$$\left(p_t^* - p_{t-1}\right) = (1-\beta\theta)E_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\frac{(1-\alpha)}{(1-\alpha+\alpha\varepsilon)}\hat{mc}_{t+k} + \sum_{s=0}^{k}\pi_{t+s}\right\} . \tag{23}$$

In the appendix below, we show that $\sum_{k=0}^{\infty}\left[(\beta\theta)^k\sum_{s=0}^{k}\pi_{t+s}\right] = \frac{1}{1-\beta\theta}\sum_{k=0}^{\infty}\left[(\beta\theta)^k\pi_{t+k}\right]$.

Substitute in (23) to get (we are almost there…).

$$\left(p_t^* - p_{t-1}\right) = (1-\beta\theta)E_t\sum_{k=0}^{\infty}(\beta\theta)^k\left\{\frac{(1-\alpha)}{(1-\alpha+\alpha\varepsilon)}\hat{mc}_{t+k} + \frac{1}{(1-\beta\theta)}\pi_{t+k}\right\} \tag{24}$$

$\Rightarrow$

$$\begin{aligned}\left(p_t^* - p_{t-1}\right) &= (1-\beta\theta)\left\{\frac{(1-\alpha)}{(1-\alpha+\alpha\varepsilon)}\hat{mc}_t + \frac{1}{(1-\beta\theta)}\pi_t\right\} \\ &\quad + (1-\beta\theta)E_t\sum_{k=1}^{\infty}(\beta\theta)^k\left\{\frac{(1-\alpha)}{(1-\alpha+\alpha\varepsilon)}\hat{mc}_{t+k} + \frac{1}{(1-\beta\theta)}\pi_{t+k}\right\} .\end{aligned} \tag{24'}$$

Note that leading equation (24) by one period taking expected values as of period t, the rightmost term in equation (24') is equal to $\beta\theta E_t\left(p^*_{t+1} - p_t\right)$, so that:

$$\left(p^*_t - p_{t-1}\right) = \left(1-\beta\theta\right)\left\{\frac{\left(1-\alpha\right)}{\left(1-\alpha+\alpha\varepsilon\right)}\overset{\wedge}{mc}_t + \frac{1}{\left(1-\beta\theta\right)}\pi_t\right\} + \beta\theta E_t\left(p^*_{t+1} - p_t\right). \quad (25)$$

Note that by moving from (24) to (25) , we changed the expression for the optimal price update, from a function of all future MCs to a function of current MC and next-period's optimal price setting. To elaborate, in (25) we just rewritten (24) using rational expectations, so the <u>entire future</u> is captured by the optimal price update in the next period, which we consider—together with <u>present</u> MC and inflation—while optimizing prices. Note further that the optimal price update is a weighted average of the future and the present, where the weights are—quite intuitively— $\beta\theta$ for the future and $\left(1-\beta\theta\right)$ for the present. So, the forward looking nature of price setting is brought about by interaction of β and θ —the degrees of patient and price rigidity, respectively!

Now, recall that—up to a log-linear approximation—inflation is a weighted average of two inflation rates: the one of those who don't update and the one of those who do. The weights are θ and $\left(1-\theta\right)$, respectively:

$$\pi_t = \theta\left(p_{t-1} - p_{t-1}\right) + \left(1-\theta\right)\left(p^*_t - p_{t-1}\right) = \left(1-\theta\right)\left(p^*_t - p_{t-1}\right). \quad (26)$$

By rational expectations it follows that

$$E_t\pi_{t+1} = \left(1-\theta\right)E_t\left(p^*_{t+1} - p_t\right). \quad (27)$$

Substituting (26) and (24) into (25), to eliminate—$\left(p^*_t - p_{t-1}\right)$ and $E_t\left(p^*_{t+1} - p_t\right)$—and rearranging, we finally get the New-Keynesian Phillips curve:

$$\pi_t = \beta \cdot E_t \pi_{t+1} + \left[\frac{(1-\beta\theta)(1-\theta)}{\theta} \cdot \frac{(1-\alpha)}{(1-\alpha+\alpha\varepsilon)} \right] \overset{\wedge}{mc}_t \quad .$$

Q.E.D.

Appendix

In this appendix we show that

$$\sum_{k=0}^{\infty} \left[(\beta\theta)^k \sum_{s=0}^{k} \pi_{t+s} \right] = \frac{1}{1-\beta\theta} \sum_{k=0}^{\infty} \left[(\beta\theta)^k \pi_{t+k} \right] \quad .$$

Proof:

Expand the LHS by writing it explicitly for every $k \in (0, \infty)$ and for every $s \in (0, k)$:

$$\begin{aligned} \sum_{k=0}^{\infty} \left[(\beta\theta)^k \sum_{s=0}^{k} \pi_{t+s} \right] = \quad & \pi_t \\ & + (\beta\theta)(\pi_t + \pi_{t+1}) \\ & + (\beta\theta)^2 (\pi_t + \pi_{t+1} + \pi_{t+2}) \\ & + (\beta\theta)^3 (\pi_t + \pi_{t+1} + \pi_{t+2} + \pi_{t+3}) \\ & \vdots \\ & + (\beta\theta)^{\infty} (\pi_t + \pi_{t+1} + \pi_{t+2} + \pi_{t+3} + \cdots + \pi_{\infty}) \quad . \end{aligned}$$

The RHS can be rearranged as:

$$\pi_t \sum_{h=0}^{\infty} (\beta\theta)^h + \pi_{t+1} \sum_{h=1}^{\infty} (\beta\theta)^h + \pi_{t+2} \sum_{h=2}^{\infty} (\beta\theta)^h + \cdots + \pi_{\infty} \sum_{h=\infty}^{\infty} (\beta\theta)^h \quad .$$

Using the formula for sum of converging geometric series, we can rewrite the last expression as:

$$\frac{1}{1-\beta\theta} \pi_t + \frac{(\beta\theta)}{1-\beta\theta} \pi_{t+1} + \frac{(\beta\theta)^2}{1-\beta\theta} \pi_{t+2} + \cdots + \frac{(\beta\theta)^{\infty}}{1-\beta\theta} \pi_{\infty} \ ,$$

or, more compactly, as $\frac{1}{1-\beta\theta} \sum_{k=0}^{\infty} \left[(\beta\theta)^k \pi_{t+k} \right]$.

Q.E.D.