July 22, 2008

Monetary Theory and Policy under Sticky Prices and Wages 2008B

Solution to homework 3

Taylor contracts:

The following questions are taken from Gali (2008), chapter 3, exercise 3.5:

During each period, a fraction $1/N$ of the firms resets prices, which then remain unchanged for the next N periods. Therefore, every N periods, the single firm solves the following problem:

$$Max \sum_{k=0}^{N-1} E_t \left\{ Q_{t,t+k} \left[P_t^* Y_{t+k|t} - \Psi_{t+k}\left(Y_{t+k|t}\right) \right] \right\} \tag{1}$$

subject to

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \tag{2}$$

$$Y_t(i) = A_t N_t(i) \; . \tag{3}$$

and

$$Q_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right) . \tag{4}$$

Given the linear production technology in (3), it follows that the marginal labor-productivity is A_t, regardless the firm output level. Additionally, recall that wages are flexible, and that the labor market is perfectly competitive. It follows that the nominal marginal cost is invariant to the firm own output:

$$\psi_{t+k|t} = \frac{W_{t+k}}{A_{t+k}} = \psi_{t+k} \; .$$

a). Show that P_t^* must satisfy the first-order condition:

$$\sum_{k=0}^{N-1} E_t \left\{ Q_{t,t+k} Y_{t+k|t}^d \left(P_t^* - \mathcal{M}\psi_{t+k} \right) \right\} = 0 \tag{5}$$

where $\psi_t \equiv \Psi'_t$ and $\mathcal{M} \equiv \dfrac{\varepsilon}{\varepsilon - 1}$.

The interpretation of (5) is that whenever revising its price, the firm considers the discounted average of its nominal marginal-cost <u>during the next N periods</u>, and sets its price as a markup over this value. There are two straightforward differences from the Calvo setup. The first difference is that under the Calvo setup, firms don't consider just the following N period, but the entire future. The second difference is that under the Taylor setup, firms are not randomly selected, and therefore the price of the **single** firm (although not the price aggregator) remains effective over the next N periods after revision, as opposed to the Calvo setup in which the firm doesn't know in advance, with certainty, when it will have the opportunity to rechange its price. Therefore, under the Taylor setup the weights of each period excludes the probability term (the Calvo parameter), and they only include the discount factor and the output level.

<u>The proof</u>:

Substituting (2) into (1) we have:

$$Max \sum_{k=0}^{N-1} E_t \left\{ Q_{t,t+k} \left(P_t^{*(1-\varepsilon)} \left(\frac{1}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} - \Psi_{t+k}\left(Y_{t+k|t} \right) \right) \right\} . \tag{6}$$

Differentiating WRT the optimal price, P_t^* and using the chain rule, we get:

$$\sum_{k=0}^{N-1} E_t \left\{ Q_{t,t+k} \left((1-\varepsilon) P_t^{*(-\varepsilon)} \left(\frac{1}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} - \psi_{t+k} \cdot \left(\frac{\partial Y_{t+k|t}}{\partial P_t^*} \right) \right) \right\} = 0 \ . \qquad (7)$$

Differentiating (2) WRT P_t^* , substituting into (7) and rearranging, we get:

$$\sum_{k=0}^{N-1} E_t \left\{ Q_{t,t+k} \left((1-\varepsilon) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} + \varepsilon \psi_{t+k|\mathrm{t}} \cdot \frac{1}{P_t^*} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \right) \right\} = 0 \ . \qquad (8)$$

Substituting (2) and rearranging, we get (5) .

Q.E.D.

b). Show that log-linearizing (5) around a zero inflation steady state, we get:

$$p_{\mathrm{t}}^* = \mu + \sum_{k=0}^{N-1} \omega_k E_t \{ \ln \psi_{t+k} \} \ , \qquad (9)$$

where $\omega_k \equiv \beta^k \dfrac{1-\beta}{1-\beta^N}$.

Note that in the book there is a mistake, where ψ is employed instead of $\ln\psi$.

Show further that when $\beta \to 1$ equation (11) approaches

$$p_{\mathrm{t}}^* = \mu + \frac{1}{N} \sum_{k=0}^{N-1} E_t \{ \ln \psi_{t+k} \} \ , \qquad (10)$$

The interpretation of (10) is straightforward: without time preference, all future periods are equally important. Therefore, all relevant N future periods will have the same weight. Consequently, the price currently chosen by the firm is a **simple** average of all the N future expected marginal-costs.

<u>The proof</u>:

First order Taylor approximation of (5) around a zero inflation steady state yields:

$$\begin{aligned}\sum_{k=0}^{N-1}\beta^k\overline{Y}\overline{P}+\Xi_t+E_t\sum_{k=0}^{N-1}\beta^k\overline{P}\left(Y_{t+k|t}^d-\overline{Y}\right)+\sum_{k=0}^{N-1}\beta^k\overline{Y}\left(P_t^*-\overline{P}\right)=\\ \sum_{k=0}^{N-1}\beta^k\overline{Y}\overline{P}+\Xi_t+E_t\sum_{k=0}^{N-1}\beta^k\overline{P}\left(Y_{t+k|t}^d-\overline{Y}\right)+\sum_{k=0}^{N-1}\beta^k\overline{Y}\mathcal{M}E_t\left(\psi_{t+k}-\overline{\psi}\right)\end{aligned} \tag{11}$$

where $\overline{Y}$, $\overline{P}$ and $\overline{\psi}$ are the steady-state output, inflation and nominal mc, respectively, and Ξ collects the terms that include the variables in $Q_{t,t+k}$.

Cancelling out terms that appear on both sides and dividing through by $\overline{Y}$ and by $\overline{P}$, we get:

$$\sum_{k=0}^{N-1}\beta^k\frac{\left(P_t^*-\overline{P}\right)}{\overline{P}}=\sum_{k=0}^{N-1}\beta^k\frac{\mathcal{M}}{\overline{P}}E_t\left(\psi_{t+k}-\overline{\psi}\right) \tag{12}$$

using the approximation

$$x_t\equiv\ln\frac{X_t}{\overline{X}}\approx\frac{X_t-\overline{X}}{\overline{X}} \tag{13}$$

we can rewrite (12) as:

$$\sum_{k=0}^{N-1}\beta^k p_t^*=p_t^*\frac{1-\beta^N}{1-\beta}=\sum_{k=0}^{N-1}\beta^k\frac{\mathcal{M}}{\overline{P}}\left(E_t\psi_{t+k}-\overline{\psi}\right). \tag{14}$$

Now, since $\overline{P}=\mathcal{M}\overline{\psi}\Rightarrow\frac{\mathcal{M}}{\overline{P}}=\frac{1}{\overline{\psi}}$,we have that:

$$p_t^*\frac{1-\beta^N}{1-\beta}=\sum_{k=0}^{N-1}\beta^k\frac{\left(E_t\psi_{t+k}-\overline{\psi}\right)}{\overline{\psi}}. \tag{15}$$

Using again the approximation in (13), we get

$$p_t^*\frac{1-\beta^N}{1-\beta}=\sum_{k=0}^{N-1}\beta^k\left[E_t\ln\psi_{t+k}-\ln\overline{\psi}\right]. \tag{16}$$

Substituting $\mu \equiv \ln M = -\ln\overline{\psi}$ (where the last equality utilizes the normalization $\overline{P} = 1$) and rearranging, we get:

$$p_t^* \frac{1-\beta^N}{1-\beta} = \mu \frac{1-\beta^N}{1-\beta} + E_t \sum_{k=0}^{N-1} \beta^k \ln\psi_{t+k} \,, \tag{17}$$

Multiply both sides by $\frac{1-\beta}{1-\beta^N}$ we get (9).

Now, rewrite the expression for ω_k as:

$$\omega_k \left(1-\beta^N\right) = \beta^k (1-\beta) \ . \tag{18}$$

First order approximation around $\beta = 1$ yields:

$$\overbrace{\omega_k (1-1)}^{0} - \omega_k N\beta^{N-1}\Delta\beta = \overbrace{1^k (1-1) + k\beta^{k-1}(1-1)\Delta\beta}^{0} - \beta^k \Delta\beta \ . \tag{19}$$

Here $\Delta\beta \equiv \beta - 1$. Dividing through by $\Delta\beta$, substituting the approximations $\lim_{\beta \to 1} \beta^{N-1} = \lim_{\beta \to 1} \beta^k = 1$ and rearranging, we get:

$$\omega_k = \frac{1}{N} \ . \tag{20}$$

Substituting in (9) we get (10).

Another way is to use the L'Hopitals rule—the derivatives ratio as an approximation when the nominator and denominator both approach zero or infinity:

$$\lim_{\beta \to 1} \omega_k \equiv \lim_{\beta \to 1} \frac{\beta^k (1-\beta)}{1-\beta^N} = \lim_{\beta \to 1} \frac{k\beta^{k-1}(1-\beta) - \beta^k}{-N\beta^{N-1}} = \frac{k \cdot 1^{k-1}(1-1) - 1^k}{-N \cdot 1^{N-1}} = \frac{1}{N}$$

Q.E.D.

c). Show that a log-linear approximation of the price aggregator

$$P_t \equiv \left[\int_0^1 P_t(i)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}} , \qquad (21)$$

is

$$p_t = \frac{1}{N}\sum_{k=0}^{N-1} p_{t-k}^{*} \ . \qquad (22)$$

The proof:

The entire economy contains N sets of firms. In each set there are firms who revise prices in the same period, and therefore set the same price, p_t^{*}. There are no two sets who reoptimizes simultaneously, so prices revision in the economy has a cycle of N periods. Since the firm index is normalize to the unit interval, it follows that each set has a mass of $1/N$. Thus, the price distribution is such, that there are N prices, each one of them is the optimal price from a different period, $j \in (t, t+1-N)$. Therefore, we can rewrite the integral in (21) as:

$$P_t = \left[\sum_{k=0}^{N-1} \frac{1}{N} P_{t-k}^{*\ (1-\varepsilon)}\right]^{\frac{1}{1-\varepsilon}} . \qquad (23)$$

We can rewrite (23) as:

$$P_t^{1-\varepsilon} = \frac{1}{N}\sum_{k=0}^{N-1} P_{t-k}^{*\ (1-\varepsilon)} . \qquad (24)$$

Now, take a first order approximation around a zero-inflation steady state:

$$\overline{P}^{(1-\varepsilon)} + (1-\varepsilon)\overline{P}^{-\varepsilon}\left(P_t - \overline{P}\right) = \overbrace{\frac{1}{N}\sum_{k=0}^{N-1}\overline{P}^{(1-\varepsilon)}}^{\overline{P}^{1-\varepsilon}} + \frac{1}{N}(1-\varepsilon)\sum_{k=0}^{N-1}\overline{P}^{(-\varepsilon)}\left(P_{t-k}^{*} - \overline{P}\right) . \qquad (25)$$

Cancelling out terms that appear on both sides and dividing through by $(1-\varepsilon)\overline{P}^{1-\varepsilon}$ we get

$$\frac{\left(P_t - \overline{P}\right)}{\overline{P}} = \frac{1}{N}\sum_{k=0}^{N-1}\frac{\left(P_{t-k}^{*} - \overline{P}\right)}{\overline{P}} \; . \tag{26}$$

Substituting once again the log-linear approximation $p_t \equiv \ln P_t \approx \dfrac{P_t - \overline{P}}{\overline{P}}$, we get (22).

Q.E.D.

The interpretation here is consistent with the interpretation of equation (5): the price index is—up to a log-linear approximation—the average of the optimal prices in the past N periods.