ON THE DISCRETE MAXIMUM PRINCIPLE FOR THE BELTRAMI COLOR FLOW

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Abstract.
We analyze the discrete maximum principle for the Beltrami color flow. The Beltrami flow can display linear as well as nonlinear behavior according to the values of a parameter β, which represents the ratio between spatial and color distances. In general, the standard schemes fail to satisfy the discrete maximum principle. In this work we show that a nonnegative second order difference scheme can be built for this flow only for small β, i.e. linear-like diffusion. Since this limitation is too severe, we construct a novel finite difference scheme, which is not nonnegative and satisfies the discrete maximum principle for all values of β. Numerical results support the analysis.

1. Introduction. The extremum principle is an important axiom needed in the construction of scale spaces, as shown in the work of Alvarez et al. [1]. The simplification process of a one-dimensional signal should be preformed such that no new local maxima (minima) is created. For higher dimensional signals the non-creation of new local maxima (minima) cannot be achieved. It is usually replaced by a new demand: the non-creation of new level sets. This new principle, termed "causality", was studied in the context of scale space and denoising by Koenderinck [9]. The extremum principle is taken then as the natural generalization of the causality principle in the vectorial case.

The discrete extremum principle is important as it ensures that intensity values in the evolving image are constrained by the initial image values and do not grow without bounds. This principle is a very restrictive stability condition and guarantees that over- and under-shoots cannot appear. We also note its relevance in image analysis via the study of the deep structure, e.g. the singular points of scale-space [10]. The discrete extremum principle was studied in several works. For their nonlinear diffusion model, Perona and Malik [12] proposed a numerical scheme which satisfies the discrete extremum principle. Catte et al. [3] have proven the ill-posedness of this continuous diffusion process and gave a regularized version of the original Perona-Malik model. Weickert [21] showed that the regularized version of Perona-Malik satisfies the extremum principle at the continuous and the discrete level. In [2] Alvarez-Lions-Morel show the existence and uniqueness of the viscosity solution for their model, as well as the discrete extremum principle for their difference scheme. Weickert [19] analyzed extensively the numerical schemes of the non-homogeneous and anisotropic flows.

In this paper we treat the discrete maximum principle for the Beltrami color flow [15]. The maximum principle in the continuous setting for the parabolic nonlinear system of PDEs characterizing this flow was proven in [4]. A natural question that raises at a discrete level is whether there exist discretizations to these equations which satisfy this principle. All standard second order explicit schemes usually violate this property (see [4]).

The goal of this paper is to find the discrete approximations which reveal the same quality as their continuous counterparts. One of the possibilities of ensuring

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the discrete maximum principle is building a non-negative scheme. For anisotropic diffusion models, Weickert [19] stated in terms of the condition number of the diffusion matrix, a sufficient condition for building non-negative schemes. We further show here that a similar condition is not only sufficient but is also necessary. This rules out the possibility of finding a non-negative scheme for this type of strong non-linear diffusion equations.

The Beltrami flow can display linear as well as nonlinear behavior according to the values of the parameter β, which represents the ratio between spatial and color distances [16]. In this work we show that for this flow, a nonnegative scheme can be built only for small values of the parameter β, i.e. linear-like diffusion. Furthermore, we propose a novel finite difference scheme, which is *not non-negative* and is valid for all values of the parameter β. The new scheme is based on adding a correction term to the standard finite difference equations. This term has the role of correcting the standard scheme at the grid points where the extremum principle fails. By choosing the step time properly, we show that the modification is needed only for points of local extremum. We prove that the modified scheme is consistent and satisfies a local and a global discrete maximum principle.

This paper is organized as follows: Section 2 gives a brief summary of the Beltrami framework. In Section 3 we review the approximations based on standard finite difference scheme. For this flow, standard difference schemes usually do not obey the maximum principle. We analyze the drawbacks of these schemes in establishing the discrete maximum principle. We also show that, necessarily, a non-negative scheme for the Beltrami color flow can be build only for a limited domain of values of the parameter β. In Section 4 we introduce our new second order finite difference scheme, and prove its consistency while preserving the discrete maximum principle. In Section 5 we present numerical results, and then we conclude in Section 6.

2. The Beltrami Framework. Let us briefly review the Beltrami framework for non-linear diffusion in computer vision [6, 15, 16].

We represent an image and other local features as an embedding maps of a Riemannian manifold in a higher dimensional space. The simplest example is a gray-level image which is represented as a 2D surface embedded in $\mathbf{R}^3$. We denote the map by $U : \Sigma \to \mathbf{R}^3$, where Σ is a two-dimensional surface, and we denote the local coordinates on it by (σ^1, σ^2). The map U is given in general by $(U^1(\sigma^1, \sigma^2), U^2(\sigma^1, \sigma^2), U^3(\sigma^1, \sigma^2))$. In our example we represent map U as follows: $(U^1 = \sigma^1, U^2 = \sigma^2, U^3 = I(\sigma^1, \sigma^2))$. We choose on this surface a Riemannian structure, namely, a metric. The metric is a positive definite and a symmetric 2-tensor that may be defined through the local distance measurements:

$$ds^2 = g_{11}(d\sigma^1)^2 + 2g_{12}d\sigma^1 d\sigma^2 + g_{22}(d\sigma^2)^2.$$

The canonical choice of coordinates in image processing is Cartesian. For such a choice, which we follow in the rest of the paper, we identify $\sigma^1 = x^1$ and $\sigma^2 = x^2$. We use below the Einstein summation convention in which a pair of upper and lower identical indices is summed over. So with this convention, the above equation is written as $ds^2 = g_{ij}dx^i dx^j$. We denote the elements of the inverse of the metric by superscripts $g^{ij} = (g^{-1})_{ij}$, and the determinant by $g = \det(g_{ij})$.

Once the image is defined as an embedding mapping of Riemannian manifolds it is natural to look for a measure on this space of embedding maps.

2.1. Polyakov Action: A Measure on the Space of Embedding Maps. Denote by (Σ, g) the image manifold and its metric, and by (M, h) the space-feature manifold and its metric. Then the functional $S[U]$ attaches a real number to a map $U : \Sigma \to M$:

$$S[U^a, g_{ij}, h_{ab}] = \int dV ||d\vec{U}||^2_{g,h},$$

where dV is a volume element that is expressed in a local coordinate system as $dV = \sqrt{g}dxdy$. This functional, for $m = 2$ and $h_{ab} = \delta_{ab}$, was first proposed by Polyakov [13] in the context of high energy physics, and the theory is known as the *string theory*.

Let us formulate the Polyakov action in matrix form: (Σ, G) is the image manifold and its metric as before. Similarly, (M, H) is the spatial-feature manifold and its metric. Define $A^{ab} = (\vec{\nabla} U^a)^t G^{-1} \vec{\nabla} U^b$.

The map $U : \Sigma \to M$ has a weight $S[U, G, H] = \int d^m\sigma \sqrt{g}\ \mathrm{Tr}(AH)$, where m is the dimension of Σ and $g = \det(G)$.

Using standard methods in the calculus of variations, the Euler-Lagrange equations with respect to the embedding (assuming Euclidean embedding space) are (see [15] for explicit derivation):

$$0 = -\frac{1}{2\sqrt{g}} h^{ab} \frac{\delta S}{\delta U^b} = \underbrace{\frac{1}{\sqrt{g}} \mathrm{div}\left(D \nabla U^a\right)}_{\Delta_g U^a}. \tag{2.1}$$

(where the matrix $D = (d^{ij})_{i,j=1,2} = \sqrt{g} G^{-1}$).

The operator that acts on U^a is the natural generalization of the Laplacian from flat spaces to manifolds and is called the Laplace-Beltrami operator, denoted by Δ_g.

The extension for non-Euclidean embedding space is treated in [7, 16, 17]. The elements of the induced metric for color images with Cartesian color coordinates are

$$g_{ij} = \delta_{ij} + \beta^2 \sum_{a=1}^{3} U^a_{x_i} U^a_{x_j}, \tag{2.2}$$

where $\beta > 0$ is the ratio between the spatial and color distances. The value of the parameter β, present in the elements of the metric g_{ij}, is very important and determines the nature of the flow. In the limit $\beta \to 0$, for example, the flow degenerates to the decoupled channel by channel linear diffusion flow. In the other limit $\beta \to \infty$ we get a different nonlinear flow. The gray-value analogue of this limit is the Total Variation flow of [14] (see details in [16]).

The gradient descent method yield a non-linear diffusion equation for each component of the color vector:

$$U^a_t = \Delta_g U^a. \tag{2.3}$$

3. Standard finite difference schemes and the discrete maximum principle. Much of the research was concentrating in non-negative schemes since this property guaranties the discrete maximum principle. In this section we analyze the standard numerical schemes and point out the inherent difficulty encountered in the anisotropic flows in general and the Beltrami flow in particular.

We rewrite the equation (2.3) for the component R of the color vector (for the other components we obtain similar equations) as follows:

$$R_t = \frac{1}{\sqrt{g}} \mathrm{div}(D \nabla R). \tag{3.1}$$

The induced metric g is given in (2.2) and the diffusion matrix D contains the coupling between the channels:

$$D = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Here the coefficients are given in terms of the image metric: $a = g_{22}/\sqrt{g}$; $c = g_{11}/\sqrt{g}$; $b = -g_{12}/\sqrt{g}$.

We work on the rectangle $\Omega = (0, b_1) \times (0, b_2)$, with $\max(b_1, b_2) = 1$, which we discretize by a uniform grid of $N = n_1 \times n_2$ pixels such that $x_i = ih,\ y_j = jh,\ t_n = n\Delta t$, where $1 \leq i \leq n_1,\ 1 \leq j \leq n_2$, and $n = 0, 1, \ldots, \lfloor T/\Delta t \rfloor\}\}$. The grid size is $h = \dfrac{b_1}{n_1} = \dfrac{b_2}{n_2}$. For each channel U^a, $a = 1, 2, 3$ we define the discrete approximation $(U^a)^n_{ij}$ by

$$(U^a)(ih, jh, n\Delta t) = (U^a)^n_{ij} \approx U^a(ih, jh, n\Delta t).$$

We impose the Neumann boundary condition.

Each channel in equation (3.1) is then discretized by an explicit finite difference scheme:

$$(U^a)^{n+1}_{ij} = (U^a)^n_{ij} + \Delta t L^n_{ij}(U^a), \tag{3.2}$$

where $L^n_{ij}(U^a)$ denotes a discretization of the Laplace-Beltrami operator $\Delta_g U^a$, to be specified below.

DEFINITION 3.1. *The scheme (3.2) is said to satisfy the (local) discrete maximum principle if for each of the color channels and for all i, j and n*

$$(U^a)^{n+1}_{ij} \leq \max_{k,l=0,\pm 1} (U^a)^n_{i+k,j+l}. \tag{3.3}$$

The term $L^n_{ij}(U^a)$, is written in the central difference framework as

$$L^n_{ij}(U^a) = A^n_{ij}(U^a) + C^n_{ij}(U^a) + B^n_{ij}(U^a), \tag{3.4}$$

where

$$A^n_{ij}(U^a) = \frac{\beta}{h^2 \sqrt{g}^n_{i,j}} \Big[a^n_{i+\frac{1}{2},j} \Big((U^a)^n_{i+1,j} - (U^a)^n_{i,j} \Big) - a^n_{i-\frac{1}{2},j} \Big((U^a)^n_{i,j} - (U^a)^n_{i-1,j}) \Big) \Big]$$

$$C^n_{ij}(U^a) = \frac{\beta}{h^2 \sqrt{g}^n_{i,j}} \Big[c^n_{i,j+\frac{1}{2}} \Big((U^a)^n_{i,j+1} - (U^a)^n_{i,j} \Big) - c^n_{i,j-\frac{1}{2}} \Big((U^a)^n_{i,j} - (U^a)^n_{i,j-1}) \Big) \Big]$$

$$\begin{aligned} B^n_{ij}(U^a) = \frac{\beta}{4h^2 \sqrt{g}^n_{i,j}} \Big[& b^n_{i,j+1}((U^a)^n_{i+1,j+1} - (U^a)^n_{i-1,j+1}) - b^n_{i,j-1}((U^a)^n_{i+1,j-1} - (U^a)^n_{i-1,j-1}) \\ & + b^n_{i+1,j}((U^a)^n_{i+1,j+1} - (U^a)^n_{i+1,j-1}) - b^n_{i-1,j}((U^a)^n_{i-1,j+1} - (U^a)^n_{i-1,j-1}) \Big]. \end{aligned}$$

REMARK 3.1. *We call the explicit scheme (3.2) a standard scheme if the terms L^n_{ij} are obtained by using the central difference approximation (3.4).*

The standard scheme, in general does not satisfy the discrete maximum principle (3.3). Regardless of the sign of b_{ij}, originating from the mixed derivatives, there will be necessarily a negative weight in this scheme. Nonnegative schemes are known to lead to the discrete maximum principle. Therefore, we look next for the most general second order non-negative scheme for our problem.

4. Nonnegative schemes and the discrete maximum principle. We are interested in building nonnegative consistent schemes for the equations (2.3).

The analysis is similar for each channel, so it is enough to show it for one of the channels. For the component R eq. (2.3) can be written as

$$R_t = \frac{1}{\sqrt{g}}\big[aR_{xx} + 2bR_{xy} + cR_{yy} + (a_x + b_y)R_x + (c_y + b_x)R_y\big]. \tag{4.1}$$

We discretize it by

$$R_{ij}^{n+1} = R_{ij}^n + \frac{\Delta t}{h^2\sqrt{g_{ij}^n}} \sum_{r,s=\pm1,0} A_{ij}^{rs} R_{i+r,j+s}^n, \tag{4.2}$$

where the weights A_{ij}^{rs} will be mentioned below.

DEFINITION 4.1. *The scheme (4.2) is called non-negative if all the weights A_{ij}^{rs}, $r, s = \pm1, 0$ are nonnegative.*

Therefore, we look for nonnegative coefficients A_{ij}^{rs}, $r, s = 0, \pm1$ so that the scheme (4.2) is a nonnegative consistent second order approximation of the equation (4.1).

Assuming that for each of the channels $U^a \in C^4(\Omega)$, $a = 1, 2, 3$ we can use Taylor's expansion and get

$$\begin{aligned}
R_{i\pm1,j\pm1} =& \\
& R_{ij} \pm h(R_x)_{ij} \pm h(R_y)_{ij} + \frac{h^2}{2}(R_{xx})_{ij} \pm h^2(R_{xy})_{ij} + \frac{h^2}{2}(R_{yy})_{ij} \\
& \pm \frac{h^3}{3!}(R_{xxx})_{ij} \pm 2\frac{h^3}{3!}(R_{xyy})_{ij} \pm 2\frac{h^3}{3!}(R_{xxy})_{ij} \pm \frac{h^3}{3!}(R_{yyy})_{ij} + O(h^4). \\
R_{i\pm1,j\mp1} =& \\
& R_{ij} \pm h(R_x)_{ij} \mp h(R_y)_{ij} + \frac{h^2}{2}(R_{xx})_{ij} - h^2(R_{xy})_{ij} + \frac{h^2}{2}(R_{yy})_{ij} \\
& \pm \frac{h^3}{3!}(R_{xxx})_{ij} \pm 2\frac{h^3}{3!}(R_{xyy})_{ij} \mp 2\frac{h^3}{3!}(R_{xxy})_{ij} \mp \frac{h^3}{3!}(R_{yyy})_{ij} + O(h^4). \\
R_{i\pm1,j} =& \\
& R_{ij} \pm h(R_x)_{ij} + \frac{h^2}{2}(R_{xx})_{ij} \pm \frac{h^3}{3!}(R_{xxx})_{ij} + O(h^4). \\
R_{i,j\pm1} =& \\
& R_{ij} \pm h(R_y)_{ij} + \frac{h^2}{2}(R_{yy})_{ij} \pm \frac{h^3}{3!}(R_{yyy})_{ij} + O(h^4).
\end{aligned} \tag{4.3}$$

Demanding that approximation (4.2) be consistent, we obtain the following system of equations:

$$\begin{aligned}
&\sum_{|r|,|s|=1,0} A_{ij}^{rs} = 0, \\
&\sum_{|r|,|s|=1,0} A_{ij}^{rs} r = h(a_x + b_y)_{ij} = \frac{1}{2}[(a_{i+1,j} - a_{i-1,j}) + (b_{i,j+1} - b_{i,j-1})], \\
&\sum_{|r|,|s|=1,0} A_{ij}^{rs} s = h(b_x + c_y)_{ij} = \frac{1}{2}[(b_{i+1,j} - b_{i-1,j}) + (c_{i,j+1} - c_{i,j-1})], \\
&\sum_{|r|,|s|=1,0} A_{ij}^{rs} r^2 = 2a_{ij}, \\
&\sum_{|r|,|s|=1,0} A_{ij}^{rs} rs = 2b_{ij},
\end{aligned}$$

$$\sum_{|r|,|s|=1,0} A_{ij}^{rs} s^2 = 2c_{ij} \tag{4.4}$$

In order that the scheme (4.2) be nonnegative we need to impose the conditions:

$$1 + \frac{\Delta t}{h^2 \sqrt{g_{ij}^n}} A^{00} \geq 0, \quad \text{and} \quad A^{rs} \geq 0, \ \forall r, s = \pm 1, 0, |r| + |s| > 0.$$

The first of these inequalities can be obtained by choosing Δt small enough. It only remains to keep the other eight weights nonnegative. From these conditions we get the following theorem:

THEOREM 4.1. *A necessary and sufficient condition for building a nonnegative second order approximation for the Beltrami color flow is*

$$|b_{ij}| \leq \min\{a_{ij}, c_{ij}\} \tag{4.5}$$

Proof.

Necessity.

As we saw above, if the scheme is consistent and nonnegative, the system (4.4) is satisfied for all nonnegative $A^{rs}, \forall r, s = \pm 1, 0$. Then

$$\begin{aligned} 2|b_{ij}| \leq & \sum_{|r|,|s|=1,0} \left|A_{ij}^{rs} rs\right| \leq \sum_{|r|,|s|=1,0} A_{ij}^{rs} \min\{r^2, s^2\} \\ & \leq \min\Big(\sum_{|r|,|s|=1,0} A_{ij}^{rs} r^2, \sum_{|r|,|s|=1,0} A_{ij}^{rs} s^2\Big) = 2\min\{a_{ij}, c_{ij}\}. \end{aligned}$$

Sufficiency.

If the condition (4.5) holds, then suitable nonnegative coefficients exists. For example, the following coefficients are nonnegative and do satisfy the needed properties given by the system (4.4).

$$\begin{aligned} A_{ij}^{1,0} &= \frac{1}{h^2}(a - |b|)_{i+\frac{1}{2},j}, \quad & A_{ij}^{-1,0} &= \frac{1}{h^2}(a - |b|)_{i-\frac{1}{2},j}, \\ A_{ij}^{0,1} &= \frac{1}{h^2}(c - |b|)_{i,j+\frac{1}{2}}, \quad & A_{ij}^{0,-1} &= \frac{1}{h^2}(c - |b|)_{i,j-\frac{1}{2}}. \end{aligned} \tag{4.6}$$

(half indices are obtained by linear interpolation)

$$\begin{aligned} A_{ij}^{1,1} &= \frac{1}{4h^2}(|b_{i+1,j+1}| + b_{i+1,j+1} + |b_{ij}| + b_{ij}), \\ A_{ij}^{-1,-1} &= \frac{1}{4h^2}(|b_{i-1,j-1}| + b_{i-1,j-1} + |b_{ij}| + b_{ij}), \\ A_{ij}^{-1,1} &= \frac{1}{4h^2}(|b_{i-1,j+1}| - b_{i-1,j+1} + |b_{ij}| - b_{ij}), \\ A_{ij}^{1,-1} &= \frac{1}{4h^2}(|b_{i+1,j-1}| - b_{i+1,j-1} + |b_{ij}| - b_{ij}) . \end{aligned} \tag{4.7}$$

This concludes the proof. □

These coefficients have been used by Weickert [19], who showed that a nonnegative scheme for a nonlinear anisotropic diffusion can be guaranteed if the spectral condition number of the matrix diffusion D is bounded by the number 5.8284. Weickert's criterion does give, though, only sufficient condition. Theorem 4.1 is more general in the sense that it formulates both the sufficient and the *necessary* conditions for the existence of nonnegative consistent second order schemes for anisotropic diffusion flows. Breaking the condition (4.5), therefore makes the task of building a nonnegative scheme impossible.

FIG. 4.1. *Top: Noisy image. Left: Denoising based on the standard scheme* ($\beta = \sqrt{0.0001}$ *middle and* $\beta = \sqrt{0.002}$ *bottom). Right: Edge detection through the nonnegativity condition* ($\beta = \sqrt{0.0001}$ *middle and* $\beta = \sqrt{0.002}$ *bottom). Parameters: 25 iterations,* $h = 1/305$, $\Delta t = 0.00006$.

It is instructive to see the points where the non-negativity condition is broken. In (Fig. 4.1) we demonstrate denoising of an image corrupted by noise artifacts introduced by the JPEG lossy compression algorithm. We use the Beltrami flow implemented with the standard scheme. We mark the grid points of the image, where condition $|b_{ij}| \leq \min(a_{ij}, c_{ij})$ holds by white, and the grid points where $|g_{12}| > \min(g_{11}, g_{22})$ by black. The regions where this condition fails are positioned near the edges of the image. Fig. 4.1 bottom can be interpreted as a segmentation/edge detection process. Notice that the larger the parameter β, the better the detection. In the middle picture, with $\beta = \sqrt{0.0001}$, the segmented image contains a lot of unclosed segments, while in the bottom picture with larger β, $\beta = \sqrt{0.002}$ most of the contours are completed. Moreover, to understand the role of parameter β within the nonnegative scheme, we consider the following example:

Given a color image where the values of the gradients are $R_x = 1 = R_y$, $B_x = 1, B_y = 10$, $G_x = 0, G_y = 0$. The nonnegativity condition $|g_{12}| \leq \min(g_{22}, g_{11})$ becomes

$$1 + \beta^2(R_x^2 + G_x^2 + B_x^2) \geq \beta^2(R_xR_y + G_xG_y + B_xB_y),$$

which means $1 + 2\beta^2 \geq 11\beta^2$. In order for the nonnegative condition to be satisfied, we need to limit the parameter β, i.e. $\beta \leq 1/3$. The example illustrates the fact that a nonnegative scheme is possible only under the restriction of the parameter β. This restriction limits the amount of possible anisotropic diffusion, which is undesired.

Since the elements of the diffusion matrix depend on parameter β, and given the significance of this parameter in establishing the nature of the Beltrami flow, it is interesting to find, in terms of β, an equivalent condition to the nonnegativity condition (4.5). This relation can be found in the next theorem.

THEOREM 4.2. *The condition $|g_{12}| \leq min(g_{11}, g_{22})$ is satisfied if and only if*

$$\beta \leq \frac{2}{\sqrt{\max(R_x^2 + G_x^2 + B_x^2, R_y^2 + G_y^2 + B_y^2)}}. \tag{4.8}$$

Proof.

Without loss of generality $g_{11} \leq g_{22}$. Then $\min(g_{11}, g_{22}) = g_{11}$. We then have the following equivalencies

$$\begin{aligned}
|g_{12}| \leq \min(g_{11}, g_{22}) \Leftrightarrow & \\
& \Leftrightarrow \beta^2|R_xR_y + G_xG_y + B_xB_y| \leq 1 + \beta^2(R_x^2 + G_x^2 + B_x^2) \\
& \Leftrightarrow \beta^2\big(|R_xR_y + G_xG_y + B_xB_y| - (R_x^2 + G_x^2 + B_x^2)\big) \leq 1 \\
& \Leftrightarrow |R_xR_y + G_xG_y + B_xB_y| - (R_x^2 + G_x^2 + B_x^2)\big) \leq \frac{1}{\beta^2} \\
& \Leftrightarrow \max\{\big(|R_xR_y + G_xG_y + B_xB_y| - (R_x^2 + G_x^2 + B_x^2)\big)\} \leq \frac{1}{\beta^2}. \qquad (4.9)
\end{aligned}$$

Let us now determine the maximum of the function $S(x, y) = |R_xR_y + G_xG_y + B_xB_y| - (R_x^2 + G_x^2 + B_x^2)$. We distinguish two cases:

1. If $R_xR_y + G_xG_y + B_xB_y \geq 0$, then $S = R_x(R_y - R_x) + G_x(G_y - G_x) + B_x(B_y - B_x) \leq (R_y^2 + G_y^2 + B_y^2)/4$. The maximum of S is $(R_y^2 + G_y^2 + B_y^2)/4$ and is attained when $R_x = R_y/2$, $G_x = G_y/2$, $B_x = B_y/2$.
2. If $R_xR_y + G_xG_y + B_xB_y \leq 0$, then $S = R_x(-R_y - R_x) + G_x(-G_y - G_x) + B_x(-B_y - B_x) \leq (R_y^2 + G_y^2 + B_y^2)/4$. The maximum of S is $(R_y^2 + G_y^2 + B_y^2)/4$ and is attained when $R_x = -R_y/2$, $G_x = -G_y/2$, $B_x = -B_y/2$.

We therefore conclude that $\max S = (R_y^2 + G_y^2 + B_y^2)/4$. The last inequality in (4.9) becomes $(R_y^2 + G_y^2 + B_y^2)/4 \leq \frac{1}{\beta^2}$, i.e. $\beta \leq \frac{2}{\sqrt{R_y^2 + G_y^2 + B_y^2}}$.

In case $g_{22} \leq g_{11}$ the argument is the same with $x \leftrightarrow y$. Taking both possibilities into account, we find $\beta \leq \frac{2}{\sqrt{\max(R_x^2 + G_x^2 + B_x^2, R_y^2 + G_y^2 + B_y^2)}}$. This concludes the proof. □

In the region of the edges, the values of the gradients are large. For a given parameter β, the bound (4.8) is broken at points of high gradients. The higher the beta, the more points fail to satisfy the bound (4.8). Inversely, given the gradients, the value of β is bounded. In natural images it can easily be seen that a β that satisfies the bound is small and the resulting flow resembles a linear diffusion. The conclusion from this state of affairs is that the nonnegative scheme (4.2) is unpractical if a strong anisotropic flow, such as the Beltrami color flow, is needed.

5. Modified finite difference scheme. In this section we present our new scheme, which **is not non-negative**. For the construction of the modified scheme it is necessary to make the following assumptions:

1. The vector color is C^4 (the scheme we propose is a finite difference scheme, second order in space and first order in time).
2. The grid is fine enough in order to get a small numerical error and to resolve the extremum points in the manner explained below.

For simplicity, everywhere in this section a local maximum point will be denoted for brevity as a maximum point. We also mention that the analysis is given for maximum points only. The analysis of the minimum points is similar.

We start with some definitions: Let u be a smooth function defined on the domain $\Omega \subset \mathbf{R}^2$ and let u_{ij} be its discrete approximation on the grid $D_h = \{(ih, jh)\}$. Let $\underline{x}_0$ be a maximum point of the function u on Ω.

DEFINITION 5.1. *A point (i, j) on the grid is called a maximum point for $u_{i,j}$ if*

$$d_{ij}^{kl} = u_{i+k,j+l} - u_{ij} \leq 0 \quad \textit{and} \quad \sum_{kl} d_{ij}^{kl} < 0, \tag{5.1}$$

for all k, l so that $|k| + |l| > 0, k, l = \pm 1, 0$.

DEFINITION 5.2. *A point (i, j) on the grid is called a smooth maximum point for $u_{i,j}$ if the following holds:*

1. *It is a maximum point, and*
2. *It is "close" to the point $\underline{x}_0$, i.e. $(i, j) = \underline{x}_0 + \underline{\gamma} h$, where $\underline{\gamma} = (\alpha, \mu), 0 \leq |\alpha| < 1, 0 \leq |\mu| < 1$.*

The new approximation scheme is explicit, is based on a modification of the standard scheme and, for each channel gives

$$(U^a)_{ij}^{n+1} = (U^a)_{ij}^n + \Delta t \tilde{L}_{ij}^n (U^a) \tag{5.2}$$

where $\tilde{L}_{ij}^n$ is a discretization of the Laplace-Beltrami operator, which will be explained below.

Before describing the new scheme, let us make the following remark:

REMARK 5.1. *In order to ensure the discrete maximum principle it is not necessary to apply condition (3.3) at all points. It is enough to guarantee that all* maximum *points (i,j) of U^a satisfy the condition:*

$$(U^a)^{n+1}_{ij} \le (U^a)^n_{ij}, \tag{5.3}$$

and to take Δt small enough to ensure that no non-maximum point violates condition (3.3).

Based on Remark 5.1, it follows that we need to modify the standard scheme only at maximum points (i,j) of U^a_{ij}, where $L^n_{ij}(U^a) > 0$.

We mention that the analysis of the new scheme will be given only for one of the components of the color vector (say $U^1 = R$), and for the other channels the analysis is similar. For convenience, we abbreviate the standard scheme's operator by $L^n_{ij}(R) = L^n_{ij}$, and our modified scheme's operator by $\tilde{L}^n_{ij}(R) = \tilde{L}^n_{ij}$.

Next we describe the **modified scheme**:

- For points (i,j) which are non-maximum points for R^n_{ij} or for those that are maximum points and $L^n_{ij} \le 0$ holds, apply the standard scheme

  $$R^{n+1}_{ij} = R^n_{ij} + \Delta t L^n_{ij}. \tag{5.4}$$

- For maximum points (i,j) of R^n_{ij}, where $L^n_{ij} > 0$ apply the modified scheme

  $$R^{n+1}_{ij} = R^n_{ij} + \Delta t \tilde{L}^n_{ij}, \tag{5.5}$$

  where

  $$\tilde{L}^n_{ij} = L^n_{ij} + k^n_{ij} h^2 M^n_{ij}. \tag{5.6}$$

  Here $M^n_{ij} = \min(\Delta R^n_{ij}, \tilde{\Delta} R^n_{ij})$ and by ΔR_{ij}, $\tilde{\Delta} R_{ij}$ we denote the two second order discretizations of the Laplacian (the "cross" and the "diagonal"):

  $$\begin{aligned} \Delta R_{i,j} &= \frac{R_{i+1,j} + R_{i-1,j} - 4R_{i,j} + R_{i,j+1} + R_{i,j-1}}{h^2}, \\ \tilde{\Delta} R_{i,j} &= \frac{R_{i+1,j+1} + R_{i-1,j-1} - 4R_{i,j} + R_{i-1,j+1} + R_{i+1,j-1}}{2h^2}. \end{aligned} \tag{5.7}$$

  The parameters k^n_{ij} are arbitrary. They must be carefully chosen so that scheme (5.5) satisfies consistency and the discrete maximum principle. Our choice is specified below.

The following lemma gives the information on the sign of the term M^n_{ij}.

Lemma 5.1. *If (i,j) is a maximum point of R_{ij} we have*

$$M_{ij} < 0.$$

Proof. Suppose on the contrary, that at a maximum point (i,j) of the function R_{ij}, we have:

$$\Delta R_{ij}, \Delta \tilde{R}_{ij} \ge 0. \tag{5.8}$$

Denote $d^{kl}_{ij} = R_{i+k,j+l} - R_{i,j}$, for all $k,l \in I_s$ where $I_s = \{k,l | k,l = \pm 1, 0,\ |k|+|l| > 0\}$. According to (5.1), at a maximum point (i,j) the following relations hold

$$d^{kl}_{ij} \le 0 \ \forall k,l \in I_s \text{ and } \sum_{k,l \in I_s} d^{kl}_{ij} < 0 \ .$$

Under assumption (5.8) $\Delta\tilde{R}_{ij} = d_{ij}^{1,1} + d_{ij}^{1,-1} + d_{ij}^{-1,1} + d_{ij}^{-1,-1} \geq 0$ and $\Delta R_{ij} = d_{ij}^{1,0} + d_{ij}^{0,1} + d_{ij}^{0,-1} + d_{ij}^{-1,0} \geq 0$. Summing up the last two inequalities gives $\sum_{k,l \in I_s} d_{ij}^{k,l} \geq 0$ and contradicts the fact that (i,j) is a maximum point. □

Let us discuss the constraints of the consistency and the maximum principle.

- In order for this modified scheme to be consistent, we must have $|k_{ij}^n| \leq O(\frac{1}{h})$.
- Maximum principle implies that the parameters k_{ij} should be chosen such that (5.3) holds, i.e. at maximum points (i,j), $\tilde{L}_{ij}^n \leq 0$.

Lemma 5.2. *Let $D = (d_{\alpha\beta})$ be a two-dimensional symmetric and positive definite matrix with smooth elements. Let R be a smooth function ($R \in C^4(\Omega)$) where Ω is a bounded domain in $\mathbf{R}^2$. Then in a neighborhood W of a maximum point of the function R, the following holds*

$$\sum_{\alpha,\beta=1}^{2} (d_{\alpha\beta} R_{x_\alpha x_\beta})(\bar{x}) \leq 0, \ \forall \bar{x} \in W. \tag{5.9}$$

Proof. Let x_0 be a maximum point of the function R. We show that

$$\sum_{\alpha,\beta=1}^{2} (d_{\alpha\beta} R_{x_\alpha x_\beta})(x_0) \leq 0. \tag{5.10}$$

The matrix D is symmetric positive definite. Then there exists an orthogonal matrix O s.t. $O^t D O = diag(c_1, c_2)$, where $c_k > 0$ for $k = 1, 2$. Let z denote the transformed coordinate system $z = Ox$. At the point x_0 we then get

$$\sum_{\alpha,\beta=1}^{2} d_{\alpha\beta} R_{x_\alpha x_\beta} = \sum_{k,l=1}^{2} \sum_{\alpha,\beta=1}^{2} d_{\alpha\beta} R_{z_k z_l} O_{ki} O_{lj} = \sum_{k=1}^{2} c_k R_{z_k z_k} \ .$$

Since $c_k > 0$ and at a maximum point $R_{z_k z_k}(x_0) \leq 0$, the desired inequality $\sum_{\alpha,\beta=1}^{2} (d_{\alpha\beta} R_{x_\alpha x_\beta})(x_0) \leq 0$ is established. Based on continuity, inequality (5.10) holds as well in the neighborhood W of a maximum point □

We divide the operator L_{ij}^n into a sum of two operators $L_{ij}^n = L_{ij}^{1,n} + L_{ij}^{2,n}$, where $L_{ij}^{1,n}$ is the discrete approximation of $L^1 = \big[(a_x + b_y)R_x + (c_y + b_x)R_y\big]/\sqrt{g}$ and $L_{ij}^{2,n}$ is the discrete approximation of $L^2 = (aR_{xx} + 2bR_{xy} + cR_{yy})/\sqrt{g}$.

Based on the assumption that the grid is fine enough, and from the definition of a smooth maximum point, Lemma 5.2 has the following consequence:

Consequence 5.1. *If (i,j) is a smooth maximum point of R_{ij}^n, then*

$$L_{ij}^{2,n} = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{2} \Big(d_{\alpha\beta} R_{x_\alpha x_\beta}\Big)_{ij}^{n} \leq 0 \tag{5.11}$$

where

$$(d_{\alpha\beta}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ .$$

The **algorithm** we use is

- If (i,j) is not a maximum point or it is a maximum point s.t. $L^n_{ij} \le 0$ then

$$R^{n+1}_{ij} = R^n_{ij} + \Delta t L^n_{ij}.$$

(here $\tilde{L}^n_{ij} = L^n_{ij}$)

- If (i,j) is a maximum point s.t. $L^{1,n}_{ij} > 0$ and $L^{2,n}_{ij} \le 0$, then

$$\begin{aligned} k^n_{ij} &= \frac{-2L^{1,n}_{ij}}{h^2 M^n_{ij}}, \\ R^{n+1}_{ij} &= R^n_{ij} + \Delta t (L^n_{ij} + k^n_{ij} h^2 M^n_{ij}) \end{aligned} \tag{5.12}$$

(here $\tilde{L}^n_{ij} = L^{2,n}_{ij} - L^{1,n}_{ij}$)

- If (i,j) is a maximum point s.t. $L^{1,n}_{ij} > 0$ and $L^{2,n}_{ij} > 0$, then

$$\begin{aligned} k^n_{ij} &= \frac{-2L^{1,n}_{ij} - L^{2,n}_{ij}}{h^2 M^n_{ij}}, \\ R^{n+1}_{ij} &= R^n_{ij} + \Delta t (L^n_{ij} + k^n_{ij} h^2 M^n_{ij}) \end{aligned} \tag{5.13}$$

(here $\tilde{L}^n_{ij} = -L^{1,n}_{ij}$)

- If (i,j) is a maximum point s.t. $L^{1,n}_{ij} \le 0$ and $L^{2,n}_{ij} > 0$, then

$$\begin{aligned} k^n_{ij} &= \frac{-2L^{2,n}_{ij} - L^{1,n}_{ij}}{h^2 M^n_{ij}}, \\ R^{n+1}_{ij} &= R^n_{ij} + \Delta t (L^n_{ij} + k^n_{ij} h^2 M^n_{ij}) \end{aligned} \tag{5.14}$$

(here $\tilde{L}^n_{ij} = -L^{2,n}_{ij}$)

- end

NOTE 5.1. *The parameters k_{ij} in our algorithm are chosen in the following way:*

- *In the case of non-smooth maximum points (see eq. (5.13), (5.14)), the parameters are built such that the discrete maximum principle holds, i.e. $\tilde{L}^n_{ij} \le 0$.*
- *In the case of smooth maximum points (see eq. (5.12)), the parameters are chosen such that the scheme would satisfy the discrete maximum principle and also be consistent.*

NOTE 5.2. *It is also important to remark that the choice of k_{ij} is not unique.*

Below we prove the consistency of the modified scheme. We need to do this only for smooth maximum points. We also show that at these points $\tilde{L}^n_{ij} < 0$, i.e. the discrete maximum principle is satisfied.

LEMMA 5.3. *Let (i,j) be a smooth maximum point of the function R^n_{ij} such that the following holds*

$$L^n_{ij} > 0 \quad \textit{and} \quad L^{1,n}_{ij} > 0. \tag{5.15}$$

If the parameters k^n_{ij} are chosen such that

$$k^n_{ij} = \frac{-2L^{1,n}_{ij}}{h^2 M^n_{ij}}, \tag{5.16}$$

then

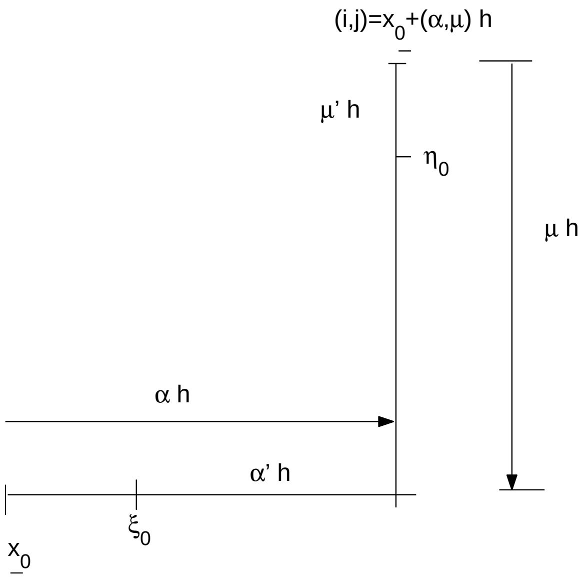

FIG. 5.1. *Maximum point*

i) $|k^n_{ij}| \leq O(\frac{1}{h})$, *(i.e. the modified scheme is consistent);*

ii) $\tilde{L}^n_{ij} < 0$, *(i.e. the discrete maximum principle holds).*

Proof. We denote the maximum point of the function R by $\underline{x}_0$ ($\underline{x}_0 \in \mathbf{R}^2$). Then according to the definition of a smooth maximum point:

$(i,j) = \underline{x}_0 + (\alpha h, \mu h)$, $(0 \leq |\alpha| < 1, \quad 0 \leq |\mu| < 1$ (note Fig. 5.1)).

We have

$$L^1_{ij} = \frac{1}{\sqrt{g_{ij}}}\Big[(a_x + b_y)_{ij}(R_x)_{ij} + (b_x + c_y)_{ij}(R_y)_{ij}\Big]. \tag{5.17}$$

Due to the assumption on the smoothness of the functions, we have

$$(a_x + b_y)_{ij}/(\sqrt{g})_{ij} = O(1). \tag{5.18}$$

Then we can write

$$|L^1_{ij}| \leq C_1|(R_x)_{ij} + (R_y)_{ij}|. \tag{5.19}$$

But

$$(R_x)_{ij} = R_x(i,j) + O(h^2), \quad (R_y)_{ij} = R_y(i,j) + O(h^2). \tag{5.20}$$

From Taylor expansion we get

$$R_x(\underline{x}_0) = (R_x)(i,j) - \alpha h R_{xx}(i,j) - \mu h R_{xy}(i,j) + O(h^2), \tag{5.21}$$

$$R_x(\underline{x}_0) = (R_y)(i,j) - \alpha h R_{yx}(i,j) - \mu h R_{yy}(i,j) + O(h^2). \tag{5.22}$$

Using the fact that $\underline{x}_0$ is a maximum point,

$$(R_x)(i,j) = \alpha h R_{xx}(i,j) + \mu h R_{xy}(i,j) + O(h^2). \tag{5.23}$$

and

$$(R_y)(i,j) = \alpha h R_{yx}(i,j) + \mu h R_{yy}(i,j) + O(h^2). \tag{5.24}$$

Relations (5.19), (5.20), (5.23), (5.24) imply

$$|L^1_{ij}| \leq C_1 h |\alpha R_{xx}(i,j) + \mu R_{yy}(i,j) + (\alpha+\mu) R_{xy}(i,j)| + O(h^2). \tag{5.25}$$

From the maximality of the point $\underline{x}_0$, $|R_{xy}(\underline{x}_0)| \leq |(R_{xx}+R_{yy})(\underline{x}_0)| = |\Delta R(\underline{x}_0)|$ and since (i,j) is a smooth maximum point, then

$$|R_{xy}(i,j)| \leq |\Delta R(i,j)| + O(h). \tag{5.26}$$

Replacing (5.26) into (5.25) we then obtain

$$|L^1_{ij}| \leq C_2 h |\Delta R(i,j)| + O(h^2), \tag{5.27}$$

where the constant $C_2 = C(C_1, \alpha, \mu)$. We have that M_{ij} is a second order approximation of the laplacian ΔR:

$$M_{ij} = \Delta R(i,j) + O(h^2). \tag{5.28}$$

Using (5.27) and (5.28), it follows that

$$|k_{ij}| = \frac{|L^1_{ij}|}{h^2 |M_{ij}|} \leq \frac{C_2 h |\Delta R(i,j)| + O(h^2)}{h^2(|\Delta R(i,j)| + O(h^2))}.$$

We thus obtained that $|k^n_{ij}| \leq O\Big(\dfrac{1}{h}\Big)$, which means consistency.

ii) Due to Consequence 5.1, at the point (i,j) we have $L^{2,n}_{ij} \leq 0$. From (5.12) and (5.16), we have

$$\tilde{L}^n_{ij} = L^{2,n}_{ij} - L^{1,n}_{ij}.$$

But we assumed in (5.15) that $L^{1,n}_{ij} > 0$. Therefore $\tilde{L}^n_{ij} < 0$, i.e. the discrete maximum principle is satisfied.
□

Remark 5.2. *Using the Mathematica computer software, we checked the accuracy of the standard finite difference scheme and we showed that it is second order in space.*

Below we show that the modified scheme is also second order accurate in space.

Lemma 5.4. *The modified scheme is second order in space.*

Proof. Without restricting the generality, suppose that the grid is quadratic and contains n^2 points. Let us denote by k the total number of local maximum points and assume that k is independent of h. From Lemma 5.3 and Remark 5.2, we then can classify the grid points in the following way: k points have first order accuracy and n^2-k points have second order accuracy. Therefore the error

$$||E||^2 = ||u - u_{ij}||^2 = \frac{1}{n^2}\sum_{ij}^{n} |E_{ij}|^2 = \frac{1}{n^2}\Big[\sum_{ij} \underbrace{(\eta_{ij} h)^2}_{k} + \sum_{ij} \underbrace{(\zeta_{ij} h^2)^2}_{n^2-k}\Big].$$

Then

$||E||^2 \leq \left[h^2\eta^2_{max}kh^2 + \frac{n^2-k}{n^2}\zeta^2_{max}h^4\right] \leq k\eta^2_{max}h^4 + \zeta^2_{max}h^4 = (k\eta^2_{max} + \zeta^2_{max})h^4$,

which implies that $||E|| = O(h^2)$.

We thus proved that the modified scheme is second order accurate in space.

□

THEOREM 5.1. *(Local maximum principle) If Δt satisfies $\Delta t = \min_{n,i,j} \Delta t^n_{ij}$, where*

$$\Delta t^n_{ij}|\tilde{L}^n_{ij}(U^a)| \leq [(\max_{(l,m),l,m=\pm1,0}(U^a)^n_{i+l,j+m}) - (U^a)^n_{ij}], \tag{5.29}$$

then the following maximum principle holds:

$$(U^a)^n_{ij} \leq \max_{(l,m),l,m=\pm1,0}(U^a)^{n-1}_{i+l,j+m}, \textit{ for all } \quad i,j. \tag{5.30}$$

Proof.

If (i,j) is a maximum point for the component U^a_{ij} then from the construction of the modified scheme, (5.30) obviously follows. If (i,j) is not a maximum point, then choosing Δt as in (5.29) also leads to (5.30). So the discrete maximum principle is satisfied. □

We also have the following global maximum principle:

COROLLARY 5.1. *(Global maximum principle) If Δt satisfies $\Delta t = \min_{n,i,j} \Delta t^n_{ij}$, where*

$$\Delta t^n_{ij}|\tilde{L}^n_{ij}| \leq [(\max_{(l,m),l,m=\pm1,0}(U^a)^n_{i+l,j+m}) - (U^a)^n_{ij}],$$

then the following maximum principle holds for each of the components of the color vector:

$$(U^a)^n_{ij} \leq \max_{i,j}(U^a)^0_{ij}\ , \quad \textit{for all} \quad i,j. \tag{5.31}$$

6. Details of the Implementation and Results. In this section we present experimental results of denoising color images by means of the two schemes (the standard and the modified). The initial data are given in three channels r, g and b in the range 0 to 255. We first transfer the images to the more perceptually adaptive coordinates $R = \log(1+r)$, $G = \log(1+g)$, $B = \log(1+b)$. These adaptive coordinates do not limit the generality of our analysis. The dynamic range of these variables is 0 to 8. For the modified scheme, the choice of the parameters k_{ij} is the one given in the algorithm presented in Section 5. The step time Δt can be fixed by hand or can be computed in an adaptive way, namely at each iteration $\Delta t^n = \min_{ij} \Delta t^n_{ij}$, and Δt^n_{ij} is computed according to (5.29). We emphasize that the criterion of comparison between the two schemes is *not necessarily the visual quality*, but rather *the fulfillment of the discrete maximum principle.*

In the first example we corrupt a given image with Gaussian random noise. We then denoise it using the two methods. The discrete maximum principle is clearly violated in the standard scheme (see Fig.6.1 bottom left). Though the visual results of the images obtained by using the two schemes are very similar, only the modified scheme enjoys the discrete maximum principle property.

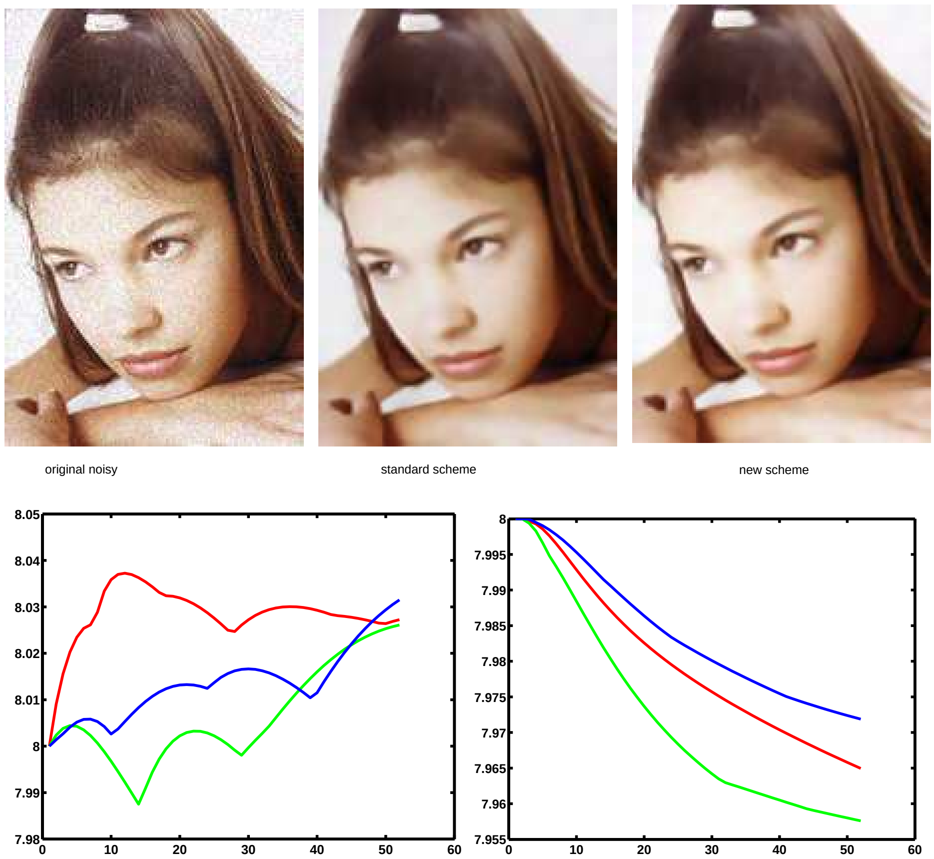

FIG. 6.1. *Top-left: Noisy Camila image. Top-middle: Result of the Beltrami flow with the standard scheme. Top-right: Result of the Beltrami flow with the modified scheme. Bottom: Plot of maximum of each of the channels versus number of iterations. Left: standard scheme. Right: modified scheme. Process parameters: 53 iterations,* $\beta = \sqrt{0.0026}$*,* $\Delta t = 0.00002$*,* $h = 1/220$*.*

In the second example we denoise an image damaged by certain artifacts of a lossy JPEG compression algorithm. In Fig. 6.3 left graph, we see a very serious violation of the discrete maximum principle in the standard scheme. All channels intensities got out of the domain of initial values ([0,8] in our new coordinates). The values of the red channel go up to 8.3, over the allowed maximum value of 8. In order to display the color image we need either to chop it or to scale it back to the initial domain. In this example, we observe the importance of the method used to display the color image. Using the chopping method, the color image seems to be better restored than using the rescaling method, which modifies the image contrast (see Fig. 6.2 bottom-right). In both images, obtained by the standard scheme, one can notice some artifacts in the eye region. These artifacts are positioned along a diagonal edge (see zoom in Fig. 6.4). In the proposed numerical scheme these artifacts are not present (see Fig. 6.2 bottom left).

FIG. 6.2. *Top-left: Noisy Cameron image. Top-right: Result of the denoising by use of the standard scheme. Display of the color image in Matlab by chopping. Bottom-left: Result of the denoising by use of the modified scheme. Bottom-right: Result of the denoising by use of the standard scheme. Display of the color image in Matlab by rescaling. Process parameters: 32 iterations,* $\beta = \sqrt{0.0012}$, Δt *adaptive,* $h = 1/159$.

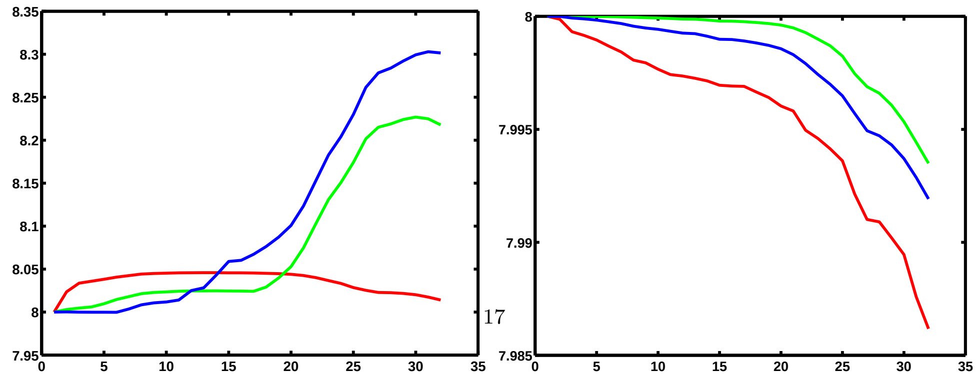

FIG. 6.3. *Plot of maximum of each of the channels versus number of iterations. Left: standard scheme. Right: modified scheme.*

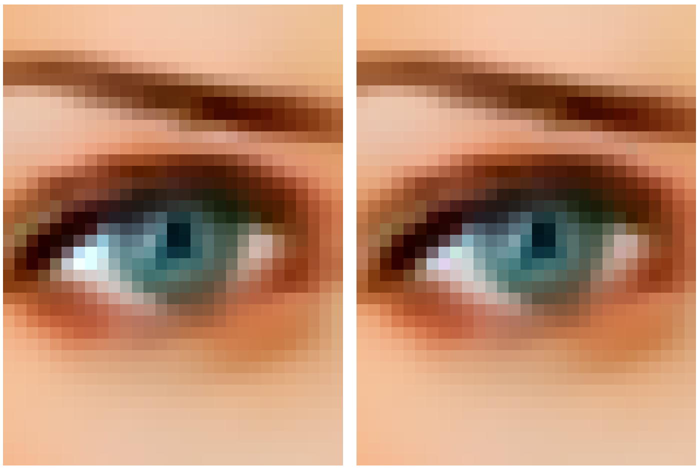

FIG. 6.4. *Zoom. Left: standard scheme. Right: modified scheme. Notice the white artifact in the left image along the diagonal edge in the eye region, and not in right image, produced by the modified scheme.*

A more severe deviation from the satisfaction of the discrete maximum principle in the standard scheme can be noticed in Fig. 6.6. The maximum of the red component goes up to the value 8.5, as illustrated in the graph below (see Fig. 6.6). The diagonal edges are better restored by the modified scheme. In Fig. 6.7 one can observe a strong artifact on the diagonal edge of the bottom part of the leaf, which is present only in the image processed by the standard scheme and not in the image processed by the modified scheme.

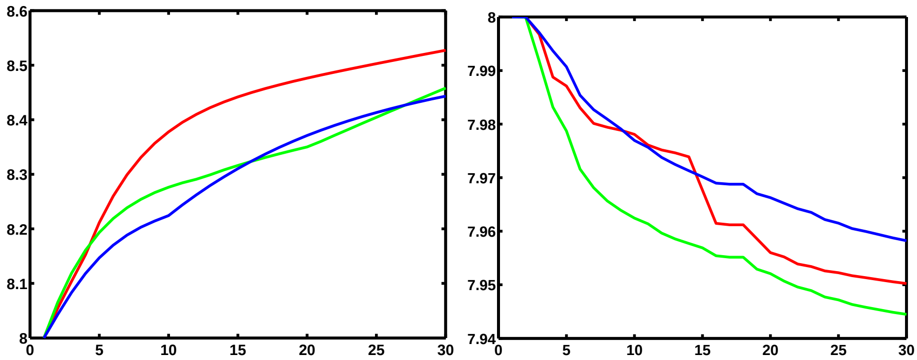

FIG. 6.6. *Plot of maximum of each of the channels versus number of iterations. Left: standard scheme. Right: modified scheme.*

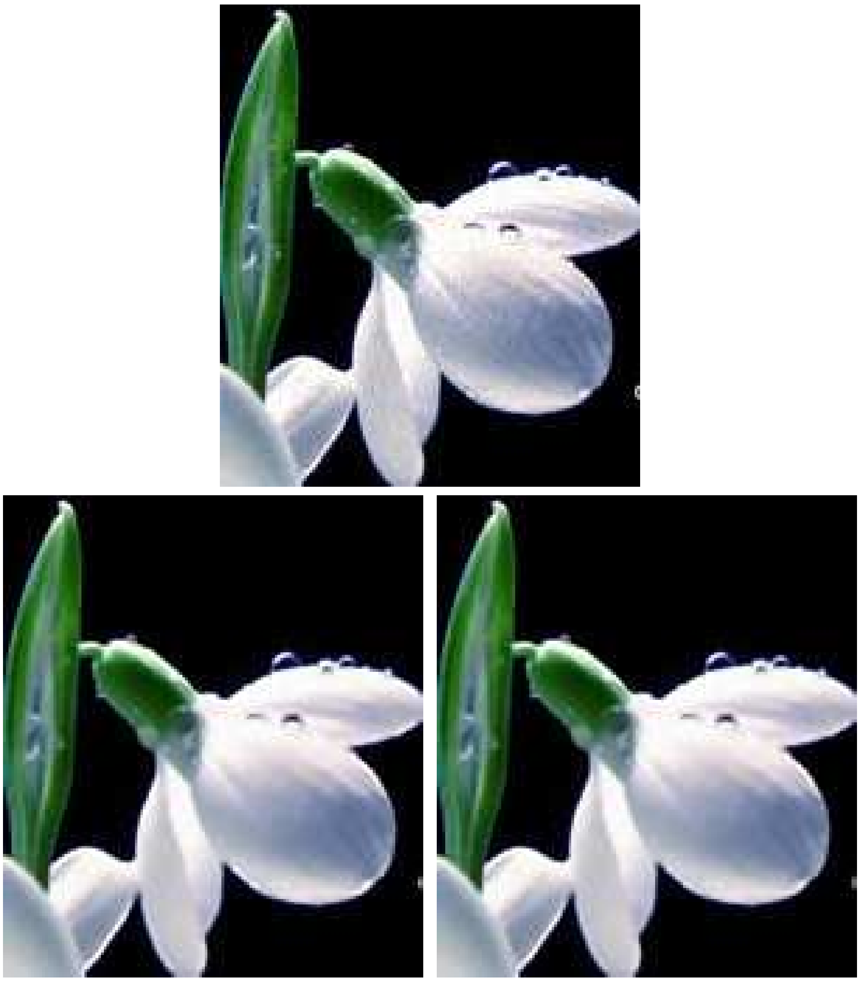

FIG. 6.5. *Top: Noisy flower image. Bottom-left: Result of the Beltrami flow with the standard scheme. Bottom-right: Result of the Beltrami flow with the modified scheme. Process parameters: 30 iterations,* $\beta = \sqrt{0.003}$, $\Delta t = 0.00001$, $h = 1/172$.

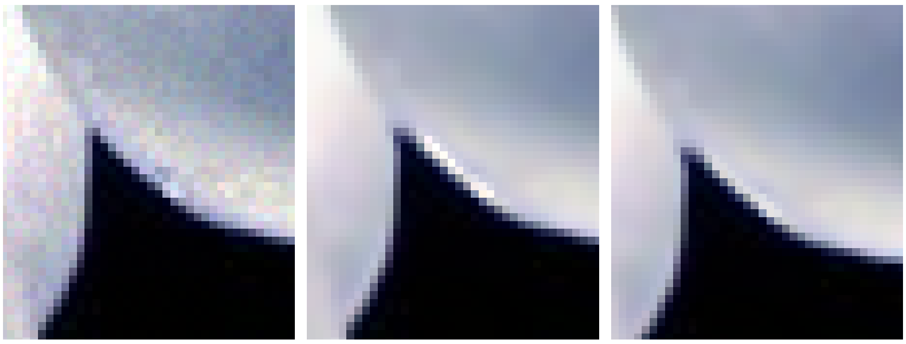

FIG. 6.7. *Zoom to the bottom part of the leaf left: Noisy image. Middle: standard scheme. Right: modified scheme. Notice the white artifact of the leaf, along the diagonal edge, present in the middle image obtained by the standard scheme and not in the right image, produced by the modified scheme.*

7. Concluding remarks. In this paper we analyze the discrete maximum principle for the Beltrami color flow. Even if often used in practice, the standard schemes in general, fail to satisfy the discrete maximum principle.

In this work we show that a nonnegative second order difference for this flow can be built for small values of the parameter β, i.e. linear-like diffusion. This limitation on the parameter β makes the nonnegative scheme unpractical. Moreover, we construct a novel second order finite difference scheme, which *is not nonnegative* and *does* satisfy the maximum principle. We illustrate its properties in numerical examples by applying this scheme to color noisy images.

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