Geometric quantum phase and angles

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The geometric phase in quantum mechanics is formulated for charged particles in a gauge-invariant, geometric manner. It is then extended to an evolution resulting from a sequence of measurements as in the work of Pancharatnam and Aharonov and Vardi. Its close connection to the Feynman formulation of quantum mechanics is pointed out. The geometric angles, which are generalizations of the classical, adiabatic angles introduced by Hannay and the quantum, adiabatic angles introduced by Anandan and Stodolsky in their group-theoretic treatment of Berry's phase, are studied in quantum and classical physics. The geometric phase for a quantum spin in a magnetic field due to a second particle is obtained using the quantum reference frame defined by the latter. The question of whether the geometric phase and angles are local or nonlocal and their relationship to the electromagnetic and gravitational phases are also discussed.

I. INTRODUCTION

The laws of physics have local and nonlocal consequences. Historically, physicists, in general, have readily recognized the local consequences, such as the acceleration of a particle subject to a given impressed force or the electromagnetic field at a point due to a given system of charges in motion. But they have been relatively slow in recognizing the nonlocal or global consequences, which, even after they are discovered, are often subject to controversy. By a nonlocal consequence, we mean an effect that cannot be predicted by a strictly local consideration. It may depend on an entire closed curve in space-time and be nonexistent or meaningless for any portion of it. An example of a nonlocal effect is the phase shift in the interference of a charged particle due to an electromagnetic field that vanishes along the interfering beams, known as the Aharonov-Bohm (AB) effect.[1]

An interesting consequence of quantum mechanics that has both local and nonlocal aspects, which remained undiscovered for almost six decades, is a geometric phase associated with any cyclic evolution. This was found by Berry[2] in adiabatic cyclic evolutions and subsequently generalized to all cyclic evolutions by Aharonov and Anandan.[3] A classical analog of the adiabatic (Berry) phase was found by Hannay[4] and Berry,[5] and is called the Hannay angle. Anandan and Stodolsky[6] have also introduced a set of angles in their geometric study of Berry's phase. While all these papers study the geometric phase when the states evolve according to an equation of motion, Pancharatnam[7] and Aharonov and Vardi[8] have considered the phase changes when measurements are made which have implicit in them the geometric phase. In this paper we shall study the relationships between all these concepts. We shall also further generalize the geometric phase, and study its physical implications.

In Sec. II we formulate the geometric phase, regarded as a consequence of Schrödinger's equation. We then treat the geometric phase, as a consequence of measurements in Sec. III and consider its relation to the path-integral formalism. In Sec. IV we study the geometric angles, which are a unification and nonadiabatic generalization of the classical adiabatic angles of Hannay and the quantum adiabatic angles by Anandan and Stodolsky,[6] due to one of us.[9] We argue in Sec. V, that while the geometric phase is nonlocal, the geometric angles are locally measurable, unlike the AB effect.

In Sec. VI we study the geometric phase for a spin precessing in a magnetic field due to another particle which is treated quantum mechanically. Following Aharonov and Kauferr,[10] we use the quantum frame of reference of the second particle. The geometric phase is acquired by the second particle from a fictitious gauge field that it interacts with in the adiabatic limit.

II. THE GEOMETRIC PHASE FOR CHARGED PARTICLES

Suppose $\mathcal{H}$ is the Hilbert space of a quantum system and $\mathcal{P}$ is the projective Hilbert space consisting of the rays of $\mathcal{H}$. The state $|\psi(t)\rangle$ of the system evolves according to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t)=[H_k+qA_0(\mathbf{x},t)]\psi(\mathbf{x},t)\ , \tag{2.1}$$

where $\psi(\mathbf{x},t)$ is the wave function of the state $|\psi(t)\rangle$, q is the charge, A_0 is the electrostatic potential, and the kinetic Hamiltonian H_k in the absence of any field other than the electromagnetic field is

$$H_k=-\frac{\hbar^2}{2m}\left[\nabla-i\frac{q}{\hbar}\mathbf{A}(\mathbf{x},t)\right]^2,$$

where $\mathbf{A}$ is the electromagnetic vector potential. Now define

$$\hat{\psi}(\mathbf{x},t)=\exp\left[i\frac{q}{\hbar}\int_{-\infty}^{t}A_0(\mathbf{x},t')dt'\right]\psi(\mathbf{x},t) \tag{2.2}$$

which is invariant under gauge transformations that tend to the identity as $t\to-\infty$, which represents an instant that is very far back in time. This gauge-invariant part of the wave function may be treated in the same way as the wave function of an uncharged system in Ref. 3. Suppose the system undergoes a cyclic evolution in the time interval $[0,\tau]$, by which we mean that the evolution of $\hat{\psi}(\mathbf{x},t)$ that is given by a curve in $\mathcal{H}$ projects to a closed curve C in $\mathcal{P}$, i.e., $\hat{\psi}(\mathbf{x},\tau)=e^{i\phi}\hat{\psi}(\mathbf{x},0)$, where ϕ is gauge invariant. Now define $\tilde{\psi}(\mathbf{x},t)=e^{-if(t)}\hat{\psi}(\mathbf{x},t)$, where the differentiable, gauge-invariant function $f(t)$ is chosen so that $\tilde{\psi}(\mathbf{x},\tau)=\tilde{\psi}(\mathbf{x},0)$. Then, on using (2.1) and (2.2),

$$\hat{\psi}(\mathbf{x},\tau)=e^{i\beta}\exp\left[-\hbar^{-1}\int_0^{\tau}\langle\hat{\psi}(t)\,|\,H_k(t)\,|\,\hat{\psi}(t)\rangle dt\right]\hat{\psi}(\mathbf{x},0)\,, \tag{2.3}$$

where[3]

$$\beta=i\int_0^t\langle\tilde{\psi}\,|\,\dot{\tilde{\psi}}\rangle dt=i\oint_C\langle\tilde{\psi}\,|\,d\,|\,\tilde{\psi}\rangle \tag{2.4}$$

with the overdot denoting differentiation with respect to time and d is the exterior differential on $\mathcal{P}$.

This treatment differs from the treatment of charged systems in Ref. 3 in that $\tilde{\psi}(\mathbf{x},t)$, which is the representative of C in $\mathcal{H}$, is gauge invariant, unlike the $\tilde{\psi}$ in Ref. 3. The present treatment has the advantage that β, given by (2.4), is electromagnetic gauge invariant and explicitly geometrical in the sense that it is the integral of the one-form $G=i\langle\tilde{\psi}\,|\,d\,|\,\tilde{\psi}\rangle$ around the closed curve C in $\mathcal{P}$. If we choose a different $|\tilde{\psi}'\rangle=e^{i\lambda}|\tilde{\psi}\rangle$, then $G\to G'=G-d\lambda$ and therefore β is invariant. Thus, β is geometrical in the sense that it is the same for the infinite class of Hamiltonians that generate the motions in $\mathcal{H}$ that project to the same closed curve C in $\mathcal{P}$. Also, the normalized states in $\mathcal{H}$ form a U(1) principal fiber bundle over $\mathcal{P}$ (Refs. 3, 11, and 12). Then $e^{i\beta}$ is the holonomy transformation along C associated with the universal connection in this Hopf bundle.

The essential difference between our approach and Berry's approach is that we regard β as a geometric phase associated with the motion of the state of the quantum system and not with the motion of the Hamiltonian as Berry did. Thus we treat β as being kinematical in origin unlike Berry for whom it is dynamical in origin. For example, in the case of the precession of a spin in a magnetic field,[3] it does not matter whether a particular motion is caused by a constant magnetic field or a large magnetic field in the direction of the spin which is changed adiabatically to cause the same motion of the spin. The geometric phase is the same in both cases. Our approach has the advantage that by disregarding which Hamiltonian caused the motion, we are able to treat nonadiabatic evolution on the same footing as adiabatic evolution.

III. GEOMETRIC PHASE AS A CONSEQUENCE OF MEASUREMENTS AND PATH INTEGRALS

The state vector $|\psi\rangle$, in the Schrödinger picture, undergoes two types of evolution. One is a continuous evolution according to the Schrödinger equation, considered in Sec. II. The other is a discontinuous change it undergoes when a measurement is performed, according to the usual Copenhagen interpretation. We shall be concerned here only with a special class of measurements called "filtering"; i.e., the apparatus interacts with $|\psi\rangle=|\phi\rangle\langle\phi\,|\,\psi\rangle+|\xi\rangle\langle\xi\,|\,\psi\rangle$, where $\langle\phi\,|\,\xi\rangle=0$, so as to select $|\phi\rangle\langle\phi\,|\,\psi\rangle$ without interacting with it, say, by absorbing $|\xi\rangle\langle\xi\,|\,\psi\rangle$. For example, the apparatus may be a polarizer that lets through light with a given polarization. Then the new state after such a measurement is

$$|\psi'\rangle=|\phi\rangle\langle\phi\,|\,\psi\rangle\,.$$

Here, $\langle\psi'\,|\,\psi'\rangle=|\langle\phi\,|\,\psi\rangle|^2$, which is in general less than 1, may be physically interpreted as the ratio of the intensity of the new beam to the old beam intensity. Since the geometric phase, as seen in Sec. II, arises entirely from the inner product, this suggests that the phases of inner products of the form $\langle\phi\,|\,\psi\rangle$ that accumulate during filtering measurements may give rise to a geometric phase.

First, following Pancharatnam,[7] define the phase difference χ between $|\psi_1\rangle$ and $|\psi_2\rangle$ which are not orthogonal by

$$\exp(i\chi)\equiv\langle\psi_1\,|\,\psi_2\rangle/|\langle\psi_1\,|\,\psi_2\rangle|\,. \tag{3.1}$$

We shall denote the projection of $|\psi\rangle$ in $\mathcal{P}$ by (ψ). We shall show that χ is the phase difference between $|\psi_2\rangle$ and $|\psi_p\rangle$ which is obtained by parallel transporting $|\psi_1\rangle$ along the shortest geodesic joining (ψ_1) to (ψ_2). It should be noted that because $|\psi_2\rangle$ and $|\psi_p\rangle$ are in the same ray, the phase difference between them is defined in the usual way, whereas an interesting aspect of the Pancharatnam phase χ is that it is defined for two states that need not be in the same ray.

To prove the theorem mentioned above, choose an orthonormal basis of an $(n+1)$-dimensional $\mathcal{H}$ in which $|\psi\rangle=(z^0,z^1,\ldots,z^n)=z^0(1,w^1,\ldots,w^n)$, where $w^i=z^i/z^0$, which are well defined everywhere except in the region for which $z^0=0$. These w^i are called inhomogeneous coordinates on $\mathcal{P}$. Then the Fubini-Study metric is

$$ds^2=\frac{(1+\bar{w}_k w^k)\delta_{ij}-\bar{w}_i w_j}{(1+\bar{w}_k w^k)^2}dw^i d\bar{w}^j\,, \tag{3.2}$$

where the overbar denotes the complex conjugate and the index is lowered using δ_{ij}. Note that (3.2) is invariant under unitary transformations of $(w^1,\ldots,w^n)$. Therefore, all "directions" in $\mathcal{P}$ are equivalent. Hence, there is no loss of generality in choosing the inhomogeneous coordinates of (ψ_1) and (ψ_2) to be $(0,0,\ldots,0)$ and

$(T,0,\ldots,0)$, where T is a positive number. By symmetry, the curve, real $t\rightarrow(t,0,\ldots,0)$, is a geodesic passing through these two points. But there are two geodesics connecting these two points. For the shorter geodesic, $0\leq t\leq T$. For the longer geodesic, t goes from 0 to negative, tends to $-\infty$, and becomes undefined at a point corresponding to $z_0=0$, and then is large positive, decreasing until $t=T$.

Also, let $|\psi_1\rangle=c(1,0,\ldots,0)$, where c is a constant, and let $|\psi(t)\rangle=c(1+t^2)^{-1/2}(1,t,0,\ldots,0)$ be a vector field defined on the geodesic. Then $\langle\psi(t)|d/dt|\psi(t)\rangle$ is pure imaginary because $\langle\psi(t)|\psi(t)\rangle$ is a constant, but it is also real because t is real. Therefore, $\langle\psi(t)|d/dt|\psi(t)\rangle=0$, which implies that $|\psi(t)\rangle$ is being parallel transported. Since, $|\psi(0)\rangle=|\psi_1\rangle$ it follows that $|\psi(T)\rangle=|\psi_p\rangle$, which is obtained by parallel transporting $|\psi_1\rangle$ along the shorter geodesic C_{12} from (ψ_1) to (ψ_2). Hence, given any $|\psi_2\rangle=r\exp(i\chi)|\psi_p\rangle$ in the same ray as $|\psi_p\rangle$, where r is positive, then χ satisfies (3.1), which proves the theorem. There is no loss of generality in choosing c to be 1. Then on defining the normalized $|\tilde{\psi}(t)\rangle$, which projects to C_{12} so that $|\tilde{\psi}(0)\rangle=|\psi_1\rangle$ and $|\tilde{\psi}(T)\rangle=|\psi_2\rangle$, it is easily seen that

$$\chi=i\int_{C_{12}}\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle dt\ . \tag{3.3}$$

Samuel and Bhandari[7] have proved previously a theorem analogous to the above one in a space $\mathcal{R}$ which is obtained from $\mathcal{H}$ by removing just the phase information. But we have also shown that χ is determined by parallel transport along the *shorter* geodesic. Also, we prefer to work in $\mathcal{P}$ because the additional information contained in $\mathcal{R}$, namely, the magnitude of a typical vectors $|\psi\rangle$, is irrelevant to the phase acquired: Even though during the measurement, in general, the state undergoes a nonunitary transformation, the magnitudes of the initial or final vectors do not affect χ, defined by (3.1), or the theorem proved above. Even for nonunitary evolutions due to a dynamical equation, it has been shown[13] that the geometric phase is the same as due to parallel transport on $\mathcal{P}$ using the above connection. Also, a cyclic evolution as we have defined it is a closed curve in $\mathcal{P}$ but not necessarily a closed curve in $\mathcal{R}$; therefore, it is not possible to obtain $\exp(i\beta)$ as a holonomy of a connection on $\mathcal{R}$, in general. We prefer $\mathcal{P}$ also because it contains a complex structure and a symplectic structure that $\mathcal{R}$ does not contain. From a physical point of view, the change in magnitude of $|\psi\rangle$ has no significance if we are describing a single quantum system and then $\mathcal{R}$ has no more relevant information than what is contained in $\mathcal{P}$.

Consider now a cyclic evolution, around a closed curve C in $\mathcal{P}$, that occurs due to a sequence of filtering measurements and dynamical evolution in between measurements. Here C is made of segments that are determined by Schrödinger's equation between measurements and shortest geodesics joining the projections of initial and final states in $\mathcal{P}$ at each measurement. Then the total phase ϕ acquired can be determined by defining a section $|\tilde{\psi}\rangle$ that is differentiable on C and keeping track of the phase changes of $|\psi(t)\rangle$ during a measurement using the above theorem and during dynamical evolution using the treatment in Sec. II. Hence $\phi=\beta+\delta$ where δ is the dynamical phase in (2.3) and $\beta=i\oint_C\langle\tilde{\psi}|d|\tilde{\psi}\rangle$, i.e., $\exp(i\beta)$ is the phase factor acquired (holonomy transformation) in parallel transporting a state vector around C with respect to the same canonical connection on the natural line bundle over $\mathcal{P}$ that we used to obtain the geometric phase when the evolution was continuous under the action of Schrödinger's equation.

In general, as mentioned above, the magnitude of $|\psi\rangle$ is reduced because of the measurements. However, as seen below, if a dense sequence of measurements is made, then the magnitude of $|\psi\rangle$ is not reduced. This means, for example, if light passes through a large number of polarizers such that the directions of polarizations of any two successive polarizers differ by an infinitesimal angle then the intensity of the beam is practically undiminished, i.e., the probability of a photon moving through any trajectory in Hilbert space can be made indefinitely close to one by a dense sequence of measurements.

If no measurement is performed in the time interval $(t,t+dt)$, then

$$|\psi_\delta(t+dt)\rangle=(1-i\hbar^{-1}H\,dt)|\psi(t)\rangle\ , \tag{3.4}$$

where the subscript δ indicates a state obtained by purely dynamical evolution. Now write $|\psi(t)\rangle=|\tilde{\psi}(t)\rangle e^{if(t)}$, where $\langle\tilde{\psi}(t)|\tilde{\psi}(t)\rangle=1$ for all t and $f(t)$ is *complex* with $f(0)=0$. Suppose a measurement is performed at time $t+dt$ leading to a projection by the operator $P_{t+dt}=|\tilde{\psi}(t+dt)\rangle\ \langle\tilde{\psi}(t+dt)|$, where $|\tilde{\psi}(t+dt)\rangle$ differs infinitesimally from $|\tilde{\psi}(t)\rangle$. Then the new state resulting from the measurement is

$$\begin{aligned}|\psi(t+dt)\rangle&=|\tilde{\psi}(t+dt)\rangle\langle\tilde{\psi}(t+dt)|\psi_\delta(t+dt)\rangle\\&=|\tilde{\psi}(t+dt)\rangle[1-i\hbar^{-1}\langle\tilde{\psi}(t)|H|\tilde{\psi}(t)\rangle dt+\langle\dot{\tilde{\psi}}(t)|\tilde{\psi}(t)\rangle dt]e^{if(t)}\end{aligned}$$

on using (3.4) and performing a Taylor expansion of the matrix element to first order in dt. Therefore

$$e^{if(t+dt)-if(t)}=[1-i\hbar^{-1}\langle\tilde{\psi}(t)|H|\tilde{\psi}(t)\rangle dt+\langle\dot{\tilde{\psi}}(t)|\tilde{\psi}(t)\rangle dt]\ . \tag{3.5}$$

If a sequence of dense measurements are performed in the interval $[0,\tau]$ such that the system undergoes a cyclic evolution then the $|\tilde{\psi}(t)\rangle s$ in the projection operators can be chosen so that $|\tilde{\psi}(\tau)\rangle=|\tilde{\psi}(0)\rangle$. Hence the final state resulting from this dense sequence of measurements is

$$|\psi(\tau)\rangle=|\psi(0)\rangle\exp\left[-i\hbar^{-1}\int_0^\tau\langle\tilde{\psi}(t)|H|\tilde{\psi}(t)\rangle dt+\int_0^\tau\langle\dot{\tilde{\psi}}(t)|\tilde{\psi}(t)\rangle dt\right]\ , \tag{3.6}$$

where the phase factor is obtained by multiplying together the infinitesimal phase factors acquired for each successive measurement given by (3.5). Hence, the phase acquired by the state is the sum of a dynamical phase and a geometrical phase, like in Sec. II where it underwent purely Schrödinger evolution.

Also, because of the normalization of $|\tilde{\psi}(t)\rangle$, $\langle\dot{\tilde{\psi}}(t)|\tilde{\psi}(t)\rangle=-\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle$ is pure imaginary. Therefore, according to (3.6), the magnitude of $|\psi(t)\rangle$ does not change because H is Hermitian, i.e., $\dot{f}(t)$ is real. We have therefore shown that the quantum state can be made to follow any trajectory in $\mathcal{P}$ with probability approaching one, by making a dense set of filtering measurements. Hence, we can associate a geometric phase with every cycle and not just the one satisfying the equation of motion. Even if H is not Hermitian, the change in the magnitude of $|\psi\rangle$ is entirely in the dynamical part of (3.6) so that the geometric phase is still the same as in the unitary evolutions considered previously.

Also, if the sequence of measurements is not dense then the change in magnitude of $|\psi\rangle$ for a given cycle C in $\mathcal{P}$ depends on the particular sequence of measurements performed, but the geometric phase factor $\exp(i\beta)$ depends only on C. This is similar to the fact that the change in magnitude due to nonunitary evolution even under a nonlinear equation is entirely in the dynamical part.[13,14] Hence the geometric phase depends only on C and is independent of how the evolution around C takes place, whether it is nonunitary, nonlinear, or what particular sequence of measurements are performed. This shows an advantage of the present treatment which uses the projective Hilbert space instead of the parameter space of the earlier treatment[2] which does not permit such a general geometric phase.

The geometric and dynamical parts can be clearly separated in the Heisenberg picture. In this representation, a state does not change except when a measurement is made. Therefore, the only way to have a nontrivial cyclic evolution is through a sequence of measurements. Then, the above analysis implies that the final Heisenberg state $|\psi(\tau)\rangle_H=\exp(i\beta)|\psi(0)\rangle_H$, where $|\psi(0)\rangle^H$ is the initial Heisenberg state and β is the geometric phase associated with the geodesic polygon in $\mathcal{P}$ determined by the measurements.

Returning to the Schrödinger picture, Aharonov and Vardi[8] considered the special case of $|\psi(t)\rangle$, above, being a Gaussian wave packet corresponding to the system following an approximate classical trajectory and showed that by subjecting a quantum system to a dense set of measurements, it can be made to follow approximately any path γ, and further, it then accumulates a phase $S_\gamma/\hbar$, where

$$S\gamma=\int_\gamma (p\,dq-H\,dt) \tag{3.7}$$

is the classical action along this path. The first term in the integration is the geometric phase[15] while the second term is the dynamical phase in this case. The first term is analogous to the $\int p\,dq$ term used by Berry[5,16] to obtain Hannay's angles,[4] as will be mentioned in Sec. V. We emphasize that in (3.7), γ need not be the trajectory satisfying the classical equation of motion. We can define a "geometric phase" for each Feynman path as the first term in (3.7). But it is only for a cyclic evolution, for which γ is a closed curve in phase space, that this term is invariant under canonical transformations.

Consider now the second quantization of the Schrödinger field for which the action used in the Feynman path integral is

$$S=\frac{i}{2}\int dt(\langle\psi|\dot{\psi}\rangle-\langle\dot{\psi}|\psi\rangle-\hbar^{-1}\langle\psi|H|\psi\rangle)\,. \tag{3.8}$$

The first two terms in the integration give the geometric phase and the last term the dynamical phase for a field that obeys Schrödinger's equation. But the above arguments suggest that we may be able to define a geometric phase more generally for any Feynman history by

$$\beta=\frac{i}{2}\int dt(\langle\psi|\dot{\psi}\rangle-\langle\dot{\psi}|\psi\rangle) \tag{3.9}$$

which may be given physical meaning by means of a dense sequence of measurements.

The above arguments also suggest that there may be a hierarchy of "geometric phases" corresponding to every level of quantization beginning with the "zeroth quantized" or classical particle theory, as long as the theory is defined by an action that has a dynamical part which can be subtracted away so that the remaining part is geometrical. So if we have a "third quantized" theory corresponding to allowing several copies of the same field (e.g., the field is a string), it appears that there should be a geometric phase in such a theory.

IV. QUANTUM AND CLASSICAL ANGLES

In our approach so far, we regarded the geometric phase as being due to the geometry of $\mathcal{P}$, which is independent of the Hamiltonian H, unlike Berry who regarded it, in the adiabatic limit, as a geometric property of the space of parameters of which H is a function. Indeed, the parameter space does not naturally have the Simon connection[12] that determines the Berry phase, unlike $\mathcal{P}$ which has a natural geometric connection whose holonomy gives the geometric phase whether or not the evolution is adiabatic.

However, $\mathcal{P}$ can be a large space that may be infinite dimensional whereas the parameter space is usually finite dimensional and may therefore appear to be more convenient in such cases. But the parameter space is not useful in the nonadiabatic case. It is therefore desirable to develop an approach that is intermediate between Berry's approach[2] and our previous approach[3] that would have the convenience of the finite-dimensional parameter space and yet provide a proper geometric description of cyclic evolutions that is unrestricted by the adiabaticity assumption. In this section we hope to demonstrate that such an intermediate approach is indeed possible.

Consider a class $\mathcal{C}$ of Hamiltonians. If there is a (1-1) map from a parameter space $\mathcal{S}$ into $\mathcal{C}$ as in the examples considered by Berry, then $\mathcal{C}$ may be taken to be the same as $\mathcal{S}$. For any time interval (t_1,t_2), $H(t)$ is a curve in $\mathcal{C}$ that determines the corresponding time-evolution operator

$$V(t_2,t_1)=P\exp\left[-i\hbar^{-1}\int_{t_1}^{t_2}H(t)dt\right].$$

$V(t_2,t_1)$, being unitary, has a complete orthonormal set of eigenvectors, each of which undergoes a cyclic evolution in the interval (t_1,t_2). Suppose that the set of all such operators generate a group $G(\mathcal{C})$ that is independent of the representation of $\mathcal{C}$. For example, for a spin with magnetic moment μ in a homogeneous magnetic field $\mathbf{B}$, $H=\mu\mathbf{J}\cdot\mathbf{B}$ where J_i, $i=1,2,3$, generate SU(2), which is our G, independently of the representation or the spin. Here the parameters are B_i so that $\mathcal{S}$ is R^3. But the parameter space is not needed in our approach.

Consider any Hamiltonian evolution $H(t)$ in an arbitrary time interval $[0,\tau]$. Let $\{|n(0)\rangle\}$ be the complete set of normalized eigenstates of $V(\tau,0)$. Then $|\psi_n(t)\rangle=V(t,0)|n(0)\rangle$ undergoes cyclic evolution in the interval $[0,\tau]$. Now choose an orthonormal basis

$$\begin{aligned}|n(t)\rangle&=U(t)|n(0)\rangle\\&=\exp[ig_n(t)]|\psi_n(t)\rangle\end{aligned}\tag{4.1}$$

such that $g(0)=0$ and

$$\begin{aligned}\langle n(t)|\dot{n}(t)\rangle&=\langle n(0)|U^{\dagger}(t)\dot{U}(t)|n(0)\rangle\\&=0\quad\text{for every } n\ .\end{aligned}\tag{4.2}$$

It can be shown that[6] there exists a unique unitary $U(t)$ that satisfies (4.1), (4.2), and $U(0)=I$.

On substituting $|\psi_n(t)\rangle$ into Schrödinger's equation, we obtain, on using (4.1) and (4.2),

$$|\psi_n(t)\rangle=\exp\left[-i\hbar^{-1}\int_0^t\langle\psi_n|H(t')|\psi_n\rangle dt'\right]|n(t)\rangle\ .\tag{4.3}$$

Hence, the geometric phase β_n arises entirely from the evolution of $|n(t)\rangle$:

$$\begin{aligned}|n(\tau)\rangle&=U(\tau)|n(0)\rangle\\&=\exp(i\beta_n)|n(0)\rangle\ .\end{aligned}\tag{4.4}$$

Suppose now that $\{|n(0)\rangle\}$ are simultaneous eigenstates of a maximal set of commuting generators $\{J_i(0): i=1,\ldots,m\}$ of G. Then $\{|n(t)\rangle\}$ and $\{|\psi_n(t)\rangle\}$ are simultaneous eigenstates of

$$J_i(t)=U(t)J_i(0)U^{\dagger}(t),\quad i=1,2,\ldots,m\ ,\tag{4.5}$$

which generate a Cartan subalgebra of the Lie algebra of G, i.e.,

$$J_i(t)|n(t)\rangle=j_i(n)|n(t)\rangle\ ,i=1,2,\ldots,m\ ,\tag{4.6}$$

where $j_j(n)$ are independent of time. Physically, this corresponds to the states $|n(t)\rangle$ being invariant under an Abelian group K of symmetries generated by the J_is. In the example of a spin precessing in a magnetic field, considered above, K is the U(1) group of rotations about the instantaneous spin states $\{|n(t)\rangle\}$ that undergo cyclic evolution.

Since $\{|n(t)\rangle\}$ undergo cyclic evolution, it follows that

$$U(\tau)=\exp\left[i\sum_{k=1}^{m}\alpha_kJ_k(0)\right].\tag{4.7}$$

These angles α_i are the generalizations to nonadiabatic motion of the angles introduced by Anandan and Stodolsky.[6] It follows from (4.4) and (4.7) that

$$\sum_{k=1}^{m}\alpha_kj_k(n)=\beta_n\ .\tag{4.8}$$

As a particular application, consider the motion of a quantum spin in a magnetic field $B(t)$ generated by the Hamiltonian $H=\mu B\cdot J$, where $J_i, i=1,2,3$, generate the SU(2) rotation group. The operator $U(t)$ would then belong to the representation of SU(2) corresponding to the spin of interest. It therefore represents a rotation. For SU(2), $m=1$ in the above treatment and we choose the element of the Cartan subalgebra for which (4.6) is satisfied to be the generator $J_z(t)$ of the instantaneous quantization axis for the spin states $|n(t)\rangle$, $n=1,2,\ldots,2j+1$.

Now, $U(t)$ can be given an interpretation analogous to the one given previously[6] in the adiabatic limit in which the magnetic field always points along the quantization axis, unlike in the present more general case. According to (4.5) and (4.6), $U(t)$ transforms the instantaneous quantization axis at time 0 to the one at time t. Also (4.2) implies that there is no rotation about the instantaneous quantization axis of $\{|n(t)\rangle\}$ at every time t, i.e., $U(t)$ is a product of infinitesimal rotations of the quantization axis, each of which is about an axis perpendicular to the instantaneous quantization axis. Hence if S is a sphere representing the set of all possible directions of the quantization axis, $U(t)$ may be regarded as parallel transporting a treibein on S with its instantaneous z axis always in the radial direction. During a cyclic evolution, this treibein moves along a closed curve on S and has rotated by an angle α about its z axis, corresponding to $U(\tau)=\exp[i\alpha J_z(0)]$ in (4.7). Since its X,Y axes are always being parallel transported,

$$\begin{aligned}\alpha&=\int_{\Sigma}(\text{Gaussian curvature})\\&=\text{solid angle subtended by }\Sigma\text{ at the center of S}\ ,\end{aligned}$$

where Σ is the portion of S that is bounded by the closed curve.

The angles α_i are independent of h. This suggests that they may have a classical analog. To investigate this, we first consider what the classical analog is for a cyclic evolution in $\mathcal{P}$. Each element in $\mathcal{P}$ may be specified as the eigenspace of a complete set of commuting observables. Therefore, the classical analog of this element may be defined by specifying the values of a complete set of observables in involution. Suppose that our $\{J_i\}$ above form a complete set of commuting observables and classically correspond to a set of observables $I_j(q,p,t)$ in involution, where q,p represent a set of generalized coordinates and momenta, i.e.,

$$\{I_i,I_j\}=0,\quad i,j=1,2,\ldots,m\tag{4.9}$$

where $\{\ ,\ \}$ represents the Poisson brackets. Every set of

values of the observables $\{I_i\}$ determine a surface in phase space, which would then be the analog of the element of $\mathcal{P}$ in quantum mechanics.

We restrict to the case when m is the number of independent q_i. Then this surface is called a Lagrangian submanifold. If it is compact then it is a torus. It may be coordinated by the conjugate set of observables $\{\theta_i(q,p,t)\}$ satisfying

$$\{\theta_i,\theta_j\}=0, \quad \{I_i,\theta_j\}=\delta_{ij} .$$

The coordinates (θ,I) are called the action angle variables. A cyclic evolution in the interval $[0,\tau]$ is defined by the conditions

$$I_j(q,p,0)=I_j(p,q,\tau)$$

and

$$\theta_i(q,p,0)=\theta_i(q,p,\tau) .$$

The Hamiltonian $H(t)$ that generates this evolution need *not* be a function of the observables $\{I_i(t)\}$.

Clearly the values of $\{I_i\}$ are the classical analogs of the eigenvalues j_i. Since j_i are constants, we must have

$$\frac{dI_i}{dt}=\frac{\partial I_i}{\partial t}+\{H,I_i\}=0 . \tag{4.10}$$

Then the corresponding classical system is said to be integrable. Semiclassically, a suitably chosen $\{I_i\}$ satisfies the Bohr-Sommerfeld quantization condition

$$I_i=\hbar j_i=\hbar(n_i+\sigma_i) , \tag{4.11}$$

where n_i integers and σ_i are constants. Thus the Lagrangian submanifolds are discrete in the semiclassical picture. But in the classical limit of large n_i, I_i and therefore the Lagrangian submanifolds vary continuously, as it is in classical physics. In this limit, (4.8) implies

$$\alpha_i=\hbar\frac{\partial}{\partial I_i}\beta . \tag{4.12}$$

It can be shown using the purely classical theory[5,16,9] that

$$\alpha_i=\frac{\partial}{\partial I_i}\int\left\langle\sum_k p_k\dot{q}_k\right\rangle dt$$
$$=\frac{\partial}{\partial I_i}\left\langle\oint\sum_k p_k dq_k\right\rangle , \tag{4.13}$$

where angular brackets denote averaging over the torus.

A special case is a system for which the Hamiltonian varies adiabatically, then the orbit lies on the Lagrangian submanifold to a good approximation. This is the special case that was considered by Hannay[4] and Berry[5] in his derivation of (4.12). But we notice that (4.8) is actually a stronger condition than (4.12) and it does not assume the adiabatic limit. Also, the geometric angles α_i already exist in the quantum theory, even before the classical limit is taken.

We now illustrate the above general ideas by considering the precession of a symmetrical top, which is the classical analog of the precession of a quantum spin considered above. The action variable I corresponding to the quantum observable J_z, above, is the angular momentum about the axis of the top and it is assumed to satisfy (4.10). The conjugate variable θ is the angle of rotation about the same axis. We choose the sphere S now to represent the set of possible directions of the axis of the top. A cyclic evolution has occurred when the axis of the top has traced out a closed curve on this sphere. As in the quantum case we can consider a treibein, with its instantaneous z axis always radial, that is parallel transported along this curve. The component ω of the angular velocity of the top along this z axis is the same as the one measured relative to the treibein. But the treibein itself rotates by the angle α, described above. Hence the total rotation of the top about its axis during the cyclic evolution is

$$\Delta\theta=\int_0^\tau\omega\,dt+\alpha . \tag{4.14}$$

We emphasize that the adiabatic approximation is not needed here, and the above example is a classical analog of the nonadiabatic precession of a quantum spin in a magnetic field considered in our earlier work.[3] Also, the geometric angle α is the same for the quantum and classical cases and should not be regarded as existing only in the classical limit.

V. ARE THE GEOMETRIC PHASE AND ANGLES NONLOCAL?

As mentioned in Sec. II only for a cyclic evolution the change in the geometric phase β can be defined invariantly. This makes it appear that β is global in the sense that it can be unambiguously defined only for a cyclic evolution. In this respect β is analogous to the AB phase.[1] But the geometric angle α differs from the AB phase in the following important respect. The AB phase, in a nonsimply connected region, is truly nonlocal and global in the sense that it cannot, in general, be expressed in terms of locally observable quantities, whereas α, at least in special cases, can be expressed as a sum of locally observable quantities.

To see this, note first that while the geometric phase is associated with a cyclic evolution of a single state, the geometric angles α_i are associated with the cyclic evolution of a complete set of orthogonal states $\{|\psi_n(t)\rangle\}$. We can observe α_i by observing the motion of superpositions of these states. This is easily illustrated by the precession of a spin-$\frac{1}{2}$ particle for which there is just one α. Suppose $|\psi(t)\rangle=a|\psi_1(t)\rangle+b|\psi_2(t)\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthonormal states that undergo cyclic evolution. There exists a continuous vector field $\mathbf{n}(t)$ such that $|\psi_i(t)\rangle$, $i=1,2$, are eigenstates of $\mathbf{n}(t)\cdot J$. We shall call the direction of $\mathbf{n}(t)$ the direction of the spin vector. It follows from (4.3) that this direction precesses about the instantaneous axis of quantization with frequency

$$\omega(t)=\hbar^{-1}|\langle\psi_1(t)|H(t)|\psi_1(t)\rangle$$
$$-\langle\psi_2(t)|H(t)|\psi_2(t)\rangle| .$$

But the total angle of precession during a cyclic evolution is given by (4.14). On the other hand if we choose a plane perpendicular to the initial direction of the quantization

axis and $\bar{\omega}(t)$ is the frequency of precession of the projection of the spin vector on this plane then $\Delta\theta=\int_0^\tau \bar{\omega}(t)dt$. Hence, $\bar{\omega}(t)$ has a dynamical part and a geometrical part $d\alpha/dt$ that give, respectively, the first and the second terms in (4.14) on integration.

Hence α is, in this special case, the integral of a locally measurable quantity and therefore, in this sense, represents a local effect. And α, understandably, survives the classical limit. But β, on the other hand, depends on the choice of $|\tilde{\psi}\rangle$, except for a cyclic evolution. Therefore, β must be regarded as a nonlocal phase. Since $\beta(n)$ can be determined from α_j by (4.8), it follows that the actions $I_j(n)$, which obey the Bohr-Sommerfeld quantization conditions, must also be regarded as being nonlocal.

It follows also from the above remarks that the experiments of Tycko[17] and Suter *et al.*[18] which measure the extra geometric precession of a superposition of spin states that undergo cyclic evolution using nuclear magnetic resonance, are really direct measurements of the geometric angle α and are only indirectly measurements of the geometric phases β_i. However, a recent experiment of Suter, Mueller, and Pines[19] does measure the geometric phase.

It appears that the basic reasons why α can be expressed as a sum of locally observable quantities in the above example are that (1) it is possible to interfere different spin states and (2) the precession of the superposed spin state can be compared with a laboratory frame that does not undergo the same precession because its motion is governed by a Hamiltonian that does not have the same type of interaction that causes the precession of the spin. This is unlike the AB phase which is observable only around a closed curve in space-time and not along any portion of it because it is not possible to interfere different charges, which is called the charge superselection rule. In this respect the geometric angles, in this case, are like the phase in a gravitational field,[20,21] with the mass playing the role analogous to the electric charge. Since two masses can be interfered as in the kaon system, it is possible to express the gravitational phase in terms of locally measurable quantities,[22] unlike the electromagnetic phase.

On the other hand, the effect of the coupling of spin to the curvature of space-time in the interference of two coherent beams in a gravitational field, that was found by one of us,[21,23] is nonlocal, like the AB effect because the curvature affects all physical systems in the same way, as implied by the principle of equivalence. So it is not possible to compare the Lorentz transformation undergone by the spin state along any subportion of the beams with some standard system, unlike in the case of the geometrical angle as mentioned in (2) above.

VI. QUANTUM FRAMES OF REFERENCE

So far, the geometric phase was treated in situations in which a quantum system was placed in an environment that was treated classically, e.g., a quantum spin in a classical magnetic field. We wish to investigate the geometric phase when the environment is also quantum mechanical.[10,24]

To this end consider a quantum spin that is in a magnetic field whose direction is determined by the position of a second quantum-mechanical particle. The Hamiltonian is

$$H=\frac{B}{r}\mathbf{L}_1\cdot\mathbf{r}+\frac{1}{2m}\mathbf{p}^2 , \tag{6.1}$$

where $\mathbf{L}_1$ is the spin or angular momentum of the first particle, whereas $\mathbf{r}$ and $\mathbf{p}$ are the position and momentum of the second particle, B and m are constants representing the magnetic field and the mass.

Now, (6.1) is written in a basis determined by the laboratory frame that is described classically. It is as if the spin is "immersed" in an external classical space that contains also the laboratory frame, which, because of its large mass, can be described classically to a good approximation. On the other hand, we can insist on strict "relativism" by giving up the laboratory axes, which are not part of the dynamical system being considered and refer the orientation of the spin to the second quantum particle. Following Aharonov and Kauferr,[8] in this description which will be further explained below, we shall call the second particle the quantum reference frame for the first particle.

Suppose, $\mathbf{r}=r(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$ and $\mathbf{L}_1=(L_{1x},L_{1y},L_{1z})$ in the laboratory frame. We wish to "diagonalize" the first term in H in (6.1) by transforming the z axis to the direction of $\mathbf{r}$. This can be achieved by the unitary operator

$$U(\theta,\phi)=\exp(iL_{1y}\theta)\exp(iL_{1z}\phi) . \tag{6.2}$$

The transformed Hamiltonian is

$$H'=UHU^\dagger=BL_{1z}+\frac{1}{2m}(\mathbf{p}-\mathbf{A})^2 , \tag{6.3}$$

where

$$\mathbf{A}=\sin\theta\,\nabla\phi L_{1x}+\nabla\theta L_{1y}+\cos\theta\,\nabla\phi L_{1z} . \tag{6.4}$$

Hence by giving up the laboratory axes and adopting the quantum frame of the second particle we have gained a "gauge potential" $\mathbf{A}$. This can be understood as follows. Before the unitary transformation was performed the laboratory frame determined a basis $\{|n\rangle\}$ for the Hilbert space V of particle 1 consisting of the eigenstates of the component of $\mathbf{L}$ along the laboratory $\mathbf{z}$ axis. For each position $\mathbf{r}$ of particle 2, V may be regarded as an "internal space." The basis $\{|n\rangle\}$ determined a "distant parallelism" between the internal spaces at any two values of $\mathbf{r}$, since two vectors in the two spaces can be regarded as parallel if their components with respect to the common basis determined by the laboratory frame are proportional.

By giving up the laboratory frame we can no longer identify vectors in vector spaces corresponding to different values of $\mathbf{r}$. We are free to choose the basis for the V that is attached each $\mathbf{r}$ in the spirit of "local gauge invariance" of a gauge theory. The transformation (6.2) chooses a particular such basis consisting of eigenstates of $\mathbf{L}\cdot\mathbf{r}$ which are determined by the direction of particle 2. The "gauge potential" $\mathbf{A}$ then enables us to parallel

transport the "internal space" vectors in this basis along curves in the **r** space so that distant parallelism can be established. This is possible because **A** still represents the flat connection of the laboratory space.

We now consider the adiabatic limit corresponding to $B \to \infty$. The transition from the ground state $|\psi_g\rangle$ to the excited states would then be negligible. It is then easy to show that the effective Hamiltonian for the second particle is $\langle \psi_g | H | \psi_g \rangle$. Then the first two terms in (6.4) "average out" to zero so that

$$\mathbf{A}_{\text{effective}} \approx \cos\theta \nabla\phi L_{1z} . \tag{6.5}$$

This is the vector potential of a "magnetic monopole" at the origin which gives the Berry phase analogous to how the vector potential due to the usual magnetic field gives the Aharonov-Bohm phase.

While Berry[2] has observed that the geometric phase in this problem arises from a "magnetic monopole" at the origin of the parameter space, our approach differs from his approach in that (a) we treat the parameter space quantum mechanically as the configuration space of a second quantum particle and (b) the Berry phase β is acquired by the second particle in our treatment, although in the product wave function, $e^{i\beta}$ can instead be attributed to the first particle. Our approach differs also from the earlier treatment of coupled systems,[24] which introduce also a "gauge potential" in that these authors use the laboratory frame instead of the quantum frame of reference as we do. The present point of view will be explored in greater detail in a future paper.[25]

Note added in proof. We have received a paper by M. G. Benedict and L. Gy. Fehér which contains some of the results of Sec. III of this paper.

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