How macroscopic properties dictate microscopic probabilities

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(Received 22 January 2002; published 7 May 2002)

We argue that the quantum probability law follows, in the large N limit, from the compatibility of quantum mechanics with classical-like properties of macroscopic objects. For a finite sample, we find that likely and unlikely measurement outcomes are associated with distinct interference effects in a sample weakly coupled to an environment.

DOI: 10.1103/PhysRevA.65.052116 PACS number(s): 03.65.Ta

Given that quantum theory is probabilistic, is there a fundamental physical principle that dictates the form of the quantum probability law? We will argue that, indeed, the assigned probabilities for a given test performed on a large sample of identical systems are dictated by the classical-like properties of macroscopic systems.

Consider a sample of N identically prepared nonentangled spins in the state $|\psi\rangle$. The state of the sample is then

$$|\Psi\rangle=|\psi\rangle_1|\psi\rangle_2\cdots|\psi\rangle_N. \tag{1}$$

Since spins carry a magnetic moment, the sample may also be viewed as a collection of magnetic moments, all pointing in the same direction, which under a suitable arrangement describe a magnetlike object. In the limit of large enough N, the sample becomes a macroscopic object, with definite collective properties like a total magnetic moment and associated magnetic field.

Suppose now that we want to measure such a collective property of our sample, say, the magnetic moment $M_x=\hat{x}\cdot\vec{M}$, in the $\hat{x}$ direction. Since the magnetic moment of each spin is proportional to the spin itself, we need to measure the $\hat{x}$ component of an observable like

$$\vec{M}=\frac{\sum_{i=1}^{N}\vec{\sigma}_i}{N}. \tag{2}$$

In the large N limit

$$\lim_{N\to\infty}[M_x,M_y]=i\lim_{N\to\infty}\frac{M_z}{N}=0. \tag{3}$$

This suggests that averaged collective observables, like $\vec{M}$, represent "macroscopic," classical-like, properties of the sample.

Our main idea is to compare two distinct methods, the "macroscopic" and "microscopic" methods, for observing the macroscopic collective magnetic moment. First, we consider a *collective* measurement of M_x, which does not probe the state of individual spins. For instance, a single charged test particle can be scattered to determine the total magnetic field of the sample. As we show below, and as Eq. (3) above indicates, in the large N limit the state is an eigenstate of M_x and the outcome given by the expectation value is deterministic. The macroscopic measurement induces a vanishing small disturbance in each individual spin. Yet the total accumulated effect on the scatterer is finite. In the second microscopic method, we measure separately on each spin the operators σ_{xi}, $i=1,\ldots,N$, and evaluate from the N outcomes the average of M_x. The microscopic measurements do disturb the spins and randomize the state of the sample according to the quantum probability law. However, since the first macroscopic method nearly does not affect the sample its outcome should agree up to $1/\sqrt{N}$ corrections with the mean result of a subsequent microscopic measurements. We shall see that in the limit of $N\to\infty$ this suffices to fix the form of the quantum probability distribution. On the other hand, for a large but finite sample, the probability law still follows if we make a further assumption on the stability of physical laws against small perturbations.

Consider the following identity. Operating with the spin operator $\sigma\equiv\hat{n}\cdot\vec{\sigma}$ on a single spin state we can express the resulting state as

$$\sigma|\psi\rangle=\bar{\sigma}|\psi\rangle+\Delta\sigma|\psi^{\perp}\rangle, \tag{4}$$

where $\bar{\sigma}$ and $\Delta\sigma$ are the expectation value and the uncertainty of σ with respect to the state ψ, and $\psi^{\perp}$ is a normalized state orthogonal to ψ:

$$\langle\psi|\psi^{\perp}\rangle=0,\quad \|\psi\|=\|\psi^{\perp}\|=1. \tag{5}$$

If σ is not an eigenoperator of $|\psi\rangle$ we have on the right hand side of Eq. (4) also a component of $|\psi^{\perp}\rangle$. However, let us now apply the relation to the collective state $M_x|\Psi\rangle$. We get

$$\begin{aligned}M_x|\Psi\rangle&=\frac{1}{N}\sum_{k=1}^{N}\bar{\sigma}_{xk}|\Psi\rangle+\frac{\Delta\sigma}{N}\sum_{k=1}^{N}|\Psi_k^{\perp}\rangle\\&=\bar{\sigma}_x|\Psi\rangle+\frac{\Delta\sigma}{N}|\Psi^{\perp}\rangle.\end{aligned} \tag{6}$$

Here $|\Psi_k^{\perp}\rangle=|\psi\rangle_1\cdots|\psi^{\perp}\rangle_k\cdots|\psi\rangle_N$. Since $\langle\Psi_i^{\perp}|\Psi_j^{\perp}\rangle=\delta_{ij}$, the norm of the second term on the right hand side is

$$\frac{\Delta\sigma}{N}\|\Psi^{\perp}\|=\frac{\Delta\sigma}{\sqrt{N}}. \tag{7}$$

Apart from the special cases where $\overline{\sigma_x}\sim O(1/\sqrt{N})$, the last term on the right hand side in Eq. (6) is a small correction, and in the large N limit,

$$\lim_{N\to\infty} M_x|\Psi\rangle=\overline{\sigma_x}|\Psi\rangle. \tag{8}$$

Similar results, which we discuss in the following, have been described in [1–3].

Next, let us see that the disturbance caused to individual spins, as a result of a collective measurement of M_x, vanishes in the large N limit. The evolution of the system under a measurement is described by the unitary operator $U=\exp(iQM_x)$, where Q is conjugate to the "pointer" P of the measuring device. Denoting by $|P\rangle$ the initial state of the measuring device (say a Gaussian centered around P), and applying U to the combined state, we have

$$U|\Psi\rangle|P\rangle=\prod_{k=1}^{N} u_k|\psi_k\rangle|P\rangle, \tag{9}$$

where $u_k=e^{i(\sigma_{xk}/N)Q}$. Using Eq. (4) we have

$$u_k|\psi_k\rangle=\left[\cos\frac{Q}{N}+i\overline{\sigma_x}\sin\frac{Q}{N}\right]|\psi_k\rangle + i\Delta\sigma\sin\frac{Q}{N}|\psi_k^{\perp}\rangle. \tag{10}$$

Expanding this equation in $1/N$ we get

$$U|\Psi\rangle=\left[1-\frac{\Delta\sigma^2Q^2}{2N}\right]e^{i\overline{\sigma_x}Q}|\Psi\rangle|P\rangle+|\delta\chi\rangle, \tag{11}$$

where

$$|\delta\chi\rangle=i\frac{\Delta\sigma Q}{N}\sum_{k=1}^{N}|\Psi_k^{\perp}\rangle|P\rangle+O(1/N^2). \tag{12}$$

For nonzero $|\delta\chi\rangle$ the measuring device is entangled with the sample. Since $|\Psi_k^{\perp}\rangle$ are mutually orthogonal, $\langle\delta\chi|\delta\chi\rangle\sim 1/N$, and the entanglement produced by this measurement is small. In particular, in the limit $N\to\infty$, we may drop the term $|\delta\chi\rangle$ above and obtain

$$\lim_{N\to\infty} U|\Psi\rangle|P\rangle=\exp(iQ\overline{\sigma_x})|\Psi\rangle|P\rangle=|\Psi\rangle|(P-\overline{\sigma_x})\rangle. \tag{13}$$

For a given initial state, $|\psi\rangle=c_+|+\rangle+c_-|-\rangle$ with $|+\rangle$ and $|-\rangle$ as the eigenstates of σ_x, the collective measurement will shift the pointer by the value

$$\overline{\sigma_x}=|c_+|^2(+1)+|c_-|^2(-1). \tag{14}$$

Next suppose that we perform on the *same* sample a "microscopic" measurement of each individual spin in the $\hat{x}$-direction. The outcome of this microscopic measurement is a string of numbers of $+1$ or -1. The numbers n_+ of $+1$s and $n_-=N-n_+$ of -1s should again allow us to evaluate the average

$$F_N=\frac{n_+}{N}(+1)+\frac{n_-}{N}(-1). \tag{15}$$

Since in this limit the disturbance caused to the system vanishes, consistency of the microscopic measurements with the macroscopic collective measurement dictates the equality

$$\lim_{N\to\infty} F_N=\overline{\sigma_x} \tag{16}$$

or

$$\lim_{N\to\infty}\left[\frac{n_+}{N}(+1)+\frac{n_-}{N}(-1)\right]=|c_+|^2(+1)+|c_-|^2(-1). \tag{17}$$

On the left hand side we have the averages obtained from the individual measurements, and on the right hand side the deterministic result for the macroscopic measurement of $\overline{\sigma_x}$. From the above equation we identify $|c_+|^2$ and $|c_-|^2$ with the usual quantum mechanical frequencies for $\sigma_x=+1$ and $\sigma_x=-1$.

Our argument can be easily generalized. For an n-level system a single macroscopic measurement is not sufficient, because the average is determined from the absolute square of n amplitudes. However, now we can measure macroscopically $n-1$ independent commuting observables (e.g., $L_x,\ldots,L_x^{n-1}$) and together with the overall normalization $\Sigma n_i=N$ evaluate the relevant probabilities.

The above result for the $N\to\infty$ limit is still unsatisfactory, when considering a finite sample, because in this case we can no longer neglect the disturbance $|\delta\chi\rangle$ in Eq. (11). One way to proceed is to make an additional assumption which seems natural for macroscopically large samples: *the results of physical experiments are stable against small perturbations*. Hence, for finite large N, the operator M_x fails to be a precise eigenoperator of $|\Psi\rangle$ in Eq. (8). However, by a small modification of the state to $|\Psi\rangle+|\delta\Psi\rangle$, with magnitude $\||\delta\Psi\rangle\|=O(1/\sqrt{N})$, the perturbed state does become an exact eigenstate of M_x. Alternatively, if after coupling to P, we first project the state in Eq. (11) by $|\Psi\rangle\langle\Psi|$, this would eliminate the term $|\delta\chi\rangle$ and the final microscopic measurement would give rise only to likely outcomes. The "stability" principle then requires that the corresponding probability law can qualitatively change at most by terms of order $O(1/\sqrt{N})$.

To get further insight into the physical meaning of this disturbance it is useful to adopt another line of consideration. Suppose that, after coupling the sample to the pointer P, we do not observe the exact value of P, but proceed to perform the microscopic measurement. Hence we now regard the pointer as an environmentlike system that couples weakly with the sample. The outcome of the microscopic measurement is described by the postselected state $|n_+,n_-\rangle$ of the sample. We can evaluate the final state of the pointer system P by projecting Eq. (11) from the left by $\langle n_+,n_-|$:

$$\langle n_+,n_-|U|\Psi\rangle|P\rangle=\left[1-\frac{\Delta\sigma^2Q^2}{2N}\right]\langle n_+,n_-|e^{i\bar{\sigma}_xQ}\rangle|\Psi\rangle|P\rangle + \langle n_+,n_-|\delta\chi\rangle. \qquad (18)$$

Noting that U is diagonal with respect to the final state of the sample, we get

$$|P-F_N\rangle=|P-\overline{\sigma_x}\rangle+|\delta P\rangle, \qquad (19)$$

where

$$|\delta P\rangle=\frac{\langle n_+,n_-|\delta\chi\rangle}{\langle n_+,n_-|\Psi\rangle}. \qquad (20)$$

The last equation states that the difference between a pointer state shifted by the frequency and by the mean is given by the correction $|\delta P\rangle$. Consider the case of a "likely" outcome, with $F_N-\overline{\sigma_x}\approx 0$. By examining Eqs. (12), (19), and (20), we find that in this case destructive interferences reduce the magnitude of the correction to $\||\delta P\rangle\|\approx 0$. On the other hand, consider now an unlikely result $n_+=N$, $n_-=0$, and hence $F_N-\overline{\sigma_x}\approx N$. In this case the equality of the two sides in Eq. (19) cannot be satisfied if $|\delta P\rangle$ is small. Indeed, we get the result that in this case the postselected environment state interferes constructively and $\||\delta P\rangle\|\sim 1$. (In fact in this case all higher orders in $1/N$ give rise to order 1 contributions.) Hence what we could have regarded in the likely case as a negligible $1/N$ correction now becomes the dominant contribution.

It is interesting that here we see, as far as we know for the first time, a fundamental difference, from a *microscopic point of view*, between likely and unlikely outcomes for a given sample. The unlikely results require a large coherent interference effect between the microscopic amplitudes in Eq. (12) that are induced by the weak interaction with the weakly coupled environment.

We further emphasis that the analysis considered here is quite general. In reality, when a sample is measured, it is always subjected to environmental effects which couple weakly with the particles of the sample (e.g., the electromagnetic dipole coupling).

Before concluding, and for completeness, let us examine the effect of a macroscopic measurement, from the point of view of the full quantum formalism. (Hence, from now on, we will assume the usual quantum probability law.) When we perform a precise measurement of M_x we must disturb other noncommuting observables. In the case of the individual microscopic measurements, we will randomly change the state of individual spins, and consequently destroy the macroscopic, magnetlike, properties of the sample: the new state may look like a collection of randomly polarized spins.

Now let us consider the collective measurement. Clearly, when we measure a macroscopic quantity (here the average magnetic moment of the magnet) we do not destroy the macroscopic state. However, to be in conformity with the uncertainty principle we must cause *some* disturbance to the spins. We will now show that the strength of the disturbance is generally such that *not even one spin* of the sample has flipped its direction. To do that we will now regard the measuring device on a quantum level as well.

The accuracy of the collective measurement is determined by the initial uncertainty of the measuring device pointer. Let us express it as

$$\Delta P=\frac{1}{\epsilon\sqrt{N}}, \qquad (21)$$

where ϵ is some real number to be fixed in the following. The disturbance to the ith spin of the sample is then induced by the evolution

$$U_i=\exp i\frac{Q\sigma_{xi}}{N}, \qquad (22)$$

which describes a rotation around the $\hat{x}$ axis of the spin, by an uncertain angle of $\Delta\theta\approx\Delta Q/N\approx\epsilon/\sqrt{N}$. The probability of a single spin remaining in its initial state (not flipping) hence varies as $\approx 1-\epsilon^2/N$. Therefore the probability for all the N spins to remain in their initial state is

$$\left(1-\frac{\epsilon^2}{N}\right)^N\approx\exp(-\epsilon^2). \qquad (23)$$

Therefore, for $\epsilon\ll 1$, the probability that even one spin of the sample has not flipped is close to unity. Nevertheless, since at the same time we can still have $N\gg 1/\epsilon^2$, the measurement becomes arbitrarily precise for large enough N.

This is in agreement with the uncertainty relation for components of $\vec{M}$ and a single spin. From the commutation relation (3) with a finite N, and from $[\sigma_{zi},M_x]=i\sigma_{yi}/N$ we have

$$\Delta M_x\Delta M_y\geqslant\frac{\langle M_z\rangle}{N} \qquad (24)$$

and

$$\Delta\sigma_i\Delta M(N)\geqslant\frac{\langle\sigma_{yi}\rangle}{N}. \qquad (25)$$

Hence we can measure simultaneously all components of $\vec{M}$ provided that we keep the accuracy as $\Delta M_x\sim\Delta M_y\sim 1/\sqrt{N}$. Although for large N this inaccuracy becomes vanishing small, we still cannot distinguish between different eigenvalues whose separations vary as $1/N$.

Finally, let us compare our approach with other related attempts to derive quantum probabilities. Hartle [1] and Graham [2] constructed the "frequency operator" $\hat{f}_N$ by considering the sum of projectors to all possible results of the measurements, which are weighted by an appropriate "degeneracy" factor, the latter corresponding to the number of different strings of possible results with the same total numbers of n_+ and n_-. In the limit of $N\to\infty$, $\hat{f}_N$ becomes an eigenoperator of $|\Psi\rangle_N$. The eigenvalue of the frequency operator is then given by the quantum probability. In our approach it is the collective magnetic operator $\vec{M}$ that becomes an eigenoperator. However, since in order to observe $\hat{f}$ one has to perform a highly nonlocal measurement of the spin sample, the frequency does not seem to be a physically realizable operator.

Instead, we argued that in *ordinary*, everyday, macroscopic observations we do measure collective operators like the operator $\vec{M}$ discussed here. Indeed, such collective operators appear naturally in the usual electromagnetic interactions of an external test particle with a sample of magnetic moments that constitute a macroscopic object. An approach similar to ours has been suggested by Farhi, Goldstone, and Gutmann [3]. The present article extends this approach by analyzing the measuring process.

In conclusion, we have demonstrated how the agreement between macroscopic and microscopic observations dictates the quantum probability law. For a finite sample we suggested that this law can be obtained if one further invokes a natural stability assumption.

Finally, we compared two definite measurement outcomes: a rare sequence where all spins are found to be in the $\uparrow$ direction, and an expected sequence with roughly an equal number of $\uparrow$ and $\downarrow$ outcomes. Classically there is no microscopic difference between the two sequences: they have the same *a priori* probability. Surprisingly, we observed that in quantum mechanics these ''likely'' and ''unlikely'' sequences do differ on a microscopic level, and are associated with distinctive interference effects.

This research was supported in part by Grant No. 62/01-1 of the Israel Science Foundation, established by the Israel Academy of Sciences and Humanities, NSF Grant No. PHY-9971005, and ONR Grant No. N00014-00-0383.

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