Qubits versus Bits for Measuring an Integral of a Classical Field

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Methods for measuring an integral of a classical field via local interaction of classical bits or local interaction of qubits passing through the field one at a time are analyzed. A quantum method, which has an exponentially better precision than any classical method we could see, is described.

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Although the retrievable information content of a number of qubits is essentially equal to the information content that can be stored in the same number of bits [1], qubits are more efficient for many specifically tailored tasks. Recently, Galvão and Hardy (GH) [2] found one such task: pinpointing a particular property of an integral of a classical field is possible using a single qubit, but impossible with a classical bit, or even with many bits. This motivated the current work in which we found that qubits are more efficient than bits for a more general task, namely, the measurement of the integral itself.

The task is to measure the integral of a classical field,

$$I = \int_A^B \phi(x)\, dx, \tag{1}$$

via local interactions along the path from A to B. In one case, N bits go one at a time through the path interacting with the field, and, in the other case, N qubits pass through the field instead. In this task there are no constraints on the complexity of local interactions between the bits (qubits) and the field. The limitations that we investigate are due to the complexity of the carrier of the information about the field which goes along the path.

In order to make the comparison between quantum and classical methods simpler, we consider a non-negative classical field and assume that it is known that the order of magnitude of I is M. The classical method is as follows. The bit starts at A in the state 0, and it flips with a probability proportional to the strength of the field. Once flipped to 1, it cannot flip back. The probability of a flip along the path is

$$p = 1 - e^{-\lambda I}, \tag{2}$$

where λ has to be optimized for getting the best precision [3]. The exact optimization depends on our prior information about the probability for different values of I and the choice of the particular aspect of the precision of the measurement we wish to optimize. The precision of measuring p achieved after sending N bits can be estimated as $\Delta p = \sqrt{p(1-p)}/\sqrt{N}$. Thus, essentially for all reasonable approaches, the uncertainty of measuring I is of the order of $M/\sqrt{N}$ [4].

In the quantum method, we arrange an interaction that leads to a qubit "rotation" proportional to the strength of the field, for example, a spin precession in a magnetic field. The method works equally well for negative and general classical fields. The only important parameter is the range of possible values of the integral. The strength of the coupling should be such that for every value of the integral we will get a different final state of the qubit. Thus, we have to choose the strength of the coupling to the field such that a rotation of more than 2π will not be probable.

When we send N spin-$\frac{1}{2}$ particles through the field, they all rotate by the same angle. If we start with all spins pointing in, say, the x direction and the magnetic field in the z axis, the direction of the spin in the x-y plane will yield the value of the integral.

The precision of measuring the direction of N parallel spins is proportional to $N^{-1/2}$. This is the same dependence as in the probabilistic methods with bits. However, Peres and Scudo [5] (see also Bagan *et al.* [6]) showed that by taking N entangled spins one can reduce the uncertainty to be proportional to N^{-1}. They optimized defining the direction in three dimensions with N spin-$\frac{1}{2}$ particles, while in the present problem we need to find the direction in only two dimensions. So, some further optimization is possible, but the uncertainty remains proportional to N^{-1}. This concludes the description of our quantum method I for the measurement of integral I.

A crucial requirement for the advantage of qubits versus bits is that we are allowed to send bits and qubits only one at a time. If we send all bits together, we can build a probabilistic counter that can register up to 2^N counts. The counter makes counts while moving in the field with the probability for a count proportional to the field. The uncertainty in the total number of counts is of the order of the square root of this number. Therefore, adjusting the strength of the interaction such that the expectation value of the number of counts is of the order of 2^N, we can get the uncertainty in the measurement of I proportional to

$2^{-N/2}$. The uncertainty is exponentially smaller than in the quantum method described above.

Of course, if qubits are allowed to pass along the field together, the precision that can be achieved in the measurement of the integral is not smaller than that in the classical case. But can exponentially small uncertainty be achieved with qubits passing through the field one at a time? As we show below, the answer is yes, if we allow various strengths of the interaction between the qubits and the field. On the other hand, we do not see how this freedom might lead to a significant advantage in the classical case.

Indeed, it is intuitively clear (especially after the GH proof for a particular case) that a bit passing through the field cannot have a deterministic information about the integral of the field. Then, the only way we can imagine for storing information in the bit is in the value of its probability to be 1. When we are given N bits, the uncertainty in the measurement of the probability decreases as $N^{-1/2}$. It seems to us highly implausible that there is a classical method that can do better than this.

Now we turn to the description of the method (referred to hereafter as quantum method II) that employs qubits interacting with various strengths with the field. The strengths depend only on the number of the qubit passing through the field and are fixed before the experiment. No additional structure is required for the qubit.

The basis of this method is the result of Galvão and Hardy [2]. They considered a particular case in which

$$I = m\alpha, \tag{3}$$

where α is known and m is an integer. GH found a method that allows a single qubit to answer the question: Is m even or odd [7]? They achieved the goal by tuning the strength of the interaction in such a way that $I = \alpha$ yields a rotation by π. Thus, for an odd integer, the spin flips, and, for an even integer, it returns to its initial state.

If we send a number of qubits, one after the other, we can modify this procedure to find m itself. To this end it is arranged that the qubits we send interact with the field with different strengths: the first as in the GH protocol, the second with half of the strength of the first, the next with half of the strength of the preceding, etc. In the first step we find the last digit of m written in the binary way. In the next step we find the preceding digit, and thus, after the kth step we find $m(\mathrm{mod}\, 2^k)$.

The procedure works in the following way. If the last digit is zero, then in the second step we repeat the GH protocol with half of the strength of the interaction. Since now we know that $I = m'2\alpha$, the protocol determines the last digit of m', which is the second digit from the end of m. If the last digit of m is 1, in the second step we should modify the procedure by additional rotation of the spin by the angle $\theta_2 = -\frac{\pi}{2}$. In the kth step we should compensate for all nonzero digits, as follows:

$$\theta_k = -\sum_{i=1}^{k-1} \frac{\pi d(i)}{2^{(k-i)}}, \tag{4}$$

where $d(i)$ is the value of digit number i from the end. The method yields one digit for each qubit and yields zeros once the whole number is written.

In a general case, when our only prior information about I is its order of magnitude, we can combine the two methods we described above. We chose α and, using method I, we find the remainder β of the division of I by α:

$$I = m\alpha + \beta. \tag{5}$$

After measuring β with good precision, we apply method II for finding m. The remainder β requires additional correction angles in the application of the second method. For the first step, the correction angle is $\theta_1 = -\frac{\pi\beta}{\alpha}$ and, in general, for the kth step, the correction angle is

$$\theta_k = -\sum_{i=1}^{k-1} \frac{\pi d(i)}{2^{(k-i)}} - \frac{\pi\beta}{2^{k-1}\alpha}. \tag{6}$$

The requirement for choosing α is that we have enough qubits to find all digits of m. We get high probability for that if

$$\alpha = \frac{10M}{2^{N-N_0}}, \tag{7}$$

where N_0 is the number of qubits used in the measurement of β. If the probability of error in the measurement of m is negligible (when β is measured with high precision), then the uncertainty in the value of I is, essentially, the uncertainty in the measurement of β, which, in the best case (Peres-Scudo method), is of the order of $\frac{\alpha}{N_0}$.

However, the error in the measurement of m turns out to be not too large even if there is a large error in the measurement of β. In fact, it is more effective not to "waste" qubits on measurement of β. The quantum method II works well by itself even in a general case. Now α becomes much smaller:

$$\alpha = \frac{10M}{2^N}. \tag{8}$$

In order to estimate the precision of the quantum method, we calculate the probability of a particular reading m given the actual value of I. We show that it is a rapidly decreasing function of the difference between the actual value of the integral I and the readout of the device $\tilde{I} = m\alpha$. In our procedure, the last digit of m is specified by the spin measurement. The spin rotates by the angle $\Theta_1 = \frac{I\pi}{\alpha}$ and is found in the direction specified by the angle $\tilde{\Theta}_1 = \frac{\tilde{I}\pi}{\alpha}$. Thus, the probability for this particular outcome of the spin measurement that specifies the last

digit is $p_1 = \cos^2 \frac{\Theta_1 - \tilde{\Theta}_1}{2} = \cos^2 \frac{(I-\tilde{I})\pi}{2\alpha}$. The digit k from the end, $k > 1$, is specified by the measurement of the spin that is rotated by the angle

$$\Theta_k = \frac{I\pi}{2^{k-1}\alpha} + \theta_k \tag{9}$$

and found in the direction specified by the angle

$$\tilde{\Theta}_k = \frac{\tilde{I}\pi}{2^{k-1}\alpha} + \theta_k. \tag{10}$$

Therefore, the probability for this particular outcome of the spin measurement that specifies the kth digit is $p_k = \cos^2 \frac{(I-\tilde{I})\pi}{2^k\alpha}$. Thus, the probability for readout $\tilde{I}$ given the actual value of the integral I is

$$p(\tilde{I}|I) = \prod_{k=1}^{N} \cos^2 \frac{(I-\tilde{I})\pi}{2^k\alpha}. \tag{11}$$

Denoting the error δI and substituting the value of α from (8), the probability of the error is

$$p(\delta I) = \prod_{k=1}^{N} \cos^2 \frac{\delta I \pi}{2^{k-N}10M}. \tag{12}$$

This function vanishes for $\delta I = n\alpha$ for integer n, except for $n = 0$, where it has maxima that are equal to 1 because the method yields no errors for $I = m\alpha$. The function has local maxima at $\delta I = (n + \frac{1}{2})\alpha$, except for $n = -1, 0$.

The readout value of the integral, $\tilde{I}$, might have only discrete values, $m\alpha$. Therefore, the error of the order of α is unavoidable. In the worse case, $I \bmod \alpha = \frac{\alpha}{2}$, but even in this case we have only a small probability to get the error that is an order of magnitude larger than α. For large N, this probability has almost no dependence on N; we calculate it for $N = 30$:

$$\begin{aligned} p(\delta I > 10\alpha) &= 1 - p(\delta I < 10\alpha) \\ &= 1 - \sum_{n=-9}^{10} \prod_{k=1}^{30} \cos^2 \frac{(\frac{1}{2}+n)\alpha\pi}{2^{k-N}10M} \\ &= 1 - \sum_{n=-9}^{10} \prod_{k=1}^{30} \cos^2 \frac{(\frac{1}{2}+n)\pi}{2^k} \simeq 0.019. \end{aligned} \tag{13}$$

Thus, we should expect an error of the order of α (it is of the order of 10^{-7} in our case) and not significantly larger.

In order to illustrate our result we performed computer simulation of classical (30 bits) and quantum (30 qubits) measurements of the integral for ten different fields. We took $I_n = (n\pi) \bmod 10$, $n = 1, 2, \ldots, 10$. We assumed that the order of magnitude of the integral is given, $M = 5$, and we chose parameter λ of the classical method such that the precision of the measurement of I for $I = M$ is optimized. The uncertainty in classical measurement can be estimated as

$$\Delta I = \frac{\sqrt{e^{\lambda I} - 1}}{\lambda\sqrt{N}}. \tag{14}$$

For $I = M$ it has a minimum around $\lambda = \frac{1.2}{M}$ and for the parameter we chose, the uncertainty ΔI is of the order of 1.

TABLE I. The results of simulation of classical and quantum measurements of ten values of I, $I_n = (n\pi) \bmod 10$. The quantum method uses 30 qubits and the classical method uses 30 bits.

n	$I = n\pi \bmod(10)$	Quantum	Classical
1	3.141 592 654	3.141 592 494	3.175 583 382
2	6.283 185 307	6.283 185 389	4.577 585 162
3	9.424 777 961	9.424 777 867	9.594 107 747
4	2.566 370 614	2.566 370 611	1.689 440 418
5	5.707 963 268	5.707 963 268	5.016 553 197
6	8.849 555 922	8.849 555 813	9.594 105 057
7	1.991 148 575	1.991 148 466	2.619 198 463
8	5.132 741 229	5.132 741 166	8.395 441 798
9	8.274 333 882	8.274 333 865	6.706 033 821
10	1.415 926 536	1.415 926 495	0.929 766 618

The results are shown in Table I. We see that the error in the classical method is, indeed, of the order of 1 and the quantum error is of the order of 10^{-7}.

The technology today is far from getting this exponential advantage of the quantum method, which requires stability and high precision for a very large range of the interaction strength. Also, preparation of initial entangled states and complicated collective measurements of Peres-Scudo measurements are difficult for experimental implementation. However, recent progress in quantum information experiments allows us to believe in the prospects of at least partial implementation of our proposal, which will manifest advantages of the quantum method.

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[1] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).

[2] E. F. Galvão and L. Hardy, Phys. Rev. Lett. **90**, 087902 (2003).

[3] This method and the quantum method, which is described below, are also applicable for measuring the sum of the field values at discrete points, $\sum_i \phi(x_i)$. In this case, the interaction at each point should be such that

the probability of the flip (if it has not been flipped already) is $p_i = 1 - e^{-\lambda\phi(x_i)}$.

[4] The method has to be modified if the field is allowed to be negative as well as positive. In this case, we can send half of the bits for measuring the positive contribution of the field and half of the bits for measuring the negative contribution of the field. (An alternative method is to add to the field a constant that will make all values positive.) This, of course, decreases the precision (especially when the positive and negative contributions are much larger than M), but the overall dependence of the uncertainty on N remains (for large N) proportional to $N^{-1/2}$.

[5] A. Peres and P. F. Scudo, Phys. Rev. Lett. **86**, 4160 (2001).

[6] E. Bagan *et al.*, Phys. Rev. A **63**, 052309 (2001).

[7] Note that our classical probabilistic method allows one to answer the GH question with an almost vanishing probability of an error when we are given N bits, such that $\sqrt{N} \gg M$, where M (assumed to be known) is the estimation of the bound for m.