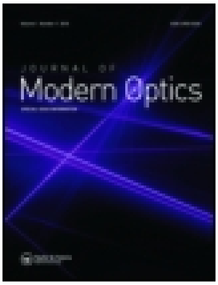

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Berry Groisman [a] , Benni Reznik [a] & Lev Vaidman [a]
[a] School of Physics and Astronomy, Tel Aviv University , Tel Aviv 69978, Israel E-mail:
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Instantaneous measurements of nonlocal variables

BERRY GROISMAN, BENNI REZNIK and LEV VAIDMAN
School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
Email: groisman@post.tau.ac.il

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Abstract. Relativistic causality imposes rigid restrictions on nondemolition (repeatable) measurements of nonlocal variables. We show that there are no causal restrictions on demolition (nonrepeatable) measurements: all Hermitian operators of multipartite quantum system can be measured instantaneously, provided unlimited supply of entanglement resources.

1. Introduction

In 1931 L. Landau and R. Peierls [1] claimed that any nonlocal property of quantum system at a given time cannot be measured in relativistic quantum theory. Only in 1980 Y. Aharonov and D. Albert found [2] that this is not true and some nonlocal properties can be measured instantaneously. Later, in 1986 some explicit methods for performing instantaneous nonlocal measurement were found [3]. In particular it was shown how $A + B$, $(A + B) \bmod a$ and the Bell-operator can be measured. However, it was shown that measurement of some other operators, like the one-side twisted operator with product states eigenstates and the operator with nonmaximally entangled eigenstates, contradicts relativistic causality [4]. The above causal restrictions were derived under the assumption that the measurement is nondemolition, i.e. if, before measurement, the observed system was in an eigenstate of the observable then after the measurement is completed, the system's state will not change. The nondemolition measurement plays a dual role. First of all, it observes the unknown quantity. Second, it prepares the system in the eigenstate of this observable. Nevertheless, roles of observation and preparation are logically independent. We can relax the preparation role of the measurement and let the measurement be a verification measurement. Verification measurement has a very clear physical meaning: if the system was prepared in one of the eigenstates of the observable, the verification measurement will confirm this with certainty. But the final state of the system will not necessarily be an eigenstate of this observable.

Here we show that verification measurements can always be performed instantaneously given unlimited resources of entanglement and arbitrary local interactions.

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2. 2 × 2 operators with twisted product eigenstates

Consider a bipartite system, held by Alice and Bob, in a 2×2 dimensional Hilbert space, spanned by the local basis vectors $|0_A\rangle$, $|1_A\rangle$ and $|0_B\rangle$, $|1_B\rangle$, respectively. In the forthcoming, we shall handle the basis $|0\rangle$, $|1\rangle$ as spin $\frac{1}{2}$-like states, $|0\rangle \equiv |\uparrow_z\rangle$, $|1\rangle \equiv |\downarrow_z\rangle$, and define accordingly the Pauli operators.

The first example which we discuss is a nondegenerate operator, defined by the four mutually orthogonal direct-product eigenstates:

$$\begin{aligned}
|\Psi^1_{AB}\rangle &= |0_A\rangle|0_B\rangle, \\
|\Psi^2_{AB}\rangle &= |0_A\rangle|1_B\rangle, \\
|\Psi^3_{AB}\rangle &= |1_A\rangle\frac{1}{\sqrt{2}}(|0_B\rangle + |1_B\rangle), \\
|\Psi^4_{AB}\rangle &= |1_A\rangle\frac{1}{\sqrt{2}}(|0_B\rangle - |1_B\rangle).
\end{aligned} \tag{1}$$

This operator was shown to be unmeasurable instantaneously, without any exchange of information during the local interactions, by the *nondemolition* method [4]. We show that the *demolition* instantaneous measurement of this operator may be performed by utilizing one ebit of shared entanglement.

We propose two techniques. One of them is based on the instantaneous probabilistic remote rotations [5]. The main idea is that Alice measures σ_{z_A} of her particle and performs a probabilistic $\pm\pi/2$ rotation of Bob's state when her state is $|1_A\rangle$. The $\pm$ signs of the angle of rotation appear with equal probability $\frac{1}{2}$, that keeps the protocol from supporting superluminal signalling. This rotation maps the twisted basis $\{|0_B\rangle + |1_B\rangle,\ |0_B\rangle - |1_B\rangle\}$ to the basis $\{|0_B\rangle, |1_B\rangle\}$ on Bob's side. On the other hand, no such mapping will be performed if Alice's state turns out to be $|0_A\rangle$. Alice controls the remote rotation. Bob on the other hand will perform a fixed set of manipulations independently of Alice's choice, and finally measures σ_{z_B}. Having done so, Alice and Bob obtain local records that when combined allow them to distinguish with certainty between the elements of the basis (1). The final state of Alice's and Bob's spins is always given by a unit density matrix. Hence, although the whole process takes place instantaneously, no violations of causality occur.

Let us now consider the process in some detail. Here we follow the state-operator (stator) method of [6] in order to perform remote rotation. The initial state at the hands of Alice and Bob is

$$\frac{1}{\sqrt{2}}(|0_a\rangle \otimes |0_b\rangle + |1_a\rangle \otimes |1_b\rangle) \otimes |\Psi_{AB}\rangle \tag{2}$$

where by the small letters, a and b, we denote the particles of ancillary ebit, shared by Alice and Bob, respectively. Bob starts by performing a local CNOT interaction (with respect to σ_{y_B}) between the entangled qubit b and his state B, described by the unitary transformation

$$U_{bB} = |0_b\rangle\langle 0_b| \otimes I_B + |1_b\rangle\langle 1_b| \otimes \sigma_{y_B} \tag{3}$$

and performs a measurement of σ_{x_b} of the entangled qubit b and keeps the result

$v(\sigma_{x_b})$. The resulting state is now

$$\begin{aligned}(|0\rangle_a \otimes I_B \pm |1\rangle_a \otimes \sigma_{y_B})|\Psi_{AB}\rangle \\ \equiv \mathcal{S}|\Psi_{AB}\rangle\end{aligned} \tag{4}$$

where the $\pm$ above corresponds to $v(\sigma_{x_b})$. Let us consider the state-operator (stator), $\mathcal{S}$, defined in [6]. The stator satisfies the eigenoperator equation:

$$\sigma_{x_a}\mathcal{S} = v(\sigma_{x_b})\sigma_{y_B}\mathcal{S}. \tag{5}$$

This equation captures the correlations between a unitary transformation which is acted by Alice on a and the equivalent rotation on an arbitrary state of Bob. Particularly, if Alice acts with the unitary transformation $\exp(i\alpha\sigma_{x_a})$ on a, it is equivalent to the unitary transformation $\exp(i\alpha\sigma_{y_B})$ on Bob's qubit (up to a trivial π rotation around the y-axis).

Having 'prepared' the above stator, Alice measures σ_{z_A}. If it turns out that $|\Psi_A\rangle = |0_A\rangle$, she next measures σ_{z_a} and keeps the result $v(\sigma_{z_a})$. If, on the other hand, she gets $|\Psi_A\rangle = |1_A\rangle$, she acts by $e^{i\frac{\pi}{4}\sigma_{x_a}}$ on a in order to untwist Bob's qubit, and then measures σ_{z_a} of a. The four possible outcomes of this map are summarized in the following table:

$\sigma_{z_a}\backslash\sigma_{x_b}$	$v(\sigma_{x_b}) = +1$	$v(\sigma_{x_b}) = -1$
$v(\sigma_{z_a}) = +1$	$\|\Psi^1_{AB}\rangle \to \|0_A\rangle\|0_B\rangle$	$\|\Psi^1_{AB}\rangle \to \|0_A\rangle\|0_B\rangle$
	$\|\Psi^2_{AB}\rangle \to \|0_A\rangle\|1_B\rangle$	$\|\Psi^2_{AB}\rangle \to \|0_A\rangle\|1_B\rangle$
	$\|\Psi^3_{AB}\rangle \to \|1_A\rangle\|0_B\rangle$	$\|\Psi^3_{AB}\rangle \to \|1_A\rangle\|1_B\rangle$
	$\|\Psi^3_{AB}\rangle \to \|1_A\rangle\|1_B\rangle$	$\|\Psi^3_{AB}\rangle \to \|1_A\rangle\|0_B\rangle$
$v(\sigma_{z_a}) = -1$	$\|\Psi^1_{AB}\rangle \to \|0_A\rangle\|1_B\rangle$	$\|\Psi^1_{AB}\rangle \to \|0_A\rangle\|1_B\rangle$
	$\|\Psi^2_{AB}\rangle \to \|0_A\rangle\|0_B\rangle$	$\|\Psi^2_{AB}\rangle \to \|0_A\rangle\|0_B\rangle$
	$\|\Psi^3_{AB}\rangle \to \|1_A\rangle\|1_B\rangle$	$\|\Psi^3_{AB}\rangle \to \|1_A\rangle\|0_B\rangle$
	$\|\Psi^4_{AB}\rangle \to \|1_A\rangle\|0_B\rangle$	$\|\Psi^4_{AB}\rangle \to \|1_A\rangle\|1_B\rangle$

To complete the measurement, Bob measures the operator σ_{z_B} for his state $|\Psi_B\rangle$. Finally, in order to infer which eigenstate they measured, Alice and Bob use $v(\sigma_{z_a})$ and $v(\sigma_{x_b})$ to identify which of the above four blocks (maps) has been realized, and then use the results for σ_{z_A} and σ_{z_B} to isolate one of the four possible states in that block.

The second technique is based on instantaneous probabilistic teleportation [7]. According to the original proposal [8], the teleportation process has three main steps: the Bell measurement on one side, transmition of classical information from this side to another one, and local corrections on the second side. Here Alice and Bob do not perform the full teleportation (which requires a finite period of time due to classical information transmission), rather only the Bell measurement at Bob's side is performed. (We will continue to use the term 'teleportation' just for this first step.) Thus, the state of quantum particle is teleported from Bob to Alice up to trivial π rotations (known to Bob) around one of the axes $\hat{x}$, $\hat{y}$, $\hat{z}$. The most important feature here is that these rotations, when acting on spin which lies along one of these axes, will leave the spin's direction unchanged or will flip the spin. Hence, if Bob's spin was along the $\hat{z}$-axis it will arrive at Alice's site along the same axis. The situation is similar regarding the x-direction. Thus, if the state of Alice's first qubit is $|0_A\rangle$ she measures the teleported qubit in z-basis. If the state of Alice's first qubit is $|1_A\rangle$ she measures the teleported qubit in x-basis. Although, Alice distinguishes between two groups of states $|\Psi^1_{AB}\rangle, |\Psi^2_{AB}\rangle$ and $|\Psi^3_{AB}\rangle, |\Psi^4_{AB}\rangle$ on the basis of her measurement of σ_{z_A}, she cannot distinguish between the states inside these groups only on the basis of her measurement of teleported spin, because this result is unbiased. Alice needs to know the results of Bell-measurement on Bob's side in order to interpret her second measurement and, thus, the final result of nonlocal measurement.

Consider now a more general operator:

$$\begin{aligned} |\Psi^1_{AB}\rangle &= |0_A\rangle|0_B\rangle, \\ |\Psi^2_{AB}\rangle &= |0_A\rangle|1_B\rangle \\ |\Psi^3_{AB}\rangle &= |1_A\rangle\left(\cos\frac{\alpha}{2}|0_B\rangle + \sin\frac{\alpha}{2}|1_B\rangle\right), \\ |\Psi^4_{AB}\rangle &= |1_A\rangle\left(\sin\frac{\alpha}{2}|0_B\rangle - \cos\frac{\alpha}{2}|1_B\rangle\right) \end{aligned} \tag{6}$$

with $0 < \alpha < \pi$. (For simplicity, and without loss of generality, we have ignored possible relative phases in $|\Psi^3_{AB}\rangle$ and $|\Psi^4_{AB}\rangle$.) This operator can also be measured instantaneously, but for arbitrary α our methods require an infinite number of ebits. Indeed, utilizing one ebit of entanglement, probabilistic remote rotation succeeds with probability $1/2$ for $\alpha \neq (\pi/2)$. In the case of failure an additional rotation with an angle α must be performed in order correct the error. This second step again succeeds with probability $1/2$. Thus the total probability of success is now $3/4$, and the process can be repeated again until the probability of failure is sufficiently small. This method of corrections has been proposed in [9]. The correction steps require the supply of additional ebits. For the general case, n ebits allow us to perform a successful measurement with probability $1 - (1/2^n)$. The ebits are necessary even if the process truncates after a finite small number of steps once Bob obtained $v(\sigma_{x_b}) = +1$. Since we do not allow any communication, Alice must apply a fixed set of rotations $e^{i2^{n-2}\alpha\sigma_{x_a}}$ on the nth ebit. Nevertheless, an infinite number of ebits is not needed for all angles in order to get a probability infinitesimally close to one. If $\alpha = (\pi k/2^n)$ then the nth step will always succeed. Thus, for certain angles the number of ebits n is relatively small: $n = 2$ for $\alpha = \pi/4, 3\pi/4$, $n = 3$ for $\alpha = \pi/8, 3\pi/8, 5\pi/8$, etc.

The situation is similar for the teleportation method. Here Bob's spin which corresponds to $|\Psi_3^{AB}\rangle$ and $|\Psi_4^{AB}\rangle$, doesn't lie along the $\hat{x}$ axis, rather along some line $\hat{n}_\alpha$ defined by the angle α. Therefore, this spin is teleported to Alice's side either along the line of $\hat{n}_\alpha$ (corresponding to teleportation without rotation or rotation around $\hat{y}$) or along the line of $\hat{n}'_\alpha$ obtained from the line of $\hat{n}_\alpha$ by π rotation around $\hat{x}$ or $\hat{z}$.

Thus, if Alice's result is 'up' she can finish by measuring the teleported particle in z-basis. But if her result is 'down', she cannot perform a measurement on Bob's teleported particle because she doesn't know the basis. In this case, Alice teleports Bob's teleported spin back to Bob using a new teleportation axes (which defines the new eigenstates of the Bell measurement) defined by $\hat{z}' = \hat{n}_\alpha$ and $\hat{y}' = \hat{y}$.

In the third step, Bob performs an action similar to that of Alice in step 2. He knows whether the spin state in α direction was teleported to Alice along the $\hat{n}_\alpha$

line or along the $\hat{n}''_\alpha$ line. In the former case, the state teleported to him is still along the $\hat{n}_\alpha$ line, so he completes the procedure by spin measurement in this direction. In the latter case, he receives the spin either along the $\hat{n}'_\alpha$ line or along the $\hat{n}''_\alpha$ line obtained by π rotation around the $\hat{n}_\alpha$ axis. In this case, he teleports the particle back with the teleportation axes $\hat{z}'' = \hat{n}'_\alpha$ and $\hat{y}'' = \hat{y}$.

Alice and Bob continue this procedure. If $\alpha = (\pi k/2^n)$, the line $\hat{n}^{(n)}_\alpha$ coincides with the line $\hat{n}^{(n-1)}_\alpha$ because the angle between the lines is: $n^{(n)}_\alpha - n^{(n-1)}_\alpha = (2^n\alpha) \bmod \pi$. In this particular case the process is guaranteed to stop after n teleportation steps.

Finally we note that both methods consume the same amount of entanglement. Nevertheless, in principle the method of remote rotations does not require Bob to know the angle α, because Alice controls the angle of rotation (unlike the teleportation method where both Alice and Bob choose the new axis of teleportation at each step, which depends on α). But in this case the number of ebits cannot be finite for any angle.

3. General nonlocal variable of a composite multipartite quantum system

We have shown how the bipartite operator with product state eigenstates can be measured instantaneously. Our methods can easily be extended to all other operators (with entangled eigenstates) in $2 \otimes 2$ dimensional Hilbert space [5]. But we will not go into details of the specific protocols, rather we give the answer to the general question: can any nonlocal variable of multipartite quantum system be measured instantaneously?

We will start with the case of a general variable of a bipartite system. For simplicity, at the first step Alice and Bob perform unitary operations which swap the states of their systems with the states of sets of spin-$\frac{1}{2}$ particles. This allows us to implement the formalism developed in the previous sections. In this way Alice and Bob will need the teleportation procedure for spin-$\frac{1}{2}$ particles only. Teleporting the states of all individual spins teleports the state of the set, be it entangled or not.

Bob teleports his system to Alice. Again, he does not send to Alice the results of his Bell measurements, but keeps them for his further actions; we signify the possible outcomes of these measurements by $i = 1, \ldots, N$. The case $i = 1$ corresponds to finding singlets in all Bell measurements and the state of Bob's system is teleported without distortion.

Alice performs a unitary operation on the composite system of her spins and the teleported spins which, under the assumption of nondistorted teleportation, transforms the eigenstates of the nonlocal variable (which now actually are fully located in Alice's site) to product states in which each spin is either 'up' or 'down' along the z direction. Then she teleports the *complete* composite system consisting of her spins and Bob's teleported spins to Bob. From now on this is the system which will be teleported back and forth between Bob and Alice. In all these teleportations the usual z, x, y basis is used. Hence, if the state is in the one of the product states in spin z basis, then it will remain in this basis.

If, indeed, Bob happened to teleport his spins without any distortion, i.e. $i = 1$ (the probability of which is $(1/N)$), Bob gets the composite system in one of the spin z product states and his measurements in the spin z basis that he now performs complete the measurement of the nonlocal variable. If $i \neq 1$, Alice's operation does not bring the eigenstates of the nonlocal variable to the spin z basis, so Bob cannot perform the measurement and he teleports the system back to Alice following a protocol that we explain below.

Alice and Bob have numerous teleportation channels arranged in $N-1$ clusters numbered from 2 to N. Each cluster consists of two teleportation channels capable of teleporting the complete system and $M-1$ clusters of a similar type, where M is the number of possible outcomes of the Bell measurement for teleportation of the complete system. In turn, each of the $M-1$ clusters consists of two teleportation channels and $M-1$ further nested clusters, etc.

If in his first teleportation the result of Bob's Bell measurements is i, he teleports now the composite system back to Alice in the teleportation channel of cluster i. Alice does not know in which channel she gets the system back (if she gets it back at all). So she must work on all of them. She knows that if she does get the system in channel i, the result of the Bell measurement in Bob's first teleportation was i. Thus, she knows all the transformations performed on this system except for the last teleportation. Alice performs a unitary operation that transforms eigenstates of the nonlocal variable to product states under the assumption that the last teleportation was without distortion and teleports the system back to Bob.

Let us denote the result of the Bell measurement in Bob's last teleportation by i', $i' = 1, \ldots, M$. Again, for $i' = 1$, which corresponds to finding singlets for all Bell measurements, Bob performs the spin z measurement on the system which he receives in the teleportation channel of the cluster i. This then completes the nonlocal measurement. Otherwise, he teleports the system back in the channel of the sub-cluster i'. Alice and Bob continue this procedure. The nonlocal measurement is completed when, for the first time, Bob performs a teleportation without distortion. Since, conceptually, there is no limitation on the number of steps, and each step (starting form the second) has the same probability of success, $1/M$, the measurement of the nonlocal variable can be performed with probability arbitrarily close to 1.

The generalization to a system with more than two parts is more or less straightforward. Let us sketch it for a three-part system. First, Bob and Carol teleport their parts to Alice. Alice performs a unitary transformation which, under the assumption of undisturbed teleportation of both Bob and Carol, transforms the eigenstates of the nonlocal variable to product states in the spin z basis. Then she teleports the complete system to Bob. Bob teleports it to Carol in a particular channel i_B depending on the results of the Bell measurement of his first teleportation. Carol teleports all the systems from the teleportation channels from Bob back to Alice in the channels (i_B, i_C) depending on her Bell measurement result i_C. The system corresponding to $(i_B, i_C) = (1,1)$ is not teleported, but measured by Carol in spin z basis. Alice knows the transformation performed on the system which arrives in her channels (i_B, i_C) except for corrections due to the last teleportations of Bob and Carol. So she again assumes that there were no distortion in those, and teleports the system back to Bob after the unitary operation which transforms the eigenstates of the variable to product states in the spin z basis. Alice, Bob and Carol continue the procedure until the desired probability of successful measurement is achieved.

4. Discussion

We have shown that, if the preparation role of an ideal measurement is relaxed, then all Hermitian observables of multipartite quantum system become in principle, measurable instantaneously, providing unlimited entanglement resources and arbitrary local interactions.

This was proved in section 3 in most general terms, and the scheme proposed there, indeed, require very large entanglement resources. Nevertheless, it was shown in section 2, that in some specific cases the measurement protocol may be optimized and consumes a finite amount of ebits. Moreover, the resources differ from operator to operator of the same kind. For example, the measurement of the bipartite operator (6) in 2×2 Hilbert space with $\alpha = \pi/2$ require one ebit, the measurement of the operator of the same kind, but with $\alpha = \pi/4$ require two ebits. In general, if α may be presented as $\pi k/2^n$, then n ebits are sufficient to perform the measurement. This implies that the measurement of another operator of this type with infinitesimally different parameter $\alpha' = \alpha + d\alpha$, where $d\alpha \to 0$, requires an additional number of ebits $\Delta n \to \infty$! Thus, the number of ebits depends sharply on the parameter α. It will be interesting to answer the following questions: is n the minimal number of ebits needed to measure the operator (6)? If not, what is the minimum and which method of measurement achieves that? On the other hand, if n really is the minimum, which basic physical principle stands behind this?

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