

WEAK MEASUREMENTS

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1. PRE- AND POST-SELECTED QUANTUM SYSTEMS

In 1964 Aharonov, Bergmann and Lebowitz¹ considered measurements performed on a quantum system between two other measurements, results of which were given. They proposed describing the quantum system between two measurements by using two states: the usual one, evolving towards the future from the time of the first measurement, and a second state evolving backwards in time, from the time of the second measurement. If a system has been prepared at time t_1 in a state $|\Psi_1\rangle$ and is found at time t_2 in a state $|\Psi_2\rangle$, then at time t , $t_1 < t < t_2$, the system is described by

$$\langle\Psi_2|e^{i\int_{t_2}^t H dt} \quad \text{and} \quad e^{-i\int_{t_1}^t H dt}|\Psi_1\rangle.$$

For simplicity, we shall consider the free Hamiltonian to be zero; then the system at time t is described by the two states $\langle\Psi_2|$ and $|\Psi_1\rangle$, see Fig. 1. In order to obtain such a system we prepare an ensemble of systems in the state $|\Psi_1\rangle$, perform measurement of desired variable using separate measuring devices for each system in the ensemble, and perform the post-selection measurement. If the outcome of the post-selection was not the desired result, we discard the system and corresponding measuring device. We look only on measuring devices corresponding to the systems post-selected in the state $\langle\Psi_2|$.

The basic concepts of the two-state approach, weak measurement and weak values, were developed several years ago.^{2,3} The weak value of any physical variable A in the time interval between pre-selection of the state $|\Psi_1\rangle$ and post-selection of the state $|\Psi_2\rangle$ is given by

$$A_w \equiv \frac{\langle\Psi_2|A|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle} \quad (1)$$

Let us present the main idea by way of a simple example. We consider, at time t , a quantum system which was prepared at time t_1 in the state $|B=b\rangle$ and was found at time t_2 in the state $|C=c\rangle$, $t_1 < t < t_2$. The measurements at times t_1 and t_2 are

complete measurements of, in general, noncommuting variables B and C . The free Hamiltonian is zero, and therefore, the first quantum state at time t is $|B = b\rangle$. In the two-state approach we characterize the system at time t by backwards-evolving state $\langle C = c|$ as well. Our motivation for including the future state is that we know that if a measurement of C has been performed at time t then the outcome is $C = c$ with probability 1. This intermediate measurement, however, destroys our knowledge that $B = b$, since the coupling of the measuring device to the variable C can change B . The idea of weak measurements is to make the coupling with the measuring device sufficiently weak so B does not change. In fact, we require that both quantum states do not change, neither the usual one $|B = b\rangle$ evolving towards the future nor $\langle C = c|$ evolving backwards.

During the whole time interval between t_1 and t_2 , both $B = b$ and $C = c$ are true (in some sense). But then, $B + C = b + c$ must also be true. The latter statement, however, might not have meaning in the standard quantum formalism because the sum of the eigenvalues $b + c$ might not be an eigenvalue of the operator $B + C$. An attempt to measure $B + C$ using a standard measuring procedure will lead to some change of the two quantum states and thus the outcome will not be $b + c$. A weak measurement, however, will yield $b + c$.

When the "strong" value of an observable is known with certainty, i.e., we know the outcome of an ideal (infinitely strong) measurement with probability 1, the weak value equal to the strong value. Let us analyze the example above. The strong value of B is b , its eigenvalue. The strong value of C is c , as we know from *retrodiction*. From the definition (1) immediately follows: $B_w = b$ and $C_w = c$. But weak values, unlike strong values, are defined not just for B and C but for *all* operators. The strong value of the sum $B + C$ when $[B, C] \neq 0$ is not defined, but the weak value of the sum is: $(B + C)_w = b + c$.

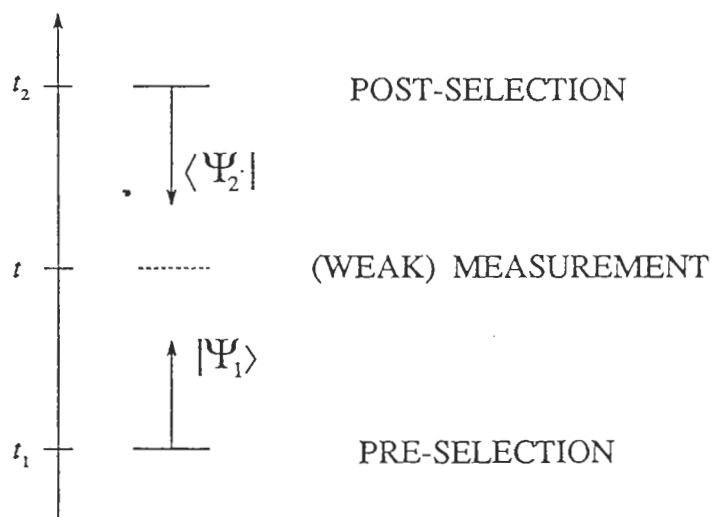


Figure 1. Pre- and post-selected quantum system. The system is considered only if it was found at time t_2 in the state $|\Psi_2\rangle$ after being pre-selected at time t_1 in the state $|\Psi_1\rangle$. The two states yield proper description of the system for analysis of various measurements at time t .

2. QUANTUM MEASURING PROCEDURE

In the standard approach to measurements in quantum theory, we measure observables which correspond to Hermitian operators. The latter have eigenvalues and a (good) measurement must yield one of these eigenvalues. If the state of a quantum system is not an eigenstate of the measured operator, then one can predict only probabilities for different outcomes of the measurement. The state of the system invariably "collapses" to an outcome corresponding to one eigenvalue. A standard measurement of a variable A is modeled in the von Neumann theory of measurement⁴ by a Hamiltonian

$$H = g(t)PA \quad , \quad (2)$$

where P is a canonical momentum, conjugate to the pointer variable Q of the measuring device. The function $g(t)$ is nonzero only for a very short time interval corresponding to the measurement, and is normalized so that $\int g(t)dt = 1$. During the time of this impulsive measurement, the Hamiltonian (2) dominates the evolution of the measured system and the measuring device. Since $[A, H] = 0$, the variable A does not change during the measuring interaction. The initial state of the pointer variable is usually modeled by a Gaussian centered at zero:

$$\Phi_{in}(Q) = (\Delta^2\pi)^{-1/4} e^{-Q^2/2\Delta^2} . \quad (3)$$

Therefore, if the initial state of the system is a superposition $|\Psi_1\rangle = \sum \alpha_i |a_i\rangle$, then after the interaction (2) the state of the system and the measuring device is:

$$(\Delta^2\pi)^{-1/4} \sum \alpha_i |a_i\rangle e^{-(Q-a_i)^2/2\Delta^2} . \quad (4)$$

If the separation between various eigenvalues a_i is much larger than the width of the Gaussian Δ , we obtain strict correlation between the values of the variable A and nearly orthogonal states of the measuring device. The measuring procedure continues with an amplification scheme which yields effective (or, according to some physicists, real) collapse to one of the pointer positions and the corresponding eigenstate $|a_i\rangle$. In this model the only possible outcomes of the measurement of the quantum variable A are the eigenvalues a_i . This fact perfectly matches the premise that the only values which can be associated with A are the a_i .

When a quantum system is in a state $|\Psi\rangle$, one can associate, mathematically, to a variable A the value $\langle\Psi|A|\Psi\rangle$. However, it was commonly believed that $\langle\Psi|A|\Psi\rangle$ is unmeasurable for a single system, and it has physical meaning only for an ensemble of identical systems prepared in the state $|\Psi\rangle$. For the ensemble, $\langle\Psi|A|\Psi\rangle$ is interpreted as statistical average over the results of measuring A on this ensemble. We, however, claim that $\langle\Psi|A|\Psi\rangle$ is more than just a statistical concept. It has physical meaning, since it can be measured directly on a single system (and not just calculated from the statistics of the results a_i).

It is clear why standard measurements cannot yield $\langle\Psi|A|\Psi\rangle$. The expectation value is a property of the quantum state and the state itself is significantly changed during the measuring interaction (2). Thus, in order to obtain $\langle\Psi|A|\Psi\rangle$ on a single system we need a *weak* coupling to the measuring device. And indeed, under certain conditions, including a weakened coupling, von Neumann procedure for a measurement of A yields $\langle\Psi|A|\Psi\rangle$. This procedure is discussed in our other lecture.⁵ Here we want to discuss weak measurements on pre- and post-selected quantum systems. The outcomes of such measurements are the weak values defined in Eq. (1).

3. THE WEAK VALUE IS THE OUTCOME OF WEAK MEASUREMENTS

The system at time t in a pre- and post-selected ensemble is defined by two states, the usual one evolving from the time of the preparation and the state evolving backwards in time from the post-selection. We may neglect the free Hamiltonian if the time between the pre-selection and the post-selection is very short. Consider a system which has been pre-selected in a state $|\Psi_1\rangle$ and shortly afterwards post-selected in a state $|\Psi_2\rangle$. The weak value of any physical variable A in the time interval between the pre-selection and the post-selection is given by Eq. (1). Let us show briefly how weak values emerge from a measuring procedure with a sufficiently weak interaction.

We consider a sequence of measurements: a pre-selection of $|\Psi_1\rangle$, a (weak) measurement interaction of the form of Eq. (2), and a post-selection measurement finding the state $|\Psi_2\rangle$. The state of the measuring device after this sequence is given (up to normalization) by

$$\Phi(Q) = \langle \Psi_2 | e^{-iPA} | \Psi_1 \rangle e^{-Q^2/2\Delta^2} \quad (5)$$

After simple algebraic manipulation we can rewrite it (in the P -representation) as

$$\begin{aligned} \tilde{\Phi}(P) = & \langle \Psi_2 | \Psi_1 \rangle e^{-iA_w P} e^{-\Delta^2 P^2/2} \\ & + \langle \Psi_2 | \Psi_1 \rangle \sum_{n=2}^{\infty} \frac{(iP)^n}{n!} [(A^n)_w - (A_w)^n] e^{-\Delta^2 P^2/2} \quad (6) \end{aligned}$$

If Δ is sufficiently large, then we can neglect the second term of (6) when we Fourier transform back to the Q -representation. Large Δ corresponds to weak measurement in the sense that the interaction Hamiltonian (2) is small. Thus, in the limit of weak measurement, the final state of the measuring device (in the Q -representation) is

$$\Phi(Q) = (\Delta^2 \pi)^{-1/4} e^{-(Q-A_w)^2/2\Delta^2} \quad (7)$$

This state represents a measuring device pointing to the weak value, A_w .

Weak measurements on pre- and post-selected ensembles yield, instead of eigenvalues, a value which might lie far outside the range of the eigenvalues. Although we have showed this result for a specific von Neumann model of measurements, the result is completely general: any coupling of a pre- and post-selected system to a variable A , provided the coupling is sufficiently weak, results in effective coupling to A_w . This weak coupling between a single system and the measuring device will not, in most cases, lead to a distinguishable shift of the pointer variable, but collecting the results of measurements on an ensemble of pre- and post-selected systems will yield the weak values of a measured variable to any desired precision.

When the strength of the coupling to the measuring device goes to zero, the outcomes of the measurement invariably yield the weak value. To be more precise, a measurement yields the real part of the weak value. Indeed, the weak value is, in general, a complex number, but its imaginary part will contribute only a phase to the wave function of the measuring device in the position representation of the pointer. Therefore, the imaginary part will not affect the probability distribution of the pointer position which is what we see in a usual measurement. However, the imaginary part of the weak value also has physical meaning. It expresses itself as a change in the conjugate momentum of the pointer variable.³

4. AN EXAMPLE: SPIN MEASUREMENT

Let us consider a simple the Stern-Gerlach experiment: measurement of a spin component of a spin-1/2 particle. We shall consider a particle prepared in the initial state spin “up” in the \hat{x} direction and post-selected to be “up” in the \hat{y} direction. At the intermediate time we measure, weakly, the spin component in the $\hat{\xi}$ direction which is bisector of \hat{x} and \hat{y} , i.e., $\sigma_{\xi} = (\sigma_x + \sigma_y)/\sqrt{2}$. Thus $|\Psi_1\rangle = |\uparrow_x\rangle$, $|\Psi_2\rangle = |\uparrow_y\rangle$, and the weak value of σ_{ξ} in this case is:

$$(\sigma_{\xi})_w = \frac{\langle \uparrow_y | \sigma_{\xi} | \uparrow_x \rangle}{\langle \uparrow_y | \uparrow_x \rangle} = \frac{1}{\sqrt{2}} \frac{\langle \uparrow_y | (\sigma_x + \sigma_y) | \uparrow_x \rangle}{\langle \uparrow_y | \uparrow_x \rangle} = \sqrt{2} . \quad (8)$$

This value is, of course, “forbidden” in the standard interpretation where a spin component can obtain the (eigen)values ± 1 only.

An effective Hamiltonian for measuring σ_{ξ} is

$$H = g(t)P\sigma_{\xi} . \quad (9)$$

Writing the initial state of the particle in the σ_{ξ} representation, and assuming the initial state (3) for the measuring device, we obtain that after the measuring interaction the quantum state of the system and the pointer of the measuring device is

$$\cos(\pi/8)|\uparrow_{\xi}\rangle e^{-(Q-1)^2/2\Delta^2} + \sin(\pi/8)|\downarrow_{\xi}\rangle e^{-(Q+1)^2/2\Delta^2} . \quad (10)$$

The probability distribution of the pointer position, if it is observed now without post-selection, is the sum of the distributions for each spin value. It is, up to normalization,

$$\text{prob}(Q) = \cos^2(\pi/8)e^{-(Q-1)^2/\Delta^2} + \sin^2(\pi/8)e^{-(Q+1)^2/\Delta^2} . \quad (11)$$

In the usual strong measurement $\Delta \ll 1$. In this case, as shown on Fig. 2a, probability distribution of the pointer is localized around -1 and $+1$ and it is strongly correlated to the values of the spin, $\sigma_z = \pm 1$.

Weak measurement corresponds to Δ which is much larger than the range of the eigenvalues, i.e., $\Delta \gg 1$. Fig. 2b shows that the pointer distribution has a large uncertainty, but it is peaked between the eigenvalues, more precisely, at the expectation value $\langle \uparrow_x | \sigma_{\xi} | \uparrow_x \rangle = 1/\sqrt{2}$. An outcome of an individual measurement usually will not be close to this number, but it can be found from an ensemble of such measurements, see Fig. 2c. Note, that we have not yet considered the post-selection.

In order to simplify the analysis of measurements on the pre- and post-selected ensemble, let us assume that we first make the post-selection of the spin of the particle and only then look on the pointer of the device that weakly measures σ_{ξ} . We must get the same result as if we first look at the outcome of the weak measurement, make the post-selection, and discard all readings of the weak measurement corresponding to the cases in which the result is not $\sigma_y = 1$. The post-selected state of the particle in the σ_{ξ} representation is $|\uparrow_y\rangle = \cos(\pi/8)|\uparrow_{\xi}\rangle - \sin(\pi/8)|\downarrow_{\xi}\rangle$. The state of the measuring device after the post-selection of the spin state is obtained by projection of (10) onto the post-selected state:

$$\Phi(Q) = \mathcal{N} \left(\cos^2(\pi/8)e^{-(Q-1)^2/2\Delta^2} - \sin^2(\pi/8)e^{-(Q+1)^2/2\Delta^2} \right) , \quad (12)$$

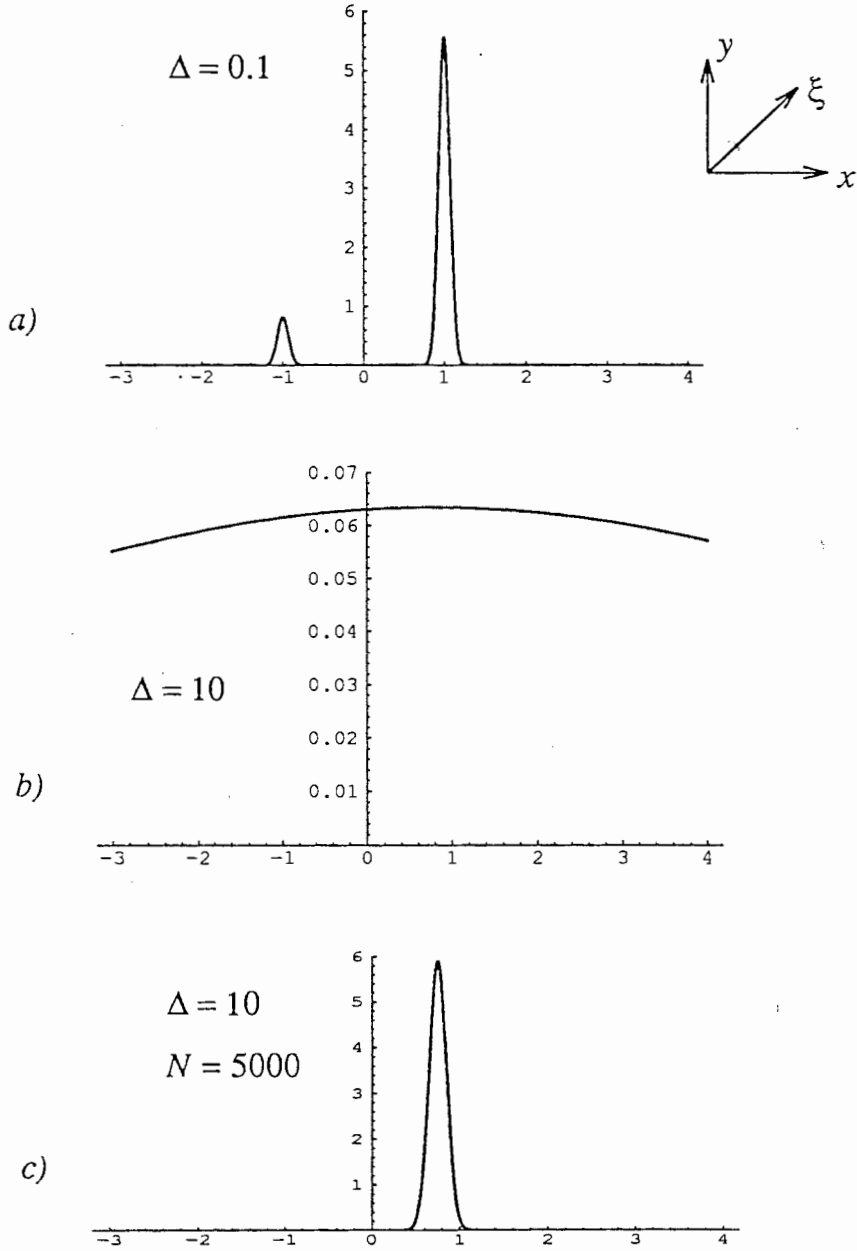


Figure 2. Spin component measurement without post-selection. Probability distribution of the pointer variable for measurement of σ_ξ when the particle is pre-selected in the state $|\uparrow_x\rangle$. (a) Strong measurement, $\Delta = 0.1$. (b) Weak measurement, $\Delta = 10$. (c) Weak measurement on the ensemble of 5000 particles. The original width of the peak, 10, is reduced to $10/\sqrt{5000} \simeq 0.14$. In the strong measurement (a) the pointer is localized around the eigenvalues ± 1 , while in the weak measurements (b) and (c) the peak is located in the expectation value $\langle \uparrow_x | \sigma_\xi | \uparrow_x \rangle = 1/\sqrt{2}$.

where \mathcal{N} is a normalization factor. The probability distribution of the pointer variable is given by

$$\text{prob}(Q) = \mathcal{N}^2 \left(\cos^2(\pi/8) e^{-(Q-1)^2/2\Delta^2} - \sin^2(\pi/8) e^{-(Q+1)^2/2\Delta^2} \right)^2. \quad (13)$$

If the measuring interaction is strong, $\Delta \ll 1$, then the distribution is localized around the eigenvalues ± 1 (mostly around 1 since the pre- and post-selected probability to find $\sigma_x = 1$ is more than 85%), see Figs. 3a, 3b. But when the strength of the coupling is weakened, i.e., Δ is increased, the distribution gradually changes to a single broad peak around $\sqrt{2}$, the weak value, see Figs. 3c – 3e.

The width of the peak is large and therefore each individual reading of the pointer usually will be pretty far from $\sqrt{2}$. The physical meaning of the weak value, in this case, can be associated only with an ensemble of pre- and post-selected particles. The accuracy of defining the center of the distribution goes as $1/\sqrt{N}$, so increasing N , the number of particles in the ensemble, we can find the weak value with any desired precision, see Fig. 3f.

In our example, the weak value of the spin component is $\sqrt{2}$, which is only slightly more than the maximal eigenvalue, 1. By appropriate choice of the pre- and post-selected states we can get pre- and post-selected ensembles with arbitrarily large weak value of a spin component. One of the first proposals⁶ was to obtain $(\sigma_x)_w = 100$. In this case the post-selected state is nearly orthogonal to the pre-selected state and, therefore, the probability to obtain appropriate post-selection becomes very small. While in the case of $(\sigma_x)_w = \sqrt{2}$ the (pre- and) post-selected ensemble was just half of the pre-selected ensemble, in the case of $(\sigma_x)_w = 100$ the post-selected ensemble will be smaller than the original ensemble by the factor of $\sim 10^{-4}$.

5. WEAK MEASUREMENTS ON A SINGLE SYSTEM

We have shown that weak measurements can yield very surprising values which are far from the range of the eigenvalues. However, the uncertainty of a single weak measurement (i.e., performed on a single system) in the above example is larger than the deviation from the range of the eigenvalues. Each single measurement separately yields almost no information and the weak value arises only from the statistical average on the ensemble. The weakness and uncertainty of the measurement goes together. Weak measurement corresponds to small value of P in the Hamiltonian (2) and, therefore, the uncertainty in P has to be small. This requires large Δ , the uncertainty of the pointer variable. Of course, we can construct measurement with large uncertainty which is not weak at all, for example, by preparing the measuring device in a mixed state instead of a Gaussian, but no precise measurement with weak coupling is possible. So, usually, a weak measurement on a single system will not yield the weak value with a good precision. However, there are special cases when it is not so. Usual strength measurement on a single pre- and post-selected system can yield “unusual” (very different from the eigenvalues) weak value with a good precision. Good precision means that the uncertainty is much smaller than the deviation from the range of the eigenvalues.

Our example above was not such a case. The weak value $(\sigma_x)_w = \sqrt{2}$ is larger than the highest eigenvalue, 1, only by ~ 0.4 , while the uncertainty, 1, is not sufficiently large for obtaining the peak of the distribution near the weak value, see Fig. 3c. Let us modify our experiment in such a way that a single experiment will yield meaningful surprising result.

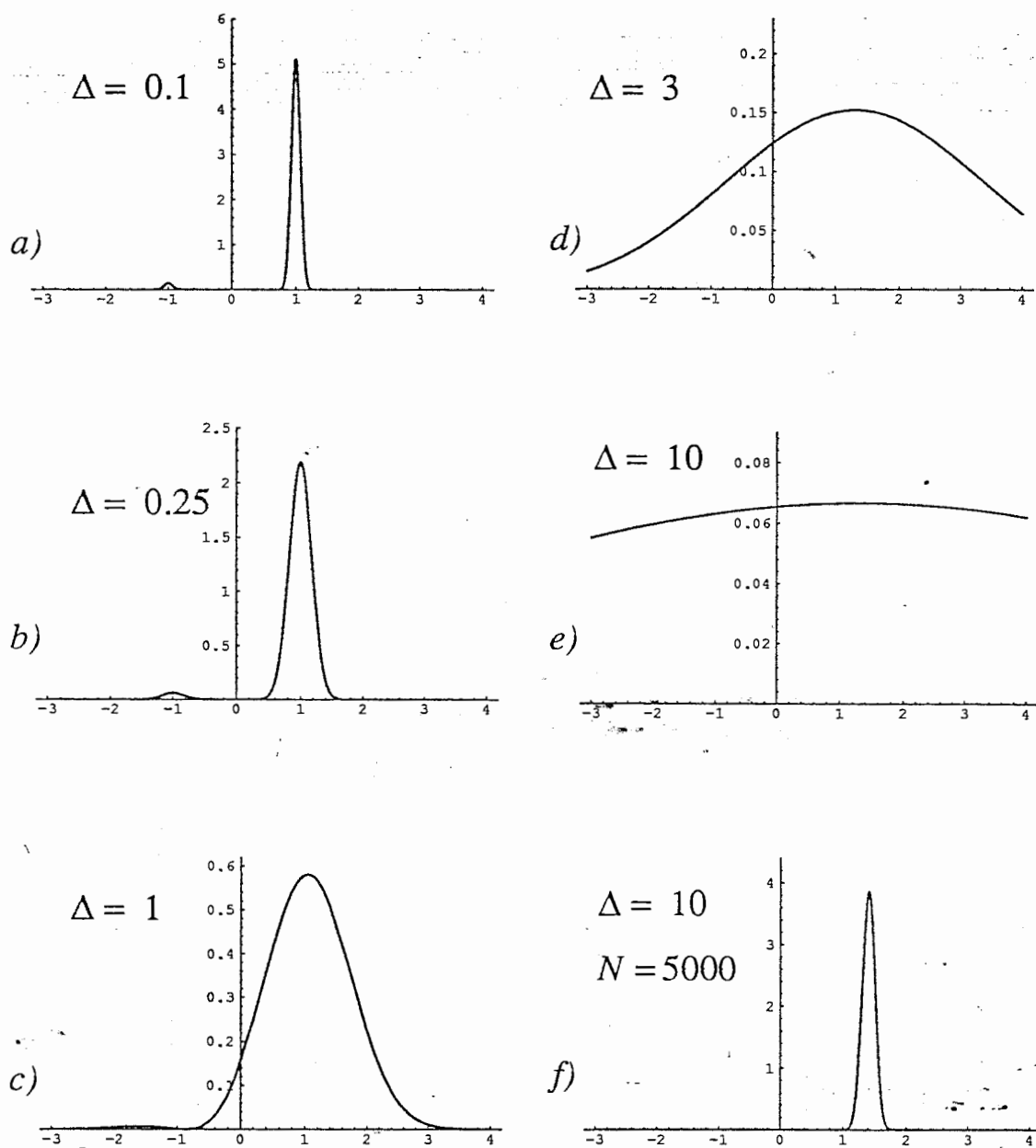


Figure 3. Measurement on pre- and post-selected ensemble. Probability distribution of the pointer variable for measurement of σ_x when the particle is pre-selected in the state $|\uparrow_x\rangle$ and post-selected in the state $|\uparrow_y\rangle$. The strength of the measurement is parameterized by the width of the distribution Δ . (a) $\Delta = 0.1$; (b) $\Delta = 0.25$; (c) $\Delta = 1$; (d) $\Delta = 3$; (e) $\Delta = 10$. (f) Weak measurement on the ensemble of 5000 particles; the original width of the peak, $\Delta = 10$, is reduced to $10/\sqrt{5000} \simeq 0.14$. In the strong measurements (a)-(b) the pointer is localized around the eigenvalues ± 1 , while in the weak measurements (d)-(f) the peak of the distribution is located in the weak value $(\sigma_x)_w = \langle \uparrow_y | \sigma_x | \uparrow_x \rangle / \langle \uparrow_y | \uparrow_x \rangle = \sqrt{2}$. The outcomes of the weak measurement on the ensemble of 5000 pre- and post-selected particles, (f), are clearly outside the range of the eigenvalues, $(-1, 1)$.

We consider a system of N spin-1/2 particles all prepared in the state $|\uparrow_x\rangle$ and post-selected in the state $|\uparrow_y\rangle$, i.e., $|\Psi_1\rangle = \prod_{i=1}^N |\uparrow_x\rangle_i$ and $|\Psi_2\rangle = \prod_{i=1}^N |\uparrow_y\rangle_i$. The variable which is measured at the intermediate time is $A \equiv (\sum_{i=1}^N (\sigma_i)_x)/N$. The operator A has $N+1$ eigenvalues equally spaced between -1 and $+1$, but the weak value of A is

$$A_w = \frac{\prod_{k=1}^N \langle \uparrow_y | k \sum_{i=1}^N ((\sigma_i)_x + (\sigma_i)_y) \prod_{j=1}^N |\uparrow_x\rangle_j}{\sqrt{2} N (\langle \uparrow_y | \uparrow_x \rangle)^N} = \sqrt{2} . \quad (14)$$

The interaction Hamiltonian is

$$H = \frac{g(t)}{N} P \sum_{i=1}^N (\sigma_i)_x . \quad (15)$$

The initial state of the measuring device defines the precision of the measurement. When we take it to be the Gaussian (3), it is characterized by the width Δ . For a meaningful experiment we have to take Δ small. Small Δ corresponds to large uncertain P , but now, the strength of the coupling to each individual spin is reduced by the factor $1/N$. Therefore, for large N , both the forward-evolving state and the backward-evolving state are essentially not changed by the coupling to the measuring device. Thus, this single measurement yields the weak value. In Ref. 7 it is proven that if the measured observable is an average on a large set of systems, $A = (\sum_i A_i)/N$, then we can always construct a single, good-precision measurement of the weak value. Here let us present just numerical calculations of the probability distribution of the measuring device for N pre- and post-selected spin-1/2 particles. The state of the pointer after the post-selection for this case is

$$\mathcal{N} \sum_{i=1}^N (-1)^i (\cos^2(\pi/8))^{N-i} (\sin^2(\pi/8))^i e^{-\left(Q - \frac{(2N-i)}{N}\right)^2 / 2\Delta^2} . \quad (16)$$

The probability distribution for the pointer variable Q is

$$\text{prob}(Q) = \mathcal{N}^2 \left(\sum_{i=1}^N (-1)^i (\cos^2(\pi/8))^{N-i} (\sin^2(\pi/8))^i e^{-\left(Q - \frac{(2N-i)}{N}\right)^2 / 2\Delta^2} \right)^2 . \quad (17)$$

The result for $N = 20$ and different values of Δ are presented in Fig. 4. We see that for $\Delta = 0.25$ and larger, the obtained results are very good: the final probability distribution of the pointer is peaked at the weak value, $((\sum_{i=1}^N (\sigma_i)_x)/N)_w = \sqrt{2}$. This distribution is very close to that of a measuring device measuring operator O on a system in an eigenstate $|O=\sqrt{2}\rangle$. For N large, the relative uncertainty can be decreased almost by a factor $1/\sqrt{N}$ without changing the fact that the peak of the distribution points to the weak value.

Although our set of particles pre-selected in one state and post-selected in another state is considered as one system, it looks very much as an ensemble. In quantum theory, measurement of the sum does not necessarily yield the same result as the sum of the results of the separate measurements, so conceptually our measurement on the set of particles differs from the measurement on an ensemble of pre- and post-selected particles. However, in our example of weak measurements, the results are the same.

Less ambiguous case is the example considered in the first work on weak measurements.² In this work a single system of a large spin N is considered. The system is pre-selected in the state $|\Psi_1\rangle = |S_x=N\rangle$ and post-selected in the state $|\Psi_2\rangle = |S_y=N\rangle$.

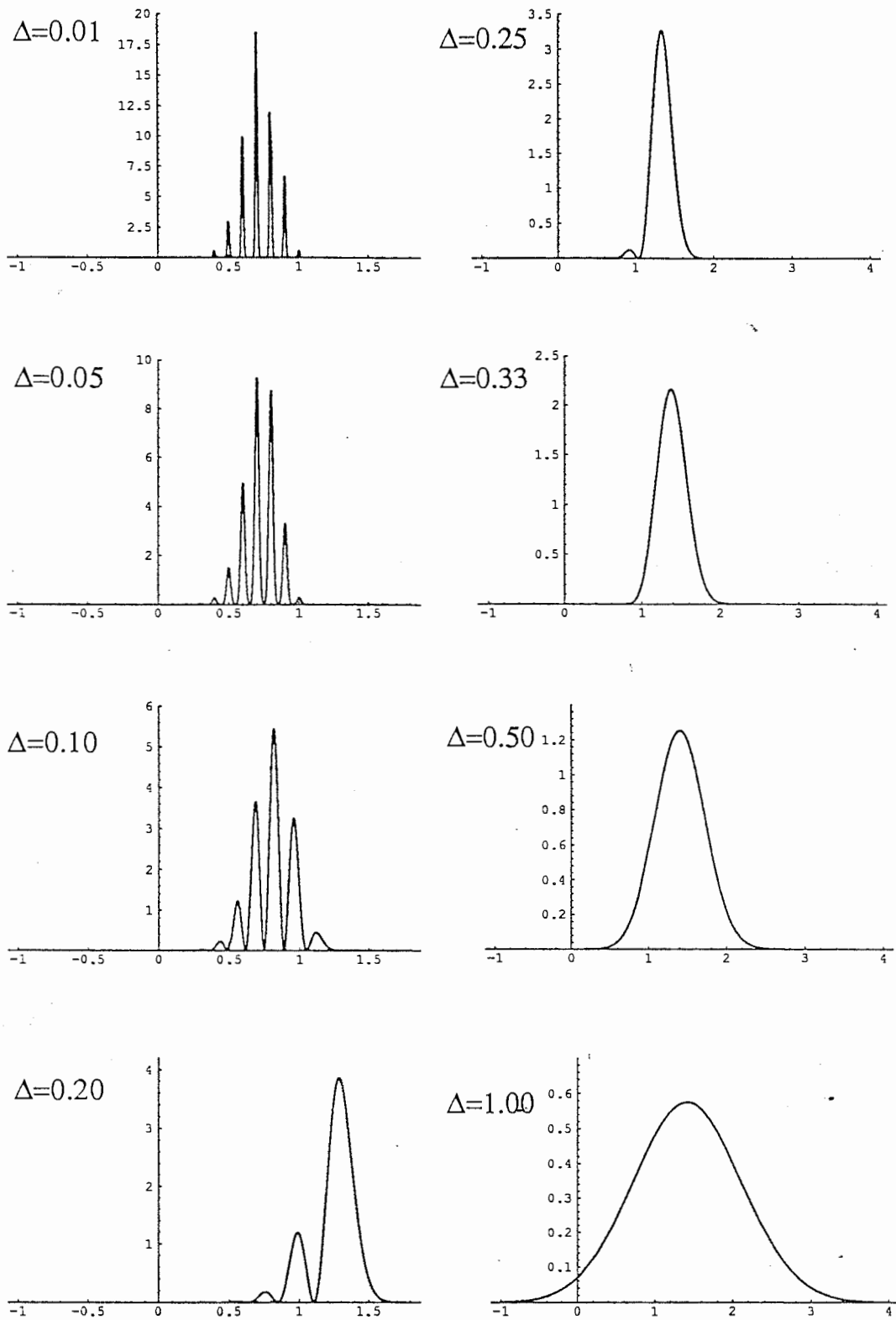


Figure 4. Measurement on a single system. Probability distribution of the pointer variable for measurement of $A = (\sum_{i=1}^{20} (\sigma_i)_x)/20$ when the system of 20 spin-1/2 particles is pre-selected in the state $|\Psi_1\rangle = \prod_{i=1}^{20} |\uparrow_x\rangle_i$ and post-selected in the state $|\Psi_2\rangle = \prod_{i=1}^{20} |\uparrow_y\rangle_i$. The strength of the measurement is parameterized by the width of the distribution Δ . While in the very strong measurements, $\Delta = 0.01 - 0.05$, the peaks of the distribution located at the eigenvalues, starting from $\Delta = 0.25$ there is essentially a single peak at the location of the weak value, $A_w = \sqrt{2}$.

At an intermediate time the spin component S_z is weakly measured and again the "forbidden" value $\sqrt{2}N$ is obtained. The uncertainty has to be only slightly larger than \sqrt{N} . The probability distribution of the results is centered around $\sqrt{2}N$, and for large N it lies clearly outside the range of the eigenvalues, $(-N, N)$. Unruh⁸ made computer calculations of the distribution of the pointer variable for this case and got results which are very similar to what is presented on Fig. 4.

An even more dramatic example is a measurement of the kinetic energy of a tunneling particle.⁹ We consider a particle prepared in a bound state of a potential well which has negative potential near the origin and vanishing potential far from the origin; $|\Psi_1\rangle = |E=E_0\rangle$. Shortly later, the particle is found far from the well, inside a classically forbidden tunneling region; this state can be characterized by vanishing potential $|\Psi_2\rangle = |U=0\rangle$. At an intermediate time a measurement of the kinetic energy is performed. The weak value of the kinetic energy in this case is

$$K_w = \frac{\langle U=0|K|E=E_0\rangle}{\langle U=0|E=E_0\rangle} = \frac{\langle U=0|E - U|E=E_0\rangle}{\langle U=0|E=E_0\rangle} = E_0 \quad (18)$$

The energy of the bound state, E_0 , is negative, so the weak value of the kinetic energy is negative. In order to obtain this negative value the coupling to the measuring device need not be too weak. In fact, for any finite strength of the measurement we can choose the post-selected state sufficiently far from the well to ensure the negative value. Therefore, for appropriate post-selection, usual measurement of a positive definite operator invariably yields negative result!

How do we get this paradoxical outcome? One can interpret it as a game of errors. Any realistic experiment must have errors. Measurement of kinetic energy must have a spread, so sometimes it might show negative outcomes. Of course, the dial of the measuring device might have a pin preventing negative readings, but we consider the device without such a pin. In our pre- and post-selection measurement a peculiar interference effect of the pointer takes place: destructive interference in the whole "allowed" region and constructive interference of the tails in the "forbidden" negative region. The initial state of the measuring device $\Phi(Q)$, due to the measuring interaction and the post-selection, transforms into a superposition of shifted wave functions. The shifts are by the (possibly small) eigenvalues, but the superposition is approximately equal to the original wave function shifted by (large and/or forbidden) weak value

$$\sum_i c_i \Phi(Q - a_i) \simeq \Phi(Q - A_w) \quad (19)$$

The example of a single weak measurement on the system of 20 pre- and post-selected spin-1/2 particles which was considered above demonstrates this effect for a Gaussian wave function of the measuring device, but we have proved⁷ that "miraculous" interference (19) occurs not just for the Gaussians, but for a large class of functions. The only requirement is that their Fourier transform must be essentially bounded.

It is possible to use this idea for constructing a quantum time machine, a device which can make a cat out of a kitten in a minute.^{7,10} The superposition of quantum states shifted by small periods of time can yield a large shift in time; and it even can be a shift to the past.

These surprising, even paradoxical effects are really gedanken experiments. The reason is that, unlike weak measurements on an ensemble, these are extremely rare events. For yielding unusual weak value, a single pre-selected system needs extremely improbable outcome of the post-selection measurement. Let us compare this with a weak measurement on an ensemble. In order to get N particles in a pre- and

post-selected ensemble which yield $(\sigma_\xi)_w = 100$, we need $\sim N10^4$ particles in the pre-selected ensemble. But, in order to get a single system of N particles yielding $(S_\xi)_w = 100N$, we need $\sim 10^{4N}$ systems of N pre-selected particles. In fact, the probability to obtain unusual value by error is much larger than the probability to obtain the proper post-selected state. We will see a negative reading of the device measuring kinetic energy much faster than we will find the particle in a deep tunneling region. What makes these rare effects interesting is that there is strong (although only one-way) correlation: every time we find in the post-selection measurement the particle outside the well, we know that the result of the kinetic energy is negative, and not just negative: it is equal to the weak value, $K_w = E_0$, with a good precision.

It is not that weak measurement on a single pre- and post-selected system cannot be measured in a laboratory. These are the experiments with very dramatic results which are not feasible to perform. But an experiment in which the weak value is only slightly outside the range of the eigenvalues, performed on particles which can be identically prepared in millions, is possible.

Although we call it a weak measurement on a single system, in practice the experiment is performed on a large (pre-selected) ensemble. We prepare many systems, couple each system to a separate measuring device, and make the post-selection measurement waiting for the desired result. This is an experiment on a pre-selected ensemble, but it is an experiment on a *single* pre- and post-selected system. Indeed, we discard everything connected to other systems. Only the reading of the measuring device of one system is considered. If we are "lucky" and the first particle gets the right result in the post-selection measurement, then the experiment is completed, and only one system has been involved. This property does not hold for usual measurement on an ensemble: even if by chance the first result yields exactly the measured expectation value, we cannot stop here because we cannot know yet that this is, indeed, the correct value.

6. EXPERIMENTAL REALIZATIONS OF WEAK MEASUREMENTS

Weak measurements have three basic elements: preparation, (weak) coupling to the measuring device, and post-selection. The preparation part is the same as in all usual experiments, so it does not require any special consideration except that we need, for getting large effects, a very good precision in the preparation of quantum state. The second stage too does not present special experimental difficulties: this is a standard measuring procedure with weakened coupling. What limits the feasibility of a weak measurement is the possibility of an effective post-selection. In order to obtain interesting results in weak measurement, the post-selection needs to be very precise, and it has to fulfill a special requirement specified below.

Realistic weak measurements (on an ensemble) involve preparation of a large pre-selection ensemble, coupling to the measuring devices of each element of the ensemble, post-selection measurement which, in all interesting cases, selects only a small fraction of the original ensemble, selection of corresponding measuring devices, and statistical analysis of their outcomes. In order to obtain good precision, this selected ensemble of the measuring devices has to be sufficiently large. Although there are significant technological developments in "marking" particles running in an experiment, clearly the most effective solution is that the particles themselves serve as measuring devices. The information about measured variable is stored, after the weak measuring interaction, in their other degree of freedom. In this case the post-selection of the

particles in the required final state automatically yield selection of measuring devices. The requirement for the post-selection measurement is, then, that there is no coupling between the variable in which the result of the weak measurement is stored and the post-selection device.

An example of such a case is the Stern-Gerlach experiment where the shift in the momentum of a particle, translated into a spatial shift, yields the outcome of the spin measurement. Post-selection measurement on a spin in a certain direction can be implemented by another (this time strong) Stern-Gerlach coupling which splits the beam of the particles. The beam corresponding to the desired value of the spin is then analyzed for the result of the weak measurement. The requirement of non-disturbance of the results of the weak measurement by post-selection can be fulfilled by arranging the shifts due to the two Stern-Gerlach devices to be orthogonal to each other. The details are spelled out in Ref. 6.

A weak measurement of a spin component is very difficult to perform in a laboratory. We need very precise pre- and post-selection of spin polarization and Stern-Gerlach experiment is very far from being precise. But analogous experiments can be performed on other two-state systems. The simplest analog of the Stern-Gerlach measurement is an optical polarization experiment. A birefringent prism splits an optical beam according to its polarization modeling the inhomogeneous magnetic field which splits the beam of particles with a spin. And high precision polarization filters serve as excellent devices for pre- and post-selection. We can define a polarization operator

$$Q \equiv |x\rangle\langle x| - |y\rangle\langle y|, \quad (20)$$

where $|x\rangle$ and $|y\rangle$ designate photon linear polarization states. The eigenstates of the polarization operator are ± 1 but if the initial state is $|\Psi_1\rangle = \cos \alpha |x\rangle + \sin \alpha |y\rangle$, and the final state is $|\Psi_2\rangle = \cos \alpha |x\rangle - \sin \alpha |y\rangle$, then the weak value of the polarization operator is

$$Q_w = \frac{(\cos \alpha \langle x| - \sin \alpha \langle y|) (|x\rangle\langle x| - |y\rangle\langle y|) (\cos \alpha |x\rangle + \sin \alpha |y\rangle)}{\cos^2 \alpha - \sin^2 \alpha} = \frac{1}{\cos(2\alpha)} \quad (21)$$

The initial and final states are chosen by placing an appropriate linear polarization filters. If the polarizations are almost orthogonal, $\alpha \simeq \pi/4$, the weak value of the polarization operator becomes arbitrary large.

An analysis of realistic experiment which can yield large weak value Q_w appears in Ref. 11. Duck, Stevenson, and Sudarshan¹² proposed slightly different optical realization which uses birefringent plate instead of a prism. In this case the measured information is stored directly in the spatial shift of the beam without being generated by the shift in the momentum. Ritchie, Story, and Hulet adopted these scheme and performed the first successful experiment measuring weak value of the polarization operator.¹³ Their results are in very good agreement with theoretical predictions. They obtained weak values which are very far from the range of the eigenvalues, $(-1, 1)$, their highest reported result is $Q_w = 100$. The discrepancy between calculated and observed weak value was 1%. The RMS deviation from the mean of 16 trials was 4.7%. The width of the probability distribution was $\Delta = 1000$ and the number of pre- and post-selected photons was $N \sim 10^8$, so the theoretical and experimental uncertainties were of the same order of magnitude. Their other run, for which they showed experimental data on graphs (which fitted very nicely theoretical graphs), has the following characteristics: $Q_w = 31.6$, discrepancy with calculated value 4%, the RMS deviation 16%, $\Delta = 100$, $N \sim 10^5$.

Suter, M. Ernst and R. Ernst reported experimental realization of quantum time-translation machine.¹⁴ We may disagree about their experiment being a model of our proposed time machine,^{7,10} but it seems that they indeed performed a weak measurement. The experiment was performed on $^{13}\text{C} - ^1\text{H}$ spin pair of chloroform. The heteronuclear J coupling of the two spins S and I is given by

$$H_{SI} = -2\pi JS_z I_z . \quad (22)$$

For a particular state of the spin I , the spin S precesses due to this spin-spin interaction. In the experiment, an appropriate pre- and post-selection on the states of spin I were performed, and it was observed that the spin S precession was 4 times faster than the one corresponding to the maximal eigenvalue of the second spin, $I_z = 1$. Rather than interpreting it as a time machine for S we see this experiment as a weak measurement of I_z , the measurement in which the weak value is 4 times larger than the maximal eigenvalue, $(I_z)_w = 4$.

There are numerous experiments on pre- and post-selected systems. Post-selection might lead to very dramatic effects pointed out as early as 1935 by Schrödinger.¹⁵ Not all measurements on pre- and post-selected systems are weak measurements. Since some "weak" measurements are not really weak, and some weak couplings are really strong measurements, it is not easy to find a rigorous definition for weak measurements. A possible criterion is that measurements yielding consistently weak values are weak measurements. Thus, another run of the experiment of Ritchie, Story, and Hulet¹³ which shows dramatic effect is not a weak measurement. They considered a post-selection to a state which is orthogonal to the initial state. The probability for this post-selection was not zero due to the intermediate weak coupling. However, since $|\Psi_2\rangle$ is orthogonal to $|\Psi_1\rangle$, the weak value is not defined in this case.

Another system which is a good candidate for weak measurements, due to a well developed technology of preparation and selection of various quantum states, is a Rydberg two-level atom. Between the pre- and post-selection the atom can have weak coupling with a resonant field in a microwave cavity.^{16,17}

There are many experiments measuring escape time of tunneling particles. Tunneling is a pre- and post-selection experiment: a particle is pre-selected inside the bounding potential and post-selected outside. Recently, Steinberg¹⁸ suggested that many of these experiments are indeed weak measurements.

We believe that the field of experimental realization of weak measurements is far from being exhausted. The next section explains the potential applications of this procedure.

6. WEAK MEASUREMENT AS AN AMPLIFICATION SCHEME

Let us consider an experiment of a weak measurement of A not as a measurement of the weak value of A but as a measurement of a certain parameter of the measuring device. Indeed, when we consider known initial state $|\Psi_1\rangle$ and known final state $|\Psi_2\rangle$, the weak value (1) is known prior to the measurement, and our experiment yields no new information. But we can perform the weak measuring procedure when the strength of the weak coupling is not known. Then, from the result of the weak measurement we can find the strength of the coupling.

The Hamiltonian of the Stern-Gerlach experiment measuring z component of a spin is

$$H = -g(t)\mu \frac{\partial B_z}{\partial z} z \sigma_z . \quad (23)$$

We can prepare the state of the spin, $\sigma_z = 1$, then, assuming that the gradient of the magnetic field is known, our experiment is a von Neumann measurement of the magnetic moment μ . (Or, if μ is known, it is a measurement of the gradient.) Indeed, Eq. (23) has the form of Eq. (2) with P replaced by $-z$ which corresponds to the pointer variable p_z . The shift in the momentum is later transformed into the shift in the position of the particle. If we perform, instead, the pre- and post-selected measurement with the initial and final states of the spin corresponding to, for example, $(\sigma_z)_w = 100$, than our procedure is a measurement of μ which is 100 times more sensitive! The shift in the peak of the pointer position distribution is 100 times larger, while the width of the peak is practically unchanged.

Of course, for increasing the shift of the pointer we have to pay some price. First, we cannot work with narrow peaks. We have to be in a regime of weak measurements, so the initial distribution of z has to be well localized around zero, and this requires wide pointer distribution (p_z). And second, we lose the intensity. For obtaining amplification by factor M we need a post-selection which will reduce the number of systems of the pre-selected ensemble by the factor of $1/M^2$, so in our example the intensity will be reduced by the factor 10^{-4} .

Still, we believe that there are measurements for which this amplification scheme might be helpful. In many experiments intensity is not a problem. If the output of the measurement is a picture on a photographic plate, then the restriction is on the total number of photons which were absorbed by the plate (before it was saturated) while the number of pre-selected photons coming from a light source is practically unlimited. This is the situation in the optical analog of the Stern-Gerlach experiment discussed above. In optical experiment, instead of the magnetic moment μ , we measure the degree of optical activity of the crystal, i.e., the difference between indices of refraction for orthogonal polarization, $n_x - n_y$. It seems that if one wants to measure this difference using a birefringent prism and the given light beam is suitable for weak measurements, then the post-selection certainly increases the precision of the measurement. If, however, our equipment allows us to make the incoming beam well collimated, then the measurement without post-selection has an advantage since it is easier to find the center of a more narrow peak. The analysis of an optimal measurement has not been performed yet. But irrespective of the results of this theoretical analysis, we are convinced that for realistic tasks when the equipment is given, the scheme of weak pre- and post-selected measurement will prove itself useful for improvement of the sensitivity of some measurements.

6. SUMMARY

Weak measurement is a certain measuring procedure which includes post-selection. There are many peculiar effects due to various post-selections, but weak measurements play a special role among them. The outcomes of weak measurements, weak values, are not just peculiar because they are very different from the outcomes of standard measurements: they are part of new simple and rich structure existing in quantum world. The concept of weak values is simple and universal. Weak values are defined for all variables and for all possible histories of quantum systems. They manifest themselves in all couplings which are sufficiently weak.

The two basic elements of our approach were investigated separately. The theory of "unsharp" measurements developed by Bush¹⁹ has the element of weakness of the interaction. Popular today, the "consistent histories" approach,²⁰ originated by

Griffiths,²¹ includes the idea of pre- and post-selection. But it is the combination of the two which created our formalism. It allowed us to see peculiar features of quantum systems. The quantum time machine, the method of increasing sensitivity using post-selection, and other surprising phenomena were inaccessible within the framework of the standard formalism. Neither consistent histories nor unsharp measurements provided tools to see these effects, although they might be helpful for analyzing these phenomena.²²

The formalism of weak measurement can also be helpful in describing existing peculiar effects. The controversy of superluminal motion of tunneling particles can be resolved by recognizing that the experiments showing superluminal motion are weak measurements.¹⁸ We have showed how, under conditions of weak measurements, the post-selection leads to superluminal motion of light wave packets (Sec. VIII of Ref. 3).

Among applications of the weak value concept is a proposal to study the back reaction of a quantum field on a particle-antiparticle pair created by the field.²³ The weak value of the field is considered between the (initial) vacuum state and the (final) state which includes the particle-antiparticle pair. The aim of this proposal is the analysis of particle creation by a black hole and the problem of what happens in the final stages of black hole evaporation.^{24,25} One key to this problem is the back reaction of the pair to the gravitational field that created it, and here the application of weak values signals the possibility of a major breakthrough.

We have shown that the concepts of weak measurements and weak values are useful tools. But one can speculate that it has more meaning than this.^{26,27} The formalism of weak values is a candidate for describing reality of quantum systems. It is well known that the usual concepts of reality developed from centuries of thoughts based on classical physics fail to describe our world which includes observed quantum phenomena. It has been shown that there are severe difficulties in defining relativistically invariant elements of reality in quantum theory. Weak values are Lorentz invariant. So, this might be the right course of action to identify weak values as elements of reality.²⁸ The weak values include the elements of reality as defined by Einstein, Podolsky and Rosen²⁹ as well as those recently defined by Redhead.³⁰ In addition to universal applicability, weak values do have desirable features such as the sum rule: if $C = A + B$ then $C_w = A_w + B_w$. However, there is no product rule: $C = AB$ does not lead to $C_w = A_w B_w$. But may be this is how our world really is.³¹

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