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Analytical solutions of the nonlinear force-free magnetic field equations

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Abstract. An analytical method for the solution of non-linear force-free magnetic field (FFF) equations is proposed. In this method, the mathematics is carried out in a special orthogonal system of coordinates in which one of the coordinates, say u_1, is taken along the (local) gradient of α, which is defined by the FFF relation $\alpha \equiv 4\pi \boldsymbol{J}/\boldsymbol{B}$ ($\boldsymbol{J}$ and $\boldsymbol{B}$ are the current and magnetic field, respectively). Thus, while generally $\alpha = \alpha(x_i)$ ($i = 1, 2, 3$), in the special system of coordinates used here, u_i ($i = 1, 2, 3$), taking the unit vector $\boldsymbol{e}_1$ as $\boldsymbol{e}_1 = \nabla\alpha/\|\nabla\alpha\|$, it follows that $\alpha = \alpha(u_1)$. Furthermore, from the FFF relation $\boldsymbol{B} \cdot \nabla\alpha = 0$, it also follows that $\boldsymbol{B}_1 = 0$ and $\boldsymbol{B} = \boldsymbol{B}_2 + \boldsymbol{B}_3$, where $\boldsymbol{B}_2$ and $\boldsymbol{B}_3$ are functions of u_1, u_2 and u_3. After the FFF equations are solved for B_2 and B_3 in this particular system of coordinates, upon transforming to a laboratory, say, cartesian system of coordinates one obtains the three component vector magnetic field as a function of α, where $\alpha = \alpha[u_1(x, y, z)]$. Thus, specification of α now provides the explicit FFF solutions of B_x, B_y, and B_z.

Several, relatively, simpler illustrative cases are considered in detail. As a particular case, the axisymmetric solution obtained by Low (1982) is recovered.

Key words: hydromagnetics – Sun (the) magnetic fields – stars: magnetic field – Sun: the corona

1. Introduction

The force-free magnetic field (FFF) model, in which the electrical current is parallel to the magnetic field (the Lorentz force is zero) represents a useful approximation to the magnetohydrodynamic description of many physical systems. Its use is adequate in the static cases characterized by low-beta values, $\beta \equiv 8\pi nkT/B^2 \ll 1$, when the effect of the pressure gradients and gravitational forces of the plasma trapped in the magnetic field can be neglected. Such situations may occur in both astrophysical and laboratory (fusion) plasma systems.

The FFF model is of great interest in the lower (solar) corona case, where situations with $\beta < 0.1$ may occur. Indeed, since only photospheric and low-chromospheric in-situ magnetic field observations can be carried out, the magnetic fields at larger heliocentric distances have to be inferred by theoretical methods.

Because of the mathematical complexity of the problem, historically, the *linear* FFF model in which the proportionality scalar function $\alpha(\boldsymbol{r})$ is taken to be space – independent was first considered ($\alpha(\boldsymbol{r}) = 4\pi \boldsymbol{J}/\boldsymbol{B}$) (see, e.g., *review papers* by Levine 1975; Low 1985; Gary 1988; also, Priest 1982).

The existing theoretical work on the *non-linear* FFF models falls into three categories:

Analytical work, applying to the cases in which the magnetic configurations are independent of one of the spatial coordinates and the solutions are given in terms of a generating function to be specified a posteriori (see, e.g. Low 1982; also Priest 1982). More specifically, in this model the field is taken to be symmetric about the axis of a spherical-coordinate system and lies on concentric spherical surfaces. The existence of a larger class of axisymmetric and nonaxisymmetric solutions has been pointed out by Chang & Carovillano (1981) and derived for the spherical case by Low (1988). The magnetic fields still lie on concentric spherical surfaces, however, the system is no longer symmetric about the axis of the spherical-coordinate system.

Numerical work, in which, starting from prescribed (or measured) boundary values at a surface ($z = 0$, say) representing the photosphere, one solves for the *vector* magnetic configuration in the half-space $z > 0$ (see, e.g., Sturrock & Woodbury 1967; Barnes & Sturrock 1972; Sakurai 1979, 1981; Yang et al. 1986; Wu et al. 1985, 1990; Cuperman et al. 1990).

Mixed analytical-numerical work. A significant extension of the axisymmetric solution of Low 1982 was recently achieved by Low & Lou (1989). Unlike the previously existing solution, the new one is such that the total energy in the half-space $z > 0$ is finite. Thus, working in spherical polar coordinates (with $\partial/\partial\phi = 0$), separating the dependence on the variables r and θ and choosing a convenient r-dependence for the field components, the authors obtained a second order, nonlinear ordinary differential equation for the function $P(\theta)$, describing the θ-dependence, which they solved numerically as an eigenvalue problem: $P(\theta = 0°) = P(\theta = 180°) = 0$.

In this work we develop an *analytical* method for solving nonlinear FFF model equations in the *non-axisymmetric case.* This is achieved by carrying out the mathematical calculations in an orthogonal system of coordinates in which one of the coordinates, say u_1, is taken along the (local) gradient of the scalar function $\alpha(\boldsymbol{r})$. Consequently, $\alpha = \alpha(\boldsymbol{u}_1)$ and $B_1 = 0$ and equations for only the field components B_2 and B_3 have to be solved. Non-axisymmetric solutions are obtained and discussed. As a particular case, the solution of Low (1982) is recovered.

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2. Solution of FFF model equations

2.1. General case

The steady state equations describing force-free magnetic field configurations are

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B} \tag{1}$$

and

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2}$$

Equation (1) states that the electric current density $\boldsymbol{J} = (4\pi)^{-1} \nabla \times \boldsymbol{B}$ is proportional to the magnetic field $\boldsymbol{B}$: $\boldsymbol{J} = \alpha \boldsymbol{B}/4\pi$. In the general case, the proportionality scalar quantity α is a function of the space coordinate, x_i $(i = 1, 2, 3)$. Upon taking the divergence of Eq. (1), by (2), one obtains

$$\boldsymbol{B} \cdot \nabla\alpha = 0, \tag{3}$$

which indicates the constancy of α along individual field lines, or, conversely, the orthogonality of the vector $\nabla\alpha$ and the magnetic field vector at any point in space. Then, in the domain in which $\nabla\alpha$ is an analytic function with a non-zero norm, we choose a local system of coordinates u_i $(i = 1, 2, 3)$ such that one of the axes, say u_1, is in direction of the vector $\nabla\alpha$. That is, we define the unit vector $\boldsymbol{e}_1$ by the equation

$$\boldsymbol{e}_1 \equiv \nabla\alpha / \| \nabla\alpha \| . \tag{4}$$

The other two unit vectors, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are taken to be orthogonal to $\boldsymbol{e}_1$ and to each other. Then, as mentioned above, the magnetic field vector reduces to[1]

$$\boldsymbol{B} = \boldsymbol{B}_2 + \boldsymbol{B}_3 \tag{5}$$

and lies in the plane orthogonal to $\boldsymbol{e}_1$ (i.e. the plane $u_1 = \text{constant}$).

Consider the new system of orthogonal coordinates (u_1, u_2, u_3) with by the Lamé coefficients (h_1, h_2, h_3), respectively. In these coordinates, by (3)–(5), Eqs. (2) and (1) provide the following relations, respectively:

$$\frac{\partial}{\partial u_2}(h_3 h_1 B_2) + \frac{\partial}{\partial u_3}(h_1 h_2 B_3) = 0, \tag{6}$$

$$\frac{\partial}{\partial u_2}(h_3 B_3) - \frac{\partial}{\partial u_3}(h_2 B_2) = 0, \tag{7}$$

$$-\frac{1}{h_1 h_3}\frac{\partial}{\partial u_1}(h_3 B_3) = \alpha B_2 \tag{8}$$

and

$$\frac{1}{h_1 h_2}\frac{\partial}{\partial u_1}(h_2 B_2) = \alpha B_3. \tag{9}$$

Thus, the solution of the initial Eqs. (1) and (2) is equivalent to the solution of the four equations (6)–(9), in which *only the magnetic field components B_2 and B_3 are present.*

For mathematical convenience we define the functions

$$\beta_2 \equiv h_2 B_2, \quad \beta_3 \equiv h_3 B_3 \tag{10}$$

and

$$H_2 \equiv h_1 h_3 / h_2, \quad H_3 \equiv h_1 h_2 / h_3. \tag{11}$$

By (10) and (11), Eqs. (6)–(9) become, respectively

$$\frac{\partial}{\partial u_2}(H_2 \beta_2) + \frac{\partial}{\partial u_3}(H_3 \beta_3) = 0, \tag{12}$$

$$\frac{\partial}{\partial u_2}(\beta_3) - \frac{\partial}{\partial u_3}(\beta_2) = 0, \tag{13}$$

$$-\frac{1}{H_2 \alpha}\frac{\partial}{\partial u_1}(\beta_3) = \beta_2 \tag{14}$$

and

$$\frac{1}{H_3 \alpha}\frac{\partial}{\partial u_1}(\beta_2) = \beta_3. \tag{15}$$

Inspection of Eqs. (12)–(15) reveals that Eqs. (12) and (13) contain only the derivatives of the field quantities β_2 and β_3 with respect to u_2 and u_3, while Eqs. (14) and (15) contain only the derivatives of the same quantities with respect to u_1. This suggests solving separately the set of Eqs. (12), (13) and the set of Eqs. (14), (15). For convenience, we denote by $\tilde{\beta}_2$ and $\tilde{\beta}_3$ the solutions of Eqs. (12), (13) and by $\hat{\beta}_2$ and $\hat{\beta}_3$ the solutions of Eqs. (14), (15).

2.1.1. Solution of the Eqs. (12) and (13) for $\tilde{\beta}_2$ and $\tilde{\beta}_3$

Inspection of Eq. (13) indicates that, provided that the derivatives of $\tilde{\beta}_2$ and $\tilde{\beta}_3$ are continuous functions of the coordinates u_2 and u_3, there exists a function F satisfying the relations

$$\tilde{\beta}_2 = \partial F / \partial u_2, \quad \tilde{\beta}_3 = \partial F / \partial u_3. \tag{16}$$

Then, substitution of Eqs. (16) into Eq. (12) provides the equation

$$\frac{\partial}{\partial u_2}\left(H_2 \frac{\partial}{\partial u_2} F\right) + \frac{\partial}{\partial u_3}\left(H_3 \frac{\partial}{\partial u_3} F\right) = 0 \tag{17}$$

or equivalently

$$\tilde{a} F_{22} + \tilde{c} F_{33} + \tilde{d} F_2 + \tilde{e} F_3 = 0 \tag{18}$$

where F_i and F_{ii} represent the first and second order derivatives of the function F with respect to the coordinate $u_i (i = 2, 3)$, respectively, and the quantities $\tilde{a} \ldots \tilde{e}$ are defined as follows:

$$\tilde{a} = H_2, \quad \tilde{c} = H_3 \quad \tilde{d} = \partial H_2 / \partial u_2, \quad \tilde{e} = \partial H_3 / \partial u_3. \tag{19}$$

Since, by (13)

$$\tilde{a}\tilde{c} = H_2 H_3 = h_1^2 > 0 \tag{20}$$

it follows that Eq. (18) is an elliptic, second order partial differential equation for the function F. The solution of Eq. (18) will be considered in the next section for various specific situations.

2.1.2. Solutions of Eqs. (14) and (15) for $\hat{\beta}_2$ and $\hat{\beta}_3$

Upon substituting the expression for $\hat{\beta}_3$ from Eq. (15) into Eq. (14) and the expression for $\hat{\beta}_2$ from Eq. (14) into Eq. (15) one obtains the following equations, respectively

$$-\frac{1}{H_2 \alpha}\frac{\partial}{\partial u_1}\left[\frac{1}{H_3 \alpha}\frac{\partial}{\partial u_1}(\hat{\beta}_2)\right] = \hat{\beta}_2 \tag{21}$$

[1] It should be mentioned that within the FFF model equations it is not possible to choose one of the axes of coordinates along the (local) vector magnetic field. Indeed, if $\boldsymbol{B} = \boldsymbol{B}_1$, say, Eq. (1) is satisfied only for the trivial case $B_1 = 0$.

and

$$-\frac{1}{H_3\alpha}\frac{\partial}{\partial u_1}\left[\frac{1}{H_2\alpha}\frac{\partial}{\partial u_1}(\hat\beta_3)\right]=\hat\beta_3. \tag{22}$$

Next, upon multiplying Eqs. (21) and (22) by $h_1^2\alpha^2$ (notice that $H_2\alpha\cdot H_3\alpha=h_1^2\alpha^2$), after some algebra, one brings them to the following form, respectively:

$$\frac{\partial^2}{\partial u_1^2}(\hat\beta_2)+\hat b_2\frac{\partial}{\partial u_1}(\hat\beta_2)+\hat c_2\hat\beta_2=0 \tag{23}$$

and

$$\frac{\partial^2}{\partial u_1^2}(\hat\beta_3)+\hat b_3\frac{\partial}{\partial u_1}(\hat\beta_3)+\hat c_3\hat\beta_3=0. \tag{24}$$

Here

$$\hat b_2=-\frac{\partial}{\partial u_1}[\ln(H_3\alpha)],\qquad \hat c_2=h_1^2\alpha^2 \tag{25}$$

and

$$\hat b_3=-\frac{\partial}{\partial u_1}[\ln(H_2\alpha)],\qquad \hat c_3=-\hat c_2. \tag{26}$$

Equations (23) and (24) are second order ordinary differential equations for $\hat\beta_2$ and $\hat\beta_3$, respectively. Unlike the Eq. (18) for $\tilde\beta_2$ and $\tilde\beta_3$, in which the coefficients $\tilde a \dots \tilde e$ are known functions of the Lamé coefficients corresponding to the particular system of coordinates chosen in Eqs. (23) and (24), the coefficients $\hat b_2$, $\hat b_3$, $\hat c_2$ and $\hat c_3$, are also dependent on the quantity $\alpha(u_1)$, the free function which determines the structure of the FFF magnetic configuration.

In the general case, Eqs. (23) and (24) can be solved, e.g., by using a series expansion (Frobenius' method). In this work, we will consider two classes of solutions namely

(i) $\beta_2(u_1, u_2, u_3)=\tilde\beta_2\cdot f(u_1)$

$\beta_3(u_1, u_2, u_3)=\tilde\beta_3\cdot f(u_1)$

and

(ii) $\beta_2(u_1, u_2, u_3)=\hat\beta_2\cdot g(u_2, u_3)$

$\beta_3(u_1, u_2, u_3)=\hat\beta_3\cdot g(u_2, u_3)$.

2.2. *A class of solutions for the case in which* $(\partial/\partial u_1)\,(h_2/h_3)=0$

A relatively simple analytical solution can be obtained in the special case in which the Lamé coefficients satisfy the condition

$$\frac{\partial}{\partial u_1}\left(\frac{h_2}{h_3}\right)=0. \tag{27}$$

In such a case, Eqs. (23) and (24) reduce to, respectively

$$\frac{\partial}{\partial u_1}\left[\frac{1}{h_1\alpha}\frac{\partial}{\partial u_1}(\hat\beta_2)\right]+h_1\alpha\hat\beta_2=0 \tag{28}$$

and

$$\frac{\partial}{\partial u_1}\left[\frac{1}{h_1\alpha}\frac{\partial}{\partial u_1}(\hat\beta_3)\right]+h_1\alpha\hat\beta_3=0. \tag{29}$$

Defining a function A by the relation

$$\alpha=\frac{1}{h_1}\frac{dA}{du_1} \tag{30}$$

we find the solutions of Eq. (28), and (29)

$$\hat\beta_2=b_1\cos A+b_2\sin A. \tag{31a}$$

and

$$\hat\beta_3=\frac{h_3}{h_2}(b_2\cos A-b_1\sin A). \tag{31b}$$

So far we considered separately the solution of the sets of Eqs. (12), (13) and (14), (15), respectively. This provided the set of solutions $\hat\beta_2$ and $\hat\beta_3$, and $\tilde\beta_2$ and $\tilde\beta_3$, respectively. The final step would be to obtain the combined consistent dependence of β_2 and β_3 on all three coordinates, $u_i, i=1, 2, 3$. This step will be carried out for the illustrative specific case considered in the next section.

Before proceeding with the consideration of particular illustrative cases, we observe that, independent of the specific situation (geometry, symmetry) discussed, if $F=F(u_2, u_3)$ and consequently $\partial\tilde\beta_2/\partial u_1=\partial\tilde\beta_3/\partial u_1=0$, there are no combined solutions of the type

$$\beta_2=\tilde\beta_2\cdot f(u_1),\qquad \beta_3=\tilde\beta_3\cdot f(u_1). \tag{32}$$

A mathematical proof of this statement is given in Appendix I.

3. Illustration of the method

3.1. *The case of spherical coordinates*[2]

In this case we take[3] the polar spherical coordinates (r, θ, ϕ) with

$$\boldsymbol{e}_1\equiv\nabla\alpha/\|\nabla\alpha\|=\boldsymbol{e}_r. \tag{4'}$$

Then, the magnetic field is

$$\boldsymbol{B}=B_\theta\boldsymbol{e}_\theta+B_\phi\boldsymbol{e}_\phi. \tag{5'}$$

Thus, the magnetic field vector lays on spherical surfaces described by $r=$constant values.

The Lamé's coefficients are $h_1=1$, $h_2=r$ and $h_3=r\sin\theta$ and the field functions, β_i, are

$$\beta_2\equiv rB_\theta,\qquad \beta_3\equiv r\sin\theta\, B_\phi \tag{10'}$$

and

$$H_2\equiv\sin\theta,\qquad H_3\equiv1/\sin\theta. \tag{11'}$$

The four equations to be solved now read, respectively,

$$\frac{\partial}{\partial\theta}(\sin\theta\,\tilde\beta_2)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\tilde\beta_3\right)=0 \tag{12'}$$

$$\frac{\partial}{\partial\theta}(\tilde\beta_3)-\frac{\partial}{\partial\phi}(\tilde\beta_2)=0 \tag{13'}$$

$$-\frac{1}{\alpha\sin\theta}\frac{\partial}{\partial r}(\hat\beta_3)=\hat\beta_2 \tag{14'}$$

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[2] The spherical coordinates are here chosen for illustration because of their familiarity. However, any other set of orthogonal unit vectors on the sphere $u_1=$const. can be equally well considered (for example, $\boldsymbol{e}_2=(\boldsymbol{e}_\theta+\boldsymbol{e}_\phi)/\sqrt{2}$, $\boldsymbol{e}_3=(\boldsymbol{e}_\theta-\boldsymbol{e}_\phi)/\sqrt{2}$)).

[3] For convenience, with the addition of the "prime" symbol, the numbering of the equations in this section is the same as that of the corresponding equations in Sect. 2.

and

$$\frac{\sin\theta}{\alpha}\frac{\partial}{\partial r}(\hat{\beta}_2)=\hat{\beta}_3. \tag{15'}$$

3.1.1. Solution of Eqs. (12′) and (13′)

The function $F(\theta,\phi)$ is related to $\tilde{\beta}_2$ and $\tilde{\beta}_3$ by the equations

$$\tilde{\beta}_2=\partial F/\partial\theta, \quad \tilde{\beta}_3=\partial F/\partial\phi. \tag{16'}$$

The equation for F then reads

$$\sin^2\theta\, F_{\theta\theta}+F_{\phi\phi}+\sin\theta\cos\theta\, F_\theta=0. \tag{18'}$$

Using the transformation

$$\eta=-\ln[\tan(\theta/2)] \tag{33}$$

one can bring Eq. (18′) to the equivalent form

$$F_{\eta\eta}+F_{\phi\phi}=0. \tag{34}$$

This is Laplace equation in the canonic form. Thus, we solve it by the method of separation of variables by assuming

$$F=H(\eta)\cdot\Phi(\phi) \tag{35}$$

By (35), Eq. (34) becomes

$$\frac{H_{\eta\eta}}{H}+\frac{\Phi_{\phi\phi}}{\Phi}=0 \tag{36}$$

where $H_{\eta\eta}$ and $\Phi_{\phi\phi}$ represent the second order derivatives of H and Φ with respect to η and ϕ, respectively.

Denoting

$$\Phi_{\phi\phi}/\Phi=-n^2 \tag{37}$$

and considering the boundary condition $\Phi(\phi)=\Phi(\phi+2\pi)$ one obtains the solution of Eq. (37), namely

$$\Phi=a_{n1}\cos(n\phi)+a_{n2}\sin(n\phi) \tag{38}$$

Next, by (36) and (37) one obtains the equation for H, namely

$$\frac{H_{\eta\eta}}{H}=n^2. \tag{39}$$

The solution of Eq. (39) is

$$H=a_{n3}\exp(n\eta)+a_{n4}\exp(-n\eta)$$
$$=a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}. \tag{40}$$

In the last part of Eq. (40), use of the transformation (33) was made.

Collecting the results, Eqs. (35), (38) and (40) one obtains the complete solution of Eq. (34), namely

$$F_n=(a_{n1}\cos n\phi+a_{n2}\sin n\phi)\left(a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\,tan^n\frac{\theta}{2}\right). \tag{41}$$

Then, by (16′) and (41) one obtains

$$\tilde{\beta}_{n2}(\theta,\phi)=(n/\sin\theta)(a_{n1}\cos n\phi+a_{n2}\sin n\phi)$$
$$\times\left(-a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right) \tag{42}$$

and

$$\tilde{\beta}_{n3}(\theta,\phi)=n(-a_{n1}\sin n\phi+a_{n2}\cos n\phi)$$
$$\times\left(a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right). \tag{43}$$

3.1.2. Solutions of Eqs. (14′) and (15′)

Since the condition (27) is satisfied, the solution of Eqs. (14′)–(15′) are

$$\hat{\beta}_2=b_1\cos A+b_2\sin A \tag{44a}$$

and

$$\hat{\beta}_3=(b_2\cos A-b_1\sin A)\sin\theta \tag{44b}$$

where the function A, defined by Eq. (30) is related to α by the equation

$$\alpha=dA/dr. \tag{30'}$$

3.1.3. Solutions $\beta_j(u_i)$, $j=2, 3$; $i=1, 2, 3$

Following the discussion at the end of Sect. 3 we consider the following class of solutions:

$$\beta_2=\hat{\beta}_2\cdot g(\theta,\phi), \quad \beta_3=\hat{\beta}_3\cdot g(\theta,\phi). \tag{45}$$

Notice that the function $g(\theta,\phi)$ in Eq. (45) is the same for both components β_2 and β_3.

Upon substituting the expressions (45) with $\hat{\beta}_2$ and $\hat{\beta}_3$ given by Eqs. (31a) and (31b) into Eqs. (12′) and (13′) one obtains, respectively

$$\frac{\partial}{\partial\phi}[g\sin\theta(b_1\cos A+b_2\sin A)]+\frac{\partial}{\partial\phi}[g(b_2\cos A-b_1\sin A)]=0 \tag{46a}$$

and

$$\frac{\partial}{\partial\phi}[g\sin\theta(b_2\cos A-b_1\sin A)]-\frac{\partial}{\partial\phi}[g(b_1\cos A+b_2\sin A)]=0. \tag{46b}$$

3.1.4. Axisymmetric case–recovery of Low (1982) solution

We start by observing that the r-independent integration constants $b_1(\theta,\phi)$ and $b_2(\theta,\phi)$ in Eqs. (46a) and (46b) are arbitrary and that α in Eq. (30) is defined up to an additive constant. To recover the axisymmetric case of Low (1982), (i) we consider b_1 and b_2 to be also θ and ϕ *independent* and (ii) take the additive constant such as to result in one of the integration constants, say b_2, being zero.

We proceed with step (i). Taking b_1 and b_2 to be coordinate independent and equating the coefficients of $\cos A$ and $\sin A$ appearing in the left and the right sides of Eqs. (46a) and (46b), one obtains the following set of equations:

$$b_1\frac{\partial}{\partial\theta}(g\sin\theta)+b_2\frac{\partial g}{\partial\phi}=0 \tag{47a}$$

$$b_2\frac{\partial}{\partial\theta}(g\sin\theta)-b_1\frac{\partial g}{\partial\phi}=0. \tag{47b}$$

From (47a) and (47b) one easily finds

$$(b_1^2+b_2^2)\frac{\partial}{\partial\theta}(g\sin\theta)=0 \quad (48)$$

and

$$(b_1^2+b_2^2)\frac{\partial}{\partial\phi}(g)=0. \quad (49)$$

Equation (49) indicates that $g=g(\theta)$. Thus, by inspection, the solution of Eq. (48) is

$$g=1/\sin\theta. \quad (50)$$

Thus, by (45), (31a), (31b′) and (50) and using also the definitions of β_2 and β_3 (Eq. (10′)), one obtains the final solutions

$$B_\theta=\frac{1}{r\sin\theta}(b_1\cos A+b_2\sin A) \quad (51)$$

and

$$B_\phi=\frac{1}{r\sin\theta}(b_2\cos A-b_1\sin A). \quad (52)$$

We now proceed with step (ii). As discussed above, we take $b_2=0$ and obtain

$$B_\theta=\frac{1}{r\sin\theta}b_1\cos A, \qquad B_\phi=-\frac{1}{r\sin\theta}\sin A. \quad (53)$$

The remaining constant, b_1, can be determined from the condition of matching the observed photospheric magnetic field, the photosphere being situated at a (radial) distance "a" from the origin:

$$B_\theta(r=a,\ \theta=90°)=B_0\cos A(a). \quad (54)$$

Then, from (51) and (54) one obtains

$$b_1=aB_0. \quad (55)$$

Finally, using the notation $\hat{\phi}=-A$, i.e., changing the definition (30′) to

$$\alpha=-\partial\hat{\phi}/\partial r \quad (30'')$$

we can write Eq. (53) as

$$B=\frac{aB_0}{r\sin\theta}[\cos\hat{\phi}(r)\cdot e_\theta+\sin\hat{\phi}(r)\cdot e_\phi] \quad (56)$$

which is just Eq. (9) of Low (1982). The transformation from the spherical to the Cartesian coordinates provides the magnetic field components B_x, B_y and B_z (Eqs. (11)–(13) in Low 1982).

3.1.5. Non-axisymmetric case

We now regard the quantities b_1 and b_2 as functions of θ and ϕ (but not on r). Equating the coefficients of $\cos A$ and $\sin A$ in Eq. (46) provides, respectively

$$\frac{\partial}{\partial\theta}(b_1\sin\theta)+\frac{\partial}{\partial\phi}(b_2)=0 \quad (57a)$$

and

$$\frac{\partial}{\partial\theta}(b_2\sin\theta)-\frac{\partial}{\partial\phi}(b_1)=0. \quad (57b)$$

Comparison of Eqs. (57a) and (57b) with Eqs. (12′) and (13′) indicates the following relationships to hold

$$b_1=\frac{\partial F}{\partial\theta}, \qquad b_2=\frac{1}{\sin\theta}\frac{\partial F}{\partial\phi}. \quad (58)$$

By (58) and (41) one obtains

$$b_{n1}=\frac{n}{\sin\theta}(a_{n1}\cos n\phi+a_{n2}\sin n\phi)\left(-a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right) \quad (59)$$

and

$$b_{n2}=\frac{n}{\sin\theta}(-a_{n1}\sin n\phi+a_{n2}\cos n\phi)\left(a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right). \quad (60)$$

Finally, by (44a), (44b), (10′), (59) and (60) one obtains the sought solutions, namely

$$B_\theta=\frac{1}{r\sin\theta}\left\{\left[a_0+\sum_{n=1}^{\infty}n(a_{n1}\cos n\phi+a_{n2}\sin n\phi)\times\left(-a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right)\right]\cos A+\sum_{n=1}^{\infty}n(-a_{n1}\sin n\phi+a_{n2}\cos n\phi)\times\left(a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right)\sin A\right\} \quad (61)$$

and

$$B_\phi=\frac{1}{r\sin\theta}\left\{\sum_{n=1}^{\infty}n(-a_{n1}\sin n\phi+a_{n2}\cos n\phi)\times\left(a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right)\cos A-\left[a_0+\sum_{n=1}^{\infty}n(a_{n1}\cos n\phi+a_{n2}\sin n\phi)\times\left(-a_{n3}\cot^n\frac{\theta}{2}+a_{n4}\tan^n\frac{\theta}{2}\right)\right]\sin A\right\}. \quad (62)$$

In Eqs. (61) and (62) the coefficients a_0, $a_{n1}\ldots a_{n4}$ $(n=1, 2\ldots)$ are real arbitrary constants; θ, ϕ-the polar coordinates; and A-the free function determining the FFF-characterizing-function α (see Eq. (30)). If $a_{ni}=0$, Eqs. (61) and (62) reduce the axisymmetric case, Eqs. (51) and (52).

Writing Eqs. (61) and (62) in the form

$$B_\theta=\frac{1}{r\sin\theta}(b_1\cos A+b_2\sin A) \quad (61')$$

$$B_\phi=\frac{1}{r\sin\theta}(b_2\cos A-b_1\sin A) \quad (62')$$

it can be easily found that the energy density $E\equiv(1/8\pi)\,(B_\theta^2+B_\phi^2)$ is

$$E=\frac{1}{8\pi}\left(\frac{1}{r\sin\theta}\right)^2(b_1^2+b_2^2)$$

where b_1^2 and b_2^2 are functions of θ and ϕ. Thus, E decreases with distance as $1/r^2$.

3.2. The case of general cylindrical coordinates

Consider a cylindrical system of coordinates (u_1, u_2, z) with $\alpha=\alpha(z)$ and choose[4]

$$e_3=\nabla\alpha/\|\nabla\alpha\|=e_z. \tag{4a}$$

Thus, e_1 and e_2 are unit vectors orthogonal to e_3 and to each other. The magnetic field vector is

$$\boldsymbol{B}=\boldsymbol{B}_1 e_1+\boldsymbol{B}_2 e_2 \tag{5a}$$

and lies on surfaces described by $z=$constant values. The following equations hold

$$\frac{\partial}{\partial u_1}(h_2h_3B_1)+\frac{\partial}{\partial u_2}(h_1h_3B_2)=0, \tag{6a}$$

$$\frac{\partial}{\partial u_1}(h_2B_2)-\frac{\partial}{\partial u_2}(h_1B_1)=0, \tag{7a}$$

$$-\frac{1}{h_2h_3}\frac{\partial}{\partial u_3}(h_2B_2)=\alpha B_1 \tag{8a}$$

and

$$\frac{1}{h_1h_3}\frac{\partial}{\partial u_3}(h_1B_1)=\alpha B_2. \tag{9a}$$

Define

$$\beta_1\equiv h_1B_1,\quad \beta_2\equiv h_2B_2 \tag{10a}$$

and

$$H_1\equiv h_2h_3/h_1,\quad H_2\equiv h_1h_3/h_2. \tag{11a}$$

Next substitute (10a) and (11a) into (6a)–(9a) to obtain, respectively

$$\frac{\partial}{\partial u_1}(H_1\beta_1)+\frac{\partial}{\partial u_2}(H_2\beta_2)=0, \tag{12a}$$

$$\frac{\partial}{\partial u_1}(\beta_2)-\frac{\partial}{\partial u_2}(\beta_1)=0, \tag{13a}$$

$$-\frac{1}{H_1\alpha}\frac{\partial}{\partial u_3}\beta_2=\beta_1 \tag{14a}$$

and

$$\frac{1}{H_2\alpha}\frac{\partial}{\partial u_3}\beta_1=\beta_2. \tag{15a}$$

For convenience, consider the circular cylindrical coordinates system (ρ, ϕ, z), with $h_1=1$, $h_2=\rho$ and $h_3=1$. Then[5]

$$e_3=e_z \tag{4''}$$

$$\boldsymbol{B}=B_\rho e_\rho+B_\phi e_\phi, \tag{5''}$$

$$\beta_1\equiv B_\rho,\quad \beta_2\equiv\rho B_\phi, \tag{10''}$$

and

$$H_1\equiv\rho,\quad H_2\equiv 1/\rho. \tag{11''}$$

[4] Comparison with Eqs. (4)–(15) indicates a cyclic permutation of the subscripts: 1, 2, 3→3, 1, 2

[5] By analogy with the numbering in Sect. IIIA, for convenience, we now use (4″), (5″) etc.

3.2.1. Solutions of the type $\tilde{\beta}_2\cdot f(z)$, $\tilde{\beta}_3\cdot f(z)$

In the present case, Eqs. (12a) and (13a) become

$$\frac{\partial}{\partial\rho}(\rho\beta_1)+\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\beta_2\right)=0 \tag{12''}$$

and

$$\frac{\partial}{\partial\rho}(\beta_2)-\frac{\partial}{\partial\phi}(\beta_1)=0. \tag{13''}$$

The equations corresponding to Eqs. (16) of Sect. III are

$$\tilde{\beta}_1=\partial F/\partial\rho,\quad \tilde{\beta}_2=\partial F/\partial\phi$$

and the equation for the function F is

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}F\right)+\frac{\partial^2}{\partial\phi^2}F=0. \tag{18''}$$

As shown in the Appendix, since F is independent of z, no solution of this type exists.

3.2.2. Solution of the type $\hat{\beta}_1 g(\rho,\phi)$, $\hat{\beta}_2 g(\rho,\phi)$

The equations corresponding to (14) and (15) read

$$-\frac{1}{\rho\alpha}\frac{\partial}{\partial z}(\beta_2)=\beta_1 \tag{14''}$$

and

$$\frac{\rho}{\alpha}\frac{\partial}{\partial z}(\beta_1)=\beta_2. \tag{15''}$$

Since in this case one has $(\partial/\partial z)(h_1/h_2)=0$ (this is the equivalent of Eq. (27) for the spherical coordinates case), the solutions for $\hat{\beta}_1$ and $\hat{\beta}_2$ are

$$\hat{\beta}_1=b_1\cos A+b_2\sin A \tag{31a''}$$

and

$$\hat{\beta}_2=\rho(b_2\cos A-b_1\sin A)$$

with

$$\alpha\equiv dA/dz. \tag{30''}$$

Upon substituting $\hat{\beta}_1 g$ and $\hat{\beta}_2 g$ into Eqs. (12″) and (13″) one obtains

$$\frac{\partial}{\partial\rho}[\rho g(b_1\cos A+b_2\sin A)]+\frac{\partial}{\partial\phi}[(b_2\cos A-b_1\sin A)]=0 \tag{46a''}$$

and

$$\frac{\partial}{\partial\rho}[\rho g(b_2\cos A-b_1\sin A)]-\frac{\partial}{\partial\phi}[(b_1\cos A+b_2\sin A)]=0. \tag{46b''}$$

Equating to zero the coefficients of $\cos A$ and $\sin A$ in (46a″) and (46b″) provides, respectively

$$b_1\frac{\partial}{\partial\rho}(\rho g)+b_2\frac{\partial}{\partial\phi}g=0 \tag{47a''}$$

and

$$b_2\frac{\partial}{\partial\rho}(\rho g)-b_1\frac{\partial}{\partial\phi}g=0. \tag{47b''}$$

Thus, one obtains

$$(b_1^2+b_2^2)\frac{\partial}{\partial\rho}(\rho g)=0 \tag{48''}$$

and

$$(b_1^2+b_2^2)\frac{\partial}{\partial\phi}g=0. \tag{49''}$$

This means

$$g=\frac{1}{\rho} \tag{50''}$$

and, therefore

$$B_\rho=(b_1\cos A+b_2\sin A)/\rho \tag{51''}$$

and

$$B_\phi=(b_2\cos A-b_1\sin A)/\rho. \tag{52''}$$

3.2.3. Non-axisymmetric solutions of the type $\hat{\beta}_1$ and $\hat{\beta}_2$

Taking $b_1=b_1(\rho,\phi)$ and $b_2=b_2(\rho,\phi)$ in Eqs. (31a″) and (31b″), substituting the resulting equations in Eqs. (12″) and (13″) and equating to zero the coefficients of $\cos A$ and $\sin A$ lead to the results

$$\frac{\partial}{\partial\rho}(\rho b_1)+\frac{\partial}{\partial\phi}b_2=0 \tag{57a''}$$

and

$$\frac{\partial}{\partial\rho}(\rho b_2)-\frac{\partial}{\partial\phi}b_1=0. \tag{57b''}$$

Define the function F by the relations

$$\rho b_2=\frac{\partial}{\partial\phi}F,\quad b_1=\frac{\partial}{\partial\rho}F. \tag{58''}$$

Upon substituting (58a″) in (57a″) provides

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}F\right)+\frac{\partial^2}{\partial\phi^2}F=0. \tag{63}$$

This equation can be solved by separation of variables and by using the boundary condition $F(\phi)=F(\phi+2\pi)$. One obtains

$$F=a_0\ln\rho+\sum_{n=1}^{\infty}(a_{n1}\cos n\phi+a_{n2}\sin n\phi)(a_{n3}\rho^n+a_{n4}\rho^{-n}). \tag{64}$$

By (51″) and (64) one obtains

$$b_1=\frac{a_0}{\rho}+\sum_{n=1} n(a_{n1}\cos n\phi+a_{n2}\sin n\phi)(a_{n3}\rho^{n-1}-a_{n4}\rho^{-n-1}) \tag{59''}$$

and

$$b_2^2=\sum_{n=1}^{\infty} n(-a_{n1}\sin n\phi+a_{n2}\cos n\phi)(a_{n3}\rho^{n-1}+a_{n4}\rho^{-n-1}). \tag{60''}$$

Finally, upon substituting these results into Eq. (31″) gives

$$B_\rho=\left[a_0/\rho+\sum_{n=1}^{\infty} n(a_{n1}\cos n\phi+a_{n2}\sin n\phi)\times(a_{n3}\rho^{n-1}-a_{n4}\rho^{-n-1})\right]\cos A+\left[\sum_{n=1}^{\infty} n(-a_{n1}\sin n\phi+a_{n2}\cos n\phi)\times(a_{n3}\rho^{n-1}+a_{n4}\rho^{-n-1})\right]\sin A \tag{61''}$$

and

$$B_\phi=\left[\sum_{n=1}^{\infty} n(-a_{n1}\sin n\phi+a_{n2}\cos n\phi)\times(a_{n3}\rho^{n-1}+a_{n4}\rho^{-n-1})\right]\cos A+\left[a_0/\rho+\sum_{n=1}^{\infty} n(a_{n1}\cos n\phi+a_{n2}\sin n\phi)\times(a_{n3}\rho^{n-1}-a_{n4}\rho^{-n-1})\right]\sin A. \tag{62''}$$

Since the properties of the FFF magnetic configuration are determined by the function α (which in the present case is $\alpha=\alpha(z)$) and not by the arbitrary choice of any particular system of coordinates, the solutions (61″) and (62″) hold for any other representation in the plane $z=$constant; for example, Cartesian coordinates, parabolic cylindrical coordinates, bipolar coordinates, etc.

4. Discussion

Figures 1–3 illustrate graphically some of the results obtained in this paper. For convenience, we choose $A(r)=\pi r^2$, from which, by (31), one obtains $\alpha=2\pi r$ (See Eq. (1)). Also, we normalize the radial distance by defining $\bar{r}=r/r_0$, where r_0 is a characteristic length of the system (for example, the solar radius). For the magnetic field components we use the normalization $\bar{B}_j=B_j(r_0/a_{ni})$ $(j=\theta,\phi)$ where a_{ni} is one of the non-zero coefficients appearing in Eqs. (61) and (62). Thus, based on Eqs. (61) and (62) we obtain the following specific illustrations:

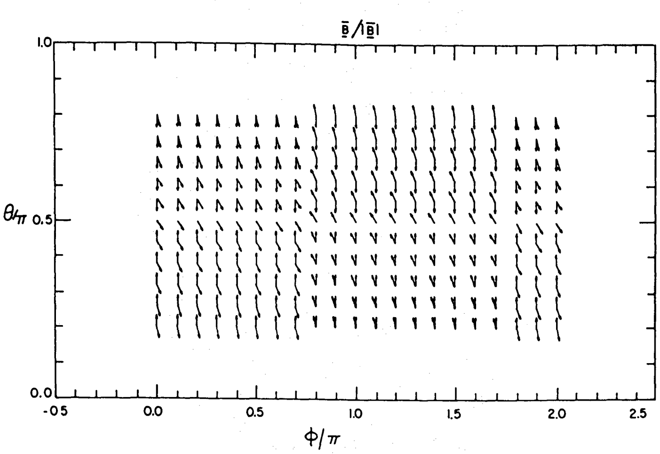

Fig. 1. Unit vector magnetic field, $\boldsymbol{B}/|\boldsymbol{B}|$ in the (θ,ϕ) plane at $\bar{r}=1$, as given by the analytical solutions Eqs. (61) and (62). The vertical arrows represent the axisymmetric case with $a_0=1$ and $a_{ni}=0$ $(n, i=1, 2, 3 - - -)$. The oblique arrows correspond to a non-axisymmetric situation for which $a_0=0$, $a_{11}=a_{12}=a_{13}=a_{14}=1$ and $a_{ni}=0$ $(n>1, i=1, 2, 3\ldots)$

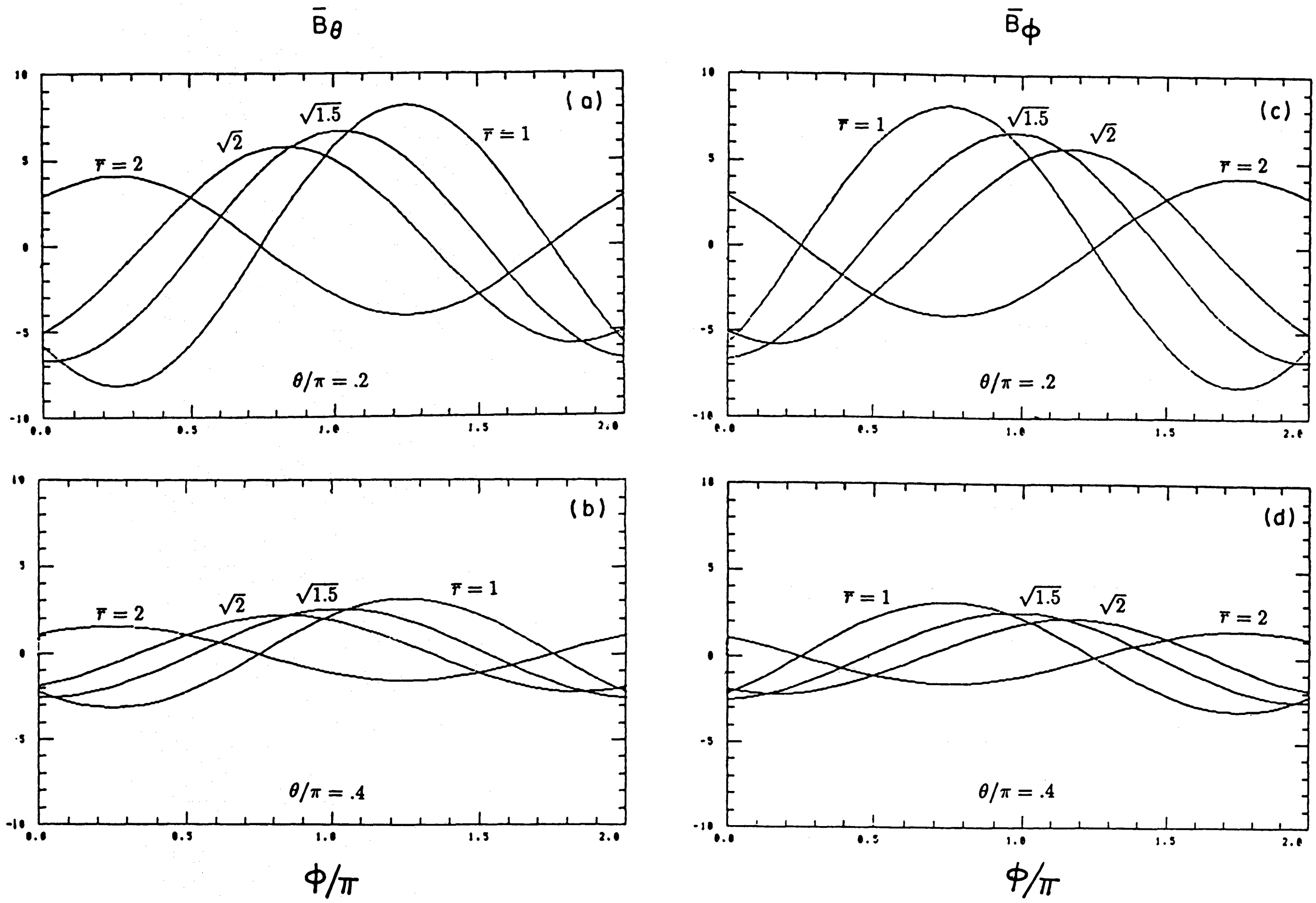

Fig. 2a–d. The magnetic field components $\bar{B}_\theta$ and $\bar{B}_\phi$ for the non-axisymmetric case and the parameters considered in Fig. 1, at four heights labelled $\bar{r}=1, \sqrt{1.5}, \sqrt{2}$ and 2. **a** Dependence of $\bar{B}_\theta$ on ϕ for fixed θ-value, namely $\phi/\pi=0.2$. **b** Same as in **a**, for $\theta/\pi=0.4$. **c** Dependence of $\bar{B}_\phi$ for $\theta/\pi=0.2$; **d** Same as in **c**, for $\theta/\pi=0.4$

Figure 1 gives the unit magnetic field vector, $\hat{\boldsymbol{B}}\equiv \boldsymbol{B}/|\boldsymbol{B}|$, $\boldsymbol{B}=\boldsymbol{B}_\theta+\boldsymbol{B}_\phi$, on the θ, ϕ surface at, $\bar{r}=1$.[6] The vertical arrows represent the axisymmetric case, $a_0=1$, $a_{ni}=0$ ($n, i=1, 2, 3 \ldots$) and consequently $\bar{B}_j=B_j r_0$. The oblique arrows correspond to a non-axisymmetric situation for which we (arbitrarily) chose $a_0=0$, $a_{11}=a_{12}=a_{13}=a_{14}=1$, and $a_{ni}=0$ for $n>1$, $i=1, 2, 3 \ldots$; then, again $\bar{B}_j=B_j r_0$. As it is seen, in the axisymmetric case, besides being independent on ϕ, the magnetic field vector has opposite signs in the domains $\theta<\pi/2$ and $\theta>\pi/2$ and vanishes at $\theta=\pi/2$. The situation is much more complex in the non-axisymmetric case – here, the direction (as well as magnitude) of the magnetic field depends on both θ and ϕ and *as a rule is quite different from that of the axisymmetric case: angle – differences of up to 180° can be found!*

Figure 2 gives the magnetic field components B_θ and B_ϕ at four height values ($\bar{r}=1, \sqrt{1.5}, \sqrt{2}$ and 2) for the non-axisymmetric case and the parameters considered in Fig. 1. For illustration, in Fig. 2a we present the dependence of B_θ on ϕ ($0\leq\phi\leq 2\pi$) for a fixed θ-value, namely $\theta/\pi=0.2$ (the same results hold for $\theta/\pi=0.8$). Figure 2b gives the same type of information for $\theta/\pi=0.4$ (or 0.6). Figure 2c presents the dependence of B_ϕ on ϕ for $\theta/\pi=0.2$ (or 0.8). Figure 2d differs form Fig. 2c by the value of θ – here $\theta/\pi=0.4$ (or 0.6).

Finally, Fig. 3 gives the θ-dependence of the field component $\bar{B}_\phi$ for fixed ϕ-values, at the heights and non-axisymmetric parameters indicated in the caption to Fig. 2. As it is seen, the component B_ϕ exhibits a cyclic behavior, as a function of ϕ. Also, referring to Eqs. (61′) and (62′) one sees that $\tilde{B}_\theta$ differs from $\tilde{B}_\phi$ by a phase $\pi/2$, entering through the functions $\sin A(r)$ and $\cos A(r)$.

To summarize, in this work we developed an analytical method for the solution of non-axisymmetric, nonlinear force-free magnetic field equations. This achievement was possible by carrying out the mathematics in a reference system in which one of the coordinates is taken along the local gradient of $\alpha(\boldsymbol{r})$, the function characterizing the FFF configuration. That is, we took $\boldsymbol{e}_1=\nabla\alpha/\|\nabla\alpha\|$, which results in $\boldsymbol{B}_1=0$, and only solutions for the other two components B_2 and B_3 have to be determined. Thus, after B_2 and B_3 are found in a conveniently chosen system of coordinates (type of coordinates as well as orientation of the axes) upon transforming to another, e.g. carthesian, reference system one obtains a three component vector magnetic field representation (B_x, B_y, B_z in the cartesian case; this corresponds to the quantities observed by spectroscopic means at the photospheric level).

[6] From Eqs. (61) and (62) it follows that the unit vector $\hat{\boldsymbol{B}}$ is characterized by the following periodicity properties: $\hat{\boldsymbol{B}}(A)=\hat{\boldsymbol{B}}(A+2\pi n)$, $n=0, \pm 1, 2 \ldots$ and $\hat{\boldsymbol{B}}(A+\pi)=-\hat{\boldsymbol{B}}(A)$

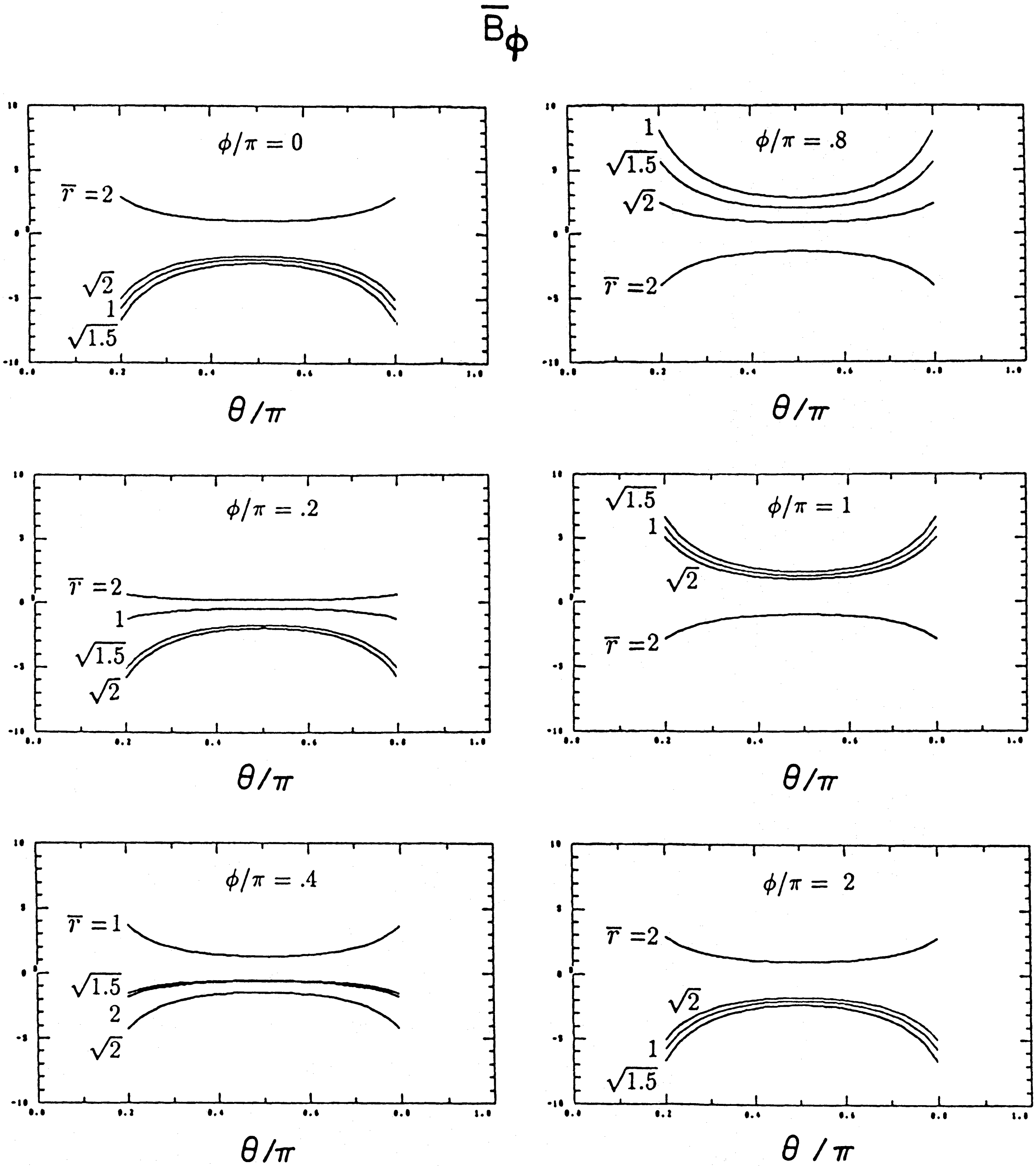

Fig. 3. θ-dependence of the field component $\bar{B}_\phi$ for the non-axisymmetric case and the parameters considered in Figs. 1 and 2, at four heights labelled $\bar{r}=1$, $\sqrt{1.5}$, $\sqrt{2}$ and 2

To obtain relatively simple analytical expressions for the solutions, we considered the case in which $(\partial/\partial u_1)(h_2/h_3)=0$, where h_2 and h_3 are the Lamé coefficients and then applied the formalism to the spherical system of coordinates. Choosing $\alpha(\boldsymbol{r}) = \alpha(r)$, i.e. taking $\boldsymbol{e}_1 = \boldsymbol{e}_r = \nabla\alpha / \|\nabla\alpha\|$, resulted in $B_r = 0$, $B_\theta \neq 0$, $B_\phi \neq 0$. A different choice of the functional dependence of α, say $\alpha = \alpha(\theta)$ or $\alpha = \alpha(\phi)$ (or a combination of r, θ, ϕ) would result into $B_r \neq 0$, etc. As a particular case, these results reduce to the Low (1982) solutions, holding for the axisymmetric $(\partial/\partial\phi = 0)$ case.

We also applied the formalism developed here to the case of general cylindrical coordinates with $\alpha = \alpha(z)$ and $\boldsymbol{B}$ laying in the plane $z = \text{constant}$.

The analytical solutions obtainable by the method developed in this work can be used, under appropriate physical conditions (see Introduction) (i) to obtain the magnetic field pattern of certain astrophysical or laboratory situations; and (ii) to provide tests cases for the numerical methods aimed to extrapolate the magnetic fields above solar (and, more general, stellar) active regions, using as boundary conditions the spectroscopically observed three component photospheric magnetic fields.

Appendix I

We here prove the statement on the inexistence of solutions of the type represented by Eq. (32).

Since by definition $F=F(u_2, u_3)$ it follows

$$\partial F/\partial u_1=0 \tag{A1}$$

and, therefore

$$\frac{\partial\tilde{\beta}_2}{\partial u_1}=\frac{\partial\tilde{\beta}_3}{\partial u_1}=0. \tag{A2}$$

Then, by Eqs. (14), (15), (32), (A1) and (A2), after some algebra, one obtains

$$(1/f)\,(\partial f/\partial u_1)=-(\tilde{\beta}_2/\tilde{\beta}_3)\,H_2\alpha \tag{A3}$$

and

$$(1/f)\,(\partial f/\partial u_1)=-(\tilde{\beta}_3/\tilde{\beta}_2)\,H_3\alpha \tag{A4}$$

or, equivalently

$$\tilde{\beta}_2=\pm ik\,\tilde{\beta}_3 \tag{A5}$$

where $k\equiv h_2/h_3$ is a real function. Denote

$$\tilde{\beta}_2(u_2, u_3)=\tilde{\beta}_{2,R}(u_2, u_3)+i\tilde{\beta}_{2,I}(u_2, u_3) \tag{A6}$$

and

$$f(u_1)=f_R(u_1)+if_I(u_1). \tag{A7}$$

In (A6) and (A7), the subscripts R and I stand for real and imaginary, respectively. Thus, the quantities $\tilde{\beta}_{2,R}$, $\tilde{\beta}_{2,I}$, f_R and f_I are real functions.

From (A5) and (A6) one has

$$\tilde{\beta}_3=-(\pm)k(\tilde{\beta}_{2,I}+i\tilde{\beta}_{2,R}). \tag{A8}$$

From (32) and (A6)–(A8) it follows

$$\beta_2=\tilde{\beta}_2 f=(\tilde{\beta}_{2,R}f_R-\tilde{\beta}_{2,I}f_I)+i(\tilde{\beta}_{2,R}f_I+\tilde{\beta}_{2,I}f_R) \tag{A9}$$

and

$$\beta_3=\tilde{\beta}_3 f=\pm k[-(\tilde{\beta}_{2,I}f_R+\tilde{\beta}_{2,R}f_I)+i(\tilde{\beta}_{2,R}f_R-\tilde{\beta}_{2,I}f_I). \tag{A10}$$

By definition, the field components B_2 and B_3 (and correspondingly, the quantities β_2 and β_3) are real functions; it follows

$$\tilde{\beta}_{2,R}f_I+\tilde{\beta}_{2,I}f_R=0 \tag{A11}$$

and

$$\tilde{\beta}_{2,R}f_R-\tilde{\beta}_{2,I}f_I=0. \tag{A12}$$

Assume $\tilde{\beta}_{2,R}\neq 0$ and $\tilde{\beta}_{2,I}\neq 0$ and divide (A11) by (A12) to obtain

$$f_I/f_R=-f_R/f_I \tag{A13}$$

or

$$f_I^2=-f_R^2. \tag{A14}$$

The Eq. (A14) is satisfied only if $f_R=f_I=0$ which implies $\beta_2=\beta_3=0$ for all u_1, u_2 and u_3 values.

Alternatively, assume $\tilde{\beta}_{2,I}=0$. Then, from (A11) and (A12) one has

$$\tilde{\beta}_{2,R}f_R=\tilde{\beta}_{2,R}f_I=0. \tag{A15}$$

The Eq. (A15) is satisfied only if $f_I=f_R=0$ or $\tilde{\beta}_{2,R}=0$. By (32) and (A5), one obtains in both cases $\beta_2=\beta_3=0$. A similar result is obtained if the assumption $\tilde{\beta}_{2,R}=0$ is made.

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